MATH 52 SAMPLE MIDTERM II

November 13, 2004

Name: ______________________________

Numeric Student ID: ____________________

Instructor's Name: ______________________

I agree to abide by the terms of the honor code:

Signature: ____________________________

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 5 questions. You have 120 minutes to do all the problems.

Question	Score	Maximum
1		20
2		10
3		10
4		10
5		10
Total		60

1. Evaluate the following line integrals:

   (a)
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$
   where $\mathbf{F}(x,y,z) = \langle xy, yz, xz \rangle$ and $C : \mathbf{r}(u) = \langle u, u^2, u^3 \rangle$ from $(-1,1,-1)$ to $(1,1,1)$.

Solution:
According to the given parametrization,
$$\mathbf{F}(\mathbf{r}(u)) = \langle u^3, u^5, u^4 \rangle.$$
Hence
$$\mathbf{F} \cdot d\mathbf{r} = u^3 + 5u^6$$
and we integrate from $u = -1$ to $u = 1$. This gives $10/7$.

(b)

$$\int_C \mathbf{F} \cdot \mathbf{T} ds$$

where $\mathbf{F}(x, y, z) = \langle \cos x, \sin y, yz \rangle$ and C is the line segment from $(0, 0, 0)$ to $(2, 3, -1)$.

Solution:

Again this is another notation for the integral of $\mathbf{F} \cdot d\mathbf{r}$, the work done over the curve C by $\mathbf{F}$. Here we are not given the parametrization, so we must determine one for the line from $(0, 0, 0)$ to $(2, 3, -1)$. There are lots of ways we can do this. Remember that a line in space is given by a point P_0 on the line and a direction vector $\mathbf{v}$ so we may write

$$x(t) = 0 + 2t, y(t) = 0 + 3t, z(t) = 0 - t$$

In general, given any two points (a_0, b_0, c_0) and (a_1, b_1, c_1) we can parametrize the line between them by

$$x(t) = a_1 t + a_0(1 - t), y(t) = b_1 t + b_0(1 - t), z(t) = c_1 t + c_0(1 - t)$$

You might think about why this is true given the equation for a line in terms of a point and a vector.

In any case,

$$\mathbf{F}(\mathbf{r}(t)) = \langle \cos 2t, \sin 3t, -3t^2 \rangle$$

and

$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = 2 \cos 2t + 3 \sin 3t + 3t^2$$

and so the integral, now from $t = 0$ to $t = 1$ according to the parametrization, is $2 + \sin 2 - \cos 3$.

(c) Compute the work done by

$$\mathbf{F}(x, y, z) = \langle 2x \ln y - yz, \frac{x^2}{y} - xz, -xy \rangle$$

along the line segment from $(1, 2, 1)$ to $(3, 2, 2)$.

Solution:

Again we find yet another way to phrase the question of integrating

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

As ever, we first check if the vector field $\mathbf{F}$ is conservative. A quick check reveals that in fact, as in 15.4.31 from homework,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{2x}{y} - z, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = -y, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = -x,$$

so it MAY be conservative. Let's try to find a potential function. The first component P is supposed to be the partial derivative of a potential function WITH RESPECT TO x. So let's integrate P with respect to x:

$$f(x, y, z) = \int P(x, y, z)dx = x^2 \ln y - xyz + C(y, z)$$

where the constant can depend on y and z. Moreover, $\partial f/\partial y$ must equal Q. From above integral of P, we know

$$\frac{\partial f(x, y, z)}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial C(y, z)}{\partial y}$$

Comparing this to Q we see

$$\frac{\partial C(y, z)}{\partial y} = 0,$$

i.e., C is a function of only z. Doing the same comparison to R, we find $\frac{\partial C(y,z)}{\partial z} = 0$ as well, so we can choose C to be any constant, like 0. Now we HAVE a potential function, so the integral is independent of path, and by the fundamental theorem for line integrals, we can just plug in the endpoints to $f(x, y, z) = x^2 \ln y - xyz$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 2, 2) - f(1, 2, 1) = 9 \ln 2 - 12 - \ln 2 + 2 = 8 \ln 2 - 10.$$

(d) Evaluate

$$\int_C \mathbf{F} \cdot \mathbf{T} ds$$

where $\mathbf{F}(x, y) = \langle 3xy + y^2, 2xy + 5x^2 \rangle$ and $C : (x-1)^2 + (y+2)^2 = 1$ is traversed in the counter-clockwise direction.

Solution:

Note that the integral is computing work done over a closed path, so we may use Green's theorem.

$$\frac{\partial P}{\partial y} = 3x + 2y, \quad \frac{\partial Q}{\partial x} = 2y + 10x$$

According to Green's theorem,

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D 7x \, dA$$

where D is, in this problem, a disc of radius 1 centered at $(1, 2)$. Changing coordinates, we first let $u = x - 1$ and $v = y - 2$. This has Jacobian 1. And now we are integrating over the unit circle (call it B)

$$\iint_B 7(u+1) du \, dv$$

Note that u is odd, so the integral of $7u$ is 0 over the unit circle. This leaves $7 \cdot \text{Area}(B) = 7\pi$. Or do a second change of coordinates to polar:

$$\int_0^{2\pi} \int_0^1 (7r \cos\theta + 7) r \, dr \, d\theta = 7\pi$$

and get the same answer.

Note: You could have gone straight to polar coordinates for the disc centered at (1,2) but you would have had to work a bit harder.

2. Compute the value of the surface integral

$$\iint_S (x^2 + y^2) dS$$

where S is the surface $z = 1 - x^2 - y^2$ above the coordinate plane.

Solution:

First note that this is NOT a flux integral, so the square root from

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u} \right| du\, dv$$

will remain in the integrand, rather than canceling.

