MATHEMATICS 52
SOLUTION SET 7

15.4.22 Denote by D the disk bounded by the given circular path C. Then the outward flux of $\mathbf{F}$ across C is obtained by using the vector form of Green's theorem:

$$\begin{aligned}\Phi &= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \nabla\cdot\mathbf{F}\,dA = \iint_D \nabla\cdot\langle x^3, y^3\rangle\,dA = \iint_D (3x^2+3y^2)\,dA \\ &= \int_0^{2\pi}\int_0^3 3r^3\,dr\,d\theta = 6\pi\left[\frac{r^4}{4}\right]_0^3 = \frac{243}{2}\pi \approx 381.70\end{aligned}$$

15.4.28 By Green's theorem:

$$\begin{aligned}\oint_C -f_y\,dx + f_x\,dy &= \iint_R (f_{xx}+f_{yy})\,dA = \iint_R \nabla\cdot\langle f_x, f_y\rangle\,dA \\ &= \iint_R \nabla\cdot(\nabla f)\,dA = \iint_R \nabla^2 f\,dx\,dy\end{aligned}$$

15.4.33 As in Problem 30 of Section 9.4, the substitution $y = tx$ in the equation $x^3+y^3 = 3xy$ of the folium yields $x^3+t^3x^3 = 3tx^2$, and thereby the parametrization

$$x(t) = \frac{3t}{1+t^3}, \quad y(t) = \frac{3t^2}{1+t^3}, \quad 0 \le t < +\infty$$

of the first-quadrant loop of the folium. If C is the half of its loop that stretches from $(0,0)$ to $(\frac{3}{2}, \frac{3}{2})$ along the lower half of the folium, then C is swept out by this parametrization as t varies from 0 to 1. Let J be the straight line segment joining $(\frac{3}{2}, \frac{3}{2})$ with $(0,0)$; parametrize J with $x = \frac{3}{2}(1-t)$, $y = \frac{3}{2}(1-t)$, $0 \le t \le 1$. Then the area of the folium is

$$A = 2\cdot\frac{1}{2}\oint_{C\cup J} x\,dy - y\,dx = \int_C x\,dy - y\,dx + \int_J x\,dy - y\,dx$$

The last integral is

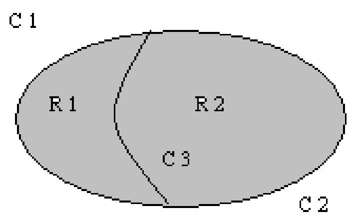
$$\frac{3}{2}\int_0^1 \left[-\frac{3}{2}(1-t) + \frac{3}{2}(1-t)\right]dt = 0$$

and hence the area of the folium is

$$\begin{aligned}A = \int_0^1 [x(t)y'(t) - y(t)x'(t)]\,dt &= \int_0^1 \left[\frac{9t(2t-t^4)}{(1+t^3)^3} - \frac{9t^2(2t^3-1)}{(1+t^3)^3}\right]dt \\ &= \int_0^1 \frac{9t^2}{(1+t^3)^2}\,dt = \left[-\frac{3}{1+t^3}\right]_0^1 = \frac{3}{2}\end{aligned}$$

15.4.37 It suffices to show the result in the case that $R = R_1 \cup R_2$ is the union of two regions, with C the boundary of R, $C_1 \cup C_3$ the boundary of R_1, and $-C_3 \cup C_2$

the boundary of R_2. Then $C = C_1 \cup C_2$.
Perhaps the figure will clarify all this.



Under the assumption that Green's theorem holds for R_1 and for R_2, we have

$$\oint_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{R_1} (Q_x - P_y)\,dA$$

and

$$\oint_{-C_3 \cup C_2} P\,dx + Q\,dy = \iint_{R_2} (Q_x - P_y)\,dA$$

Addition of these equations yields

$$\oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy = \oint_{C} P\,dx + Q\,dy = \iint_{R} (Q_x - P_y)\,dA$$

15.4.40 The proof consists of three steps. First, we use Green's theorem to express area as a line integral. Then, we determine how changing coordinates affects this line integral. Finally, we apply Green's theorem once more to express this new line integral as a double integral.

$$A = \iint_R 1\,dx\,dy = \oint_C x\,dy = \oint_J x(u,v)(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv) = \oint_J x(u,v)\frac{\partial y}{\partial u}\,du + x(u,v)\frac{\partial y}{\partial v}\,dv$$
$$= \iint_S \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + x\frac{\partial^2 y}{\partial u \partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - x\frac{\partial^2 y}{\partial u \partial v}\right] du\,dv = \iint_S |\frac{\partial(x,y)}{\partial(u,v)}|\,du\,dv$$

