MATH 52 MIDTERM

October 20, 2004

Name: ____________________________________

Numeric Student ID: ________________________

Instructor's Name: _________________________

I agree to abide by the terms of the honor code:

Signature: _________________________________

Instructions: Print your name, student ID number and instructor's name in the space provided. During the test you may not use notes, books or calculators. Read each question carefully and **show all your work**; full credit cannot be obtained without sufficient justification for your answer unless explicitly stated otherwise. Underline your final answer to each question. There are 7 questions. You have 120 minutes to do all the problems.

Question	Score	Maximum
1		15
2		10
3		10
4		10
5		10
6		5
7		10
Total		70

1. Evaluate the following double integrals:

(a)

$$\int_0^3 \int_0^2 x(4-y^2)\,dy\,dx$$

Solution:
This is a rectangle in the plane, hence the bounds are constants and we may integrate the iterated integral as a product of two single-variable integrals.

$$\begin{aligned}\int_0^3 \int_0^2 x(4-y^2)\,dy\,dx &= \int_0^3 x\,dx \int_0^2 (4-y^2)\,dy \\ &= (\frac{1}{2}(3)^2)(4(2)-\frac{1}{3}(2)^3) \\ &= 24\end{aligned}$$

(b)

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dx$$

Solution:
Using polar coordinates, we can reparametrize the integral over the quarter circle of radius 2 in terms of the polar rectangle $0 \le r \le 2$ and $0 \le \theta \le \pi/2$. Then

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dx &= \int_0^{\pi/2} \int_0^2 \frac{r^2 \sin\theta\cos\theta}{r} r\,dr\,d\theta \\ &= \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^2 r^2\,dr \\ &= (\frac{1}{2}\sin^2(\pi/2)-\frac{1}{2}\sin^2(0))(\frac{1}{3}(2)^3) \\ &= \frac{4}{3}\end{aligned}$$

(c)

$$\iint_R e^{x^2}\,dx\,dy$$

where R is the triangle in the xy-plane formed by the x-axis, $2y = x$ and $x = 2$.

Solution:

The region R is a triangle in the plane with vertices $(0,0)$, $(2,0)$ and $(2,1)$. We choose to integrate with respect to y first, since we know that the integrand cannot be integrated in terms of x first. Then

$$\begin{aligned}
\iint_R e^{x^2}\,dx\,dy &= \int_0^2 \int_0^{x/2} e^{x^2}\,dy\,dx \\
&= \int_0^2 \left[ye^{x^2}\right]_{y=0}^{y=x/2} dx \\
&= \frac{1}{2}\int_0^2 xe^{x^2}\,dx \\
&= \frac{1}{2}\left[\frac{1}{2}e^{x^2}\right]_{x=0}^{x=2} \\
&= \frac{1}{4}(e^4 - 1)
\end{aligned}$$

2. Compute the volume between the surfaces $x^2+y^2=1$ and $x^2+y^2=4$, lying below $x^2+y^2+4z^2=36$ and above the plane $z=0$ in $\mathbb{R}^3$.

Solution:

The surfaces $x^2+y^2=1$ and $x^2+y^2=4$, when considered in $\mathbb{R}^3$ are both right, regular cylinders, so we choose to integrate in cylindrical coordinates. The cylinders have equations $r=1$ and $r=2$ respectively. The ellipsoid has equation $r^2+4z^2=36$. Thus, the volume enclosed by the surfaces is

$$\begin{aligned} V &= \iiint_S r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_1^2\int_0^{\frac{1}{2}\sqrt{36-r^2}} r\,dz\,dr\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_1^2 \sqrt{36-r^2}\,r\,dr\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} d\theta \int_1^2 \sqrt{36-r^2}\,r\,dr \end{aligned}$$

Let $u=36-r^2$ so that $du=-2r\,dr$ Then

$$V = \pi\int_{32}^{35}\frac{1}{2}\sqrt{u}\,du = \frac{\pi}{3}((35)^{3/2}-(32)^{3/2})$$

3. Evaluate the following two triple integrals.

(a)

$$\iiint_S y^2 \, dV$$

where S is the solid in the first octant bounded by $2x + 3y + z = 6$, $x = 0$, $y = 0$ and $z = 0$.

Solution:

The solid S is a prism bounded above by the plane $2x+3y+z=6$ with base a triangle with vertices $(0,0),(0,2),(3,0)$. Thus we can compute the triple integral with the following iterated integral:

$$\begin{aligned}
\iiint_S y^2 \, dV &= \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} y^2 \, dz \, dy \, dx \\
&= \int_0^3 \int_0^{-\frac{2}{3}x+2} (6-2x-3y)y^2 \, dy \, dx \\
&= \int_0^3 \left[2y^3 - \frac{2}{3}xy^3 - \frac{3}{4}y^4 \right]_{y=0}^{y=\frac{-2}{3}x+2} dx \\
&= \int_0^3 \left(\frac{-2}{3}x+2\right)^3 \left(2 - \frac{2}{3}x - \frac{3}{4}\left(2 - \frac{2}{3}x\right)\right) dx \\
&= \frac{1}{4}\int_0^3 \left(\frac{-2}{3}x+2\right)^4 dx
\end{aligned}$$