As usual, our first task is to parametrize S. Here that is easy since S is already a function of x, y. Normally, we'd substitute $z = 1 - x^2 - y^2$ into the integrand for each appearance of z, but there ARE NO z's in the integrand. Recall that when we have a function of x and y, then

$$dS = \left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| dx\, dy = \sqrt{1 + \left(\frac{\partial \mathbf{r}}{\partial x}\right)^2 + \left(\frac{\partial \mathbf{r}}{\partial y}\right)^2}$$

Enough review, now we start the problem. So in our case,

$$dS = \sqrt{1 + (-2x)^2 + (-2y)^2}\, dx\, dy.$$

Now the surface lives above the region of the xy-plane given by the unit disc (we see this by finding the intersection of $z = 1 - x^2 - y^2$ and $z = 0$). Let B denote the unit disc. Then we must solve

$$\iint_B (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

Converting to polar:

$$\int_0^{2\pi} \int_0^1 r^3 \sqrt{1 + 4r^2}\, dr\, d\theta = 2\pi \int_0^1 r^3 \sqrt{1 + 4r^2}\, dr$$

Let $2r = \tan x$ so that $dr = (1/2)\sec^2 x\, dx$. Then the above integral is

$$\begin{aligned}
\frac{\pi}{8} \int_0^{\tan^{-1}(2)} \tan^3 x \sec^3 x\, dx &= \frac{\pi}{8} \int_0^{\tan^{-1}(2)} \sec^3 x (1 + \sec^2 x) \tan x\, dx \\
&= \frac{\pi}{8} \int_0^{\tan^{-1}(2)} (\sec^2 x + \sec^4 x) \sec x\ \tan x\, dx \\
= \frac{\pi}{8} \int_1^{1/\sqrt{5}} u^2 + u^4\, du &= \frac{\pi\sqrt{5}}{1875} - \frac{\pi}{15} \quad \text{(I think)}
\end{aligned}$$

where in the last step we had to draw the triangle with legs of 1 and 2 and hypotenuse $\sqrt{5}$ to get the bounds.

3. Verify the divergence theorem for the vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ and the closed surface S, a sphere of radius a centered at the origin. That is, calculate both the flux out of the surface and the integral of the divergence and note they are equal.

Solution:

The easier part is the divergence side.

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 3$$

so if B is the unit ball, we have

$$\iiint_B \nabla \cdot \mathbf{F}\, dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$

For the flux, we compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

We use a parametrization for the sphere motivated by spherical coordinates:

$$x(\phi, \theta) = a \sin\phi \cos\theta, \quad y(\phi, \theta) = a \sin\phi \sin\theta, \quad z(\phi, \theta) = a \cos\phi$$

The cross product for the normal vector is something you've computed several times in the homework. It is

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \sin\phi \cos\phi \rangle$$

Hence

$$\mathbf{F} \cdot \mathbf{n}\, dS = a^3 \left(\sin^3\phi \cos^2\theta + \sin^3\phi \sin^2\theta + \sin\phi \cos^2\phi\right)$$

And we compute

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = a^3 \int_0^{\pi} \int_0^{2\pi} \left(\sin^3\phi + \sin\phi \cos^2\phi\right) d\theta\, d\phi = 2\pi a^3 \int_0^{\pi} \left(\sin^3\phi + \sin\phi \cos^2\phi\right) d\phi$$

Now $\sin^3\phi = \sin\phi(1 - \cos^2\phi)$, so the above is just

$$= 2\pi a^3 \int_0^{\pi} \sin\phi\, d\phi = 4\pi a^3$$

4. Compute the line integral

$$\oint_C \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$$

over any closed curve C.

Solution:

If the point $(0,0)$ lies on C then the integral is undefined. Otherwise, since this is a closed curve, we may use Green's theorem to evaluate the integral provided that the vector valued function

$$\mathbf{F} = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle$$

is continuously differentiable on an open set containing C. Now

$$\frac{\partial P}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

which are continuous if C doesn't enclose $(0,0)$. Then by Green's theorem, this is the double integral

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 0\,dA$$

where D is the region inside C.

If C is any curve enclosing 0, then we must compute the line integral. As in an example we did in class, because Green's theorem holds away from $(0,0)$, any curve C enclosing the origin can be deformed to the unit circle and the line integrals of both will be the same (their difference is the double integral of the region between them, which is 0).

Over the unit circle, we have the parametrization:

$$x = \cos\theta, \qquad y = \sin\theta, \qquad 0 \le \theta \le 2\pi.$$

so the integrand $\mathbf{F}\cdot d\mathbf{r}$ is

$$\langle \cos\theta, \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle\, d\theta$$

Somewhat anticlimactically, this integrand is just $0\ d\theta$, so the line integral is ALSO equal to 0 for curves C enclosing the origin. Note this was not the case in the in-class example where we got something non-zero.

5. The Folium of Descartes is the curve given by

$$x^3 + y^3 = 3axy, \qquad a > 0 \text{ some fixed constant.}$$

If we let $t = y/x$ then the folium can be parametrized by the following set of equations:

$$x = \frac{3at}{1+t^3} \qquad y = \frac{3at^2}{1+t^3} \qquad t \neq -1$$

(a) Graph the folium by plotting a few points.

(b) Use Green's theorem to find the area of the closed loop.

(c) Compute an improper integral to show that the area of the region between the asymptote $x + y + a = 0$ and the folium equals the area of the closed loop.

Solution:

This is a difficult question. We will discuss it in class. But note that you solved part (b) in your homework, so refer to your book for a picture, and to the pdf file for a solution to the area problem. It remains to compute (c).