15.5.2 Here S is the surface $z = h(x,y) = 6 - 2x - 3y$ over the plane triangle bounded by the nonnegative coordinate axes and the line $y = (6-2x)/3$. We have that the surface area element is:

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\,dx\,dy = \sqrt{1+4+9}\,dx\,dy = \sqrt{14}\,dx\,dy$$

and therefore:

$$\begin{aligned}
\iint_S xyz\,dS &= \int_0^3 \int_0^{(6-2x)/3} xy(6-2x-3y)\sqrt{14}\,dy\,dx \\
&= \sqrt{14}\int_0^3 \left[3xy^2 - x^2y^2 - xy^3\right]_0^{(6-2x)/3}\,dx \\
&= \sqrt{14}\int_0^3 \left(4x - 4x^2 + \frac{4}{3}x^3 - \frac{4}{27}x^4\right)\,dx \\
&= \sqrt{14}\left[2x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4 - \frac{4}{135}x^5\right]_0^3 \\
&= \frac{9}{5}\sqrt{14} \approx 6.735
\end{aligned}$$

15.5.8 The surface S has equation $z = h(x,y) = xy$ and lies over (and under) the circular disk D with center $(0,0)$ and the radius 5 in the xy-plane. The surface area element is

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\,dA = \sqrt{1 + x^2 + y^2}\,dA$$

and therefore the moment of inertia of S (with constant density δ) with respect to the z-axis is

$$\begin{aligned}
I_z &= \iint_S (x^2+y^2)\delta\,dS = \iint_D \delta r^2\sqrt{1+r^2}\,dA = \int_0^{2\pi}\int_0^5 \delta r^3\sqrt{1+r^2}\,dr\,d\theta \\
&= \frac{2}{15}\pi\delta\left[(3r^4 + r^2 - 2)\sqrt{1+r^2}\right]_0^5 = \frac{4}{15}\pi\delta\left(1 + 949\sqrt{26}\right)
\end{aligned}$$

Because the mass of S is

$$m = \iint_S \delta\,dS = \int_0^{2\pi}\int_0^5 \delta r\sqrt{1+r^2}\,dr\,d\delta = \frac{2}{3}\pi\delta\left(-1 + 26\sqrt{26}\right)$$

the moment of inertia may also be expressed in the form

$$I_z = \frac{2 + 1898\sqrt{26}}{-5 + 130\sqrt{26}} \cdot m$$

15.5.12 The upper half of the surface S has the spherical coordinates parametrization:

$$\mathbf{r}(\phi,\theta) = \langle 5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi\rangle,\ \ \arccos\left(\frac{4}{5}\right) \le \phi \le \frac{\pi}{2},\ \ 0 \le \theta \le 2\pi$$

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5\cos\phi\cos\theta & 5\cos\phi\sin\theta & -5\sin\phi \\ -5\sin\phi\sin\theta & 5\sin\phi\cos\theta & 0 \end{vmatrix} \\
&= \langle 25\sin^2\phi\cos\theta, 25\sin^2\phi\sin\theta, 25\sin\phi\cos\phi\rangle
\end{aligned}$$

and therefore

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{625\sin^4\phi\cos^2\theta + 625\sin^4\phi\sin^2\theta + 625\sin^2\phi\cos^2\phi} = 25\sin\phi$$

To find the mass and moment of inertia we integrate over the top half and then double the result.

$$m = \iint_S 1\,dS = 2\int_0^{2\pi}\int_{\arccos(4/5)}^{\pi/2} 25\sin\phi\,d\phi\,d\theta = 4\pi\Big[-25\cos\phi\Big]_{\arccos(4/5)}^{\pi/2} = 80\pi$$

Next,

$$x^2 + y^2 = (5\sin\phi\cos\theta)^2 + (5\sin\phi\sin\theta)^2 = 25\sin^2\phi$$

so that the moment of inertia with respect to the z-axis is:

$$\begin{aligned} I_z &= \iint_S (x^2+y^2)\,dS = 2\int_0^{2\pi}\int_{\arccos(4/5)}^{\pi/2} 625\sin^3\phi\,d\phi\,d\theta \\ &= 4\pi\left[\frac{625}{3}\cos^3\phi - 625\cos\phi\right]_{\arccos(4/5)}^{\pi/2} = \frac{4720\pi}{3} \approx 4942.772 \end{aligned}$$