Letting $u = \frac{-2}{3}x + 2$ so $du = \frac{-2}{3}dx$ we have

$$\iiint_S y^2 \, dV = \frac{3}{8}\int_0^2 u^4 \, du = \frac{3}{8}\,\frac{1}{5}(2)^5 = \frac{12}{5}$$

(b)

$$\int_0^\pi \int_\phi^\pi \int_1^4 \frac{\sin\theta}{\theta}\, d\rho\, d\theta\, d\phi$$

Solution:

This is really a Cartesian coordinate volume in disguise. It isn't natural to consider this as a volume in spherical coordinates because the integrand is missing the $\rho^2 \sin\phi$. One option is try to insert this factor into the integrand. This causes problems when we try to integrate the resulting mess that ends up in the denominator.

Since the Greek letters are arbitrary, this integral may look simpler in the form

$$\int_0^\pi \int_x^\pi \int_1^4 \frac{\sin y}{y}\, dz\, dy\, dx$$

The inner integration leaves

$$3\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$$

which can be solved by changing the order of integration after drawing the domain in the xy-plane. This produces

$$3\int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy = 3\int_0^\pi \sin y\, dy = 6$$

4. Find BOTH the mass and center of mass (i.e. "centroid") of a ball of radius a whose density is proportional to the distance from the boundary of the ball.

5. Find the area of the surface of the paraboloid $x^2 + y^2 - z = 0$ lying below the plane $z = 2$.

Solution:

The paraboloid is cut by the plane forming a surface lying over $x^2 + y^2 = 2$, a circle C of radius $\sqrt{2}$(we see this by setting each of the equations equal to each other). We can now proceed in two ways: either setting up the surface area in rectangular coordinates and then solving the resulting iterated integral by polar coordinates OR by considering the surface in cylindrical coordinates and using that surface area integral. We do the second.

$$S.A. = \iint_C \sqrt{r^2 + (r\frac{dz}{dr})^2 + \frac{dz}{d\theta}^2}\, dr\, d\theta$$

Note this integral form was derived using the area of tangent planes over polar rectangles to approximate the surface, not the area of the polar rectangles themselves, so the integrand doesn't end up with an $r\, dr\, d\theta$.

Now using the above equation, noting that in cylindrical coordinates the surface is $z = r^2$:

$$\begin{aligned} S.A. &= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{r^2 + (r(2r))^2 + 0}\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr \end{aligned}$$

where we have used the fact that r is always non-negative on this domain, so $\sqrt{r^2} = r$.

To finish, let $u = 1 + 4r^2$ so $du = 8dr$

$$S.A. = 2\pi \int_1^9 \frac{1}{8}\sqrt{u}\, du = \pi/4(\frac{2}{3}(9)^{3/2} - \frac{2}{3}) = \frac{13}{3}\pi$$

6. Use the definition of the double integral to show that for any continuous function $f(x, y)$, if $f(x, y) \geq 0$ for all points $(x, y) \in R$, then

$$\iint_R f(x, y)\, dA \geq 0.$$

Solution:

This essentially follows immediately from the definition. Recall that

$$\iint_R f(x, y)\, dA \geq 0 = \lim_{|P| \to 0} \sum_{i=1}^{k} f(x_i^*, y_i^*) \Delta A_i$$

where the limit is taken over inner partitions of rectangles over the domain R and the rectangles in the partition must have their size all going to 0. Since the function is non-negative at all points in R and ΔA_i measures the area (a positive quantity) in each rectangle of the partition, then each contribution of each summand to the finite sum is non-negative. Now the limit of a sequence of non-negative numbers is non-negative. This takes some proof, actually, (a short proof by contradiction based on the definition of the limit) but we can assume this for the sake of the question. Hence by definition in terms of Riemann sums,

$$\iint_R f(x, y)\, dA \geq 0.$$

7. Recall that the greatest integer function $f(x) = \lfloor x \rfloor$ returns the largest integer smaller than x. So, for example, $f(3.9) = 3$. Compute

$$\int_1^3 \int_2^5 \lfloor x + y \rfloor \, dx \, dy.$$

Solution:

This is a discontinuous function, so that, in particular it has no simple-to-state anti-derivative. Instead, we note that the function is constant on the diagonal slices from $x + y = k$ to $x + y = k + 1$ for each integer value k. Drawing a picture of the rectangle R and the way it is cut by these diagonal lines, we can use the interpretation of the double integral as the volume underneath the surface defined by the integrand. These are simple geometric shapes whose volume is just the area of the base multiplied by the height.

Working this out, the region is divided into five such diagonal bands of area $1/2, 3/2, 2, 3/2, 1/2$ moving left to right. One way to see this is by adding the number of 1 by 1 triangles in each band. Figuring in the height, we compute

$$\int_1^3 \int_2^5 \lfloor x + y \rfloor \, dx \, dy = 1/2(3) + 3/2(4) + 2(5) + 3/2(6) + 1/2(7) = 30$$