15.5.14 An outward unit vector normal to S is $\mathbf{n} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$. The surface S has equation $z = h(x,y) = 3 - 2x - 2y$ so that $dS = \sqrt{1+4+4}\,dA = 3\,dA$. The surface lies over the triangle T cut off from the first quadrant of the xy-plane by the line $y = \frac{1}{2}(3-2x)$. This we get that:

$$\begin{aligned} \iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_T 3\,dy\,dx = \int_0^{3/2}\int_0^{(3-2x)/2} 3\,dy\,dx \\ &= \int_0^{3/2} \frac{3}{2}(3-2x)\,dx = \frac{1}{2}\Big[9x - 3x^2\Big]_0^{3/2} = \frac{27}{8} \end{aligned}$$

15.5.20 The hemispherical surface $z = \sqrt{4-x^2-y^2}$ has the unit normal vector $\mathbf{n} = \frac{1}{2}\langle x, y, z\rangle$ and parametrization:

$$x = 2\sin\phi\cos\theta,\ y = 2\sin\phi\sin\theta,\ z = 2\cos\phi,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi$$

The usual computation of $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$ (see, for example, some of the previous problems in this solution set) yields $dS = 4\sin\phi\,dA$ so that:

$$\begin{aligned} \mathbf{F}(x,y,z)\cdot\mathbf{n}(x,y,z) &= \langle 4\sin\phi\cos\theta, -6\sin\phi\sin\theta, 2\cos\phi\rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \\ &= 2\cos^2\phi + 4\sin^2\phi\cos^2\theta - 6\sin^2\phi\sin^2\theta \end{aligned}$$

so that the flux across the upper hemispherical surface is

$$\iint_H \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi/2}(8\cos^2\phi\sin\phi + 16\sin^3\phi\cos^2\theta - 24\sin^3\phi\sin^2\theta)\,d\phi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{2\pi}(8\cos^2\phi\sin\phi + 16\sin^3\phi\cos^2\theta - 24\sin^3\phi\sin^2\theta)\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{2\pi}(8\cos^2\phi\sin\phi + 16\sin^3\phi\frac{1+\cos 2\theta}{2} - 24\sin^3\phi\frac{1-\cos 2\theta}{2})\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{2\pi}\left(8\cos^2\phi\sin\phi + 16\sin^3\phi\cdot\frac{1}{2} - 24\sin^3\phi\cdot\frac{1}{2}\right)\,d\theta\,d\phi$$

$$=2\pi\int_0^{\pi/2}\left(8\cos^2\phi\sin\phi + 8\sin^3\phi - 12\sin^3\phi\right)\,d\phi$$

$$=2\pi\left[-\frac{8}{3}\cos^3\phi + 4(\cos\phi - \frac{1}{3}\cos^3\phi)\right]_0^{\pi/2} = 2\pi\cdot\left(\frac{8}{3} - 4 + 4\frac{1}{3}\right) = 0$$

On the circular disk D that forms the base of the hemispherical solid, we have that

$$\mathbf{F}(x,y,z)\cdot\mathbf{n}(x,y,z) = \langle 2x,-3y,0\rangle\cdot\langle 0,0,-1\rangle = 0$$

so that the flux across the disk is zero, and hence the total flux across the boundary of the solid is zero as well.

15.5.24 The surface area element for the conical surface $z=\sqrt{x^2+y^2}$ is $dS = \sqrt{1+(\frac{2x}{2(x^2+y^2)})^2+(\frac{2y}{2(x^2+y^2)})^2} = \sqrt{2}\,dA$ and the outer normal vector for the surface C is

$$\mathbf{n_1} = \frac{\sqrt{2}}{2}\langle\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}},-1\rangle$$

Let D denote the circular disk in the xy-plane centered at the origin and having radius 3. Then

$$\iint_C \mathbf{F}\cdot\mathbf{n_1}\,dS = \iint_D \frac{x^3+2y^3-3z^2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,dA = \iint_D \frac{x^3+2y^3-3(x^2+y^2)^{3/2}}{\sqrt{x^2+y^2}}\,dA$$

$$=\int_0^{2\pi}\int_0^3(r^3\cos^3\theta + 2r^3\sin^3\theta - 3r^3)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{1}{4}r^4(\cos^3\theta + 2\sin^3\theta - 3)\right]_0^3\,d\theta$$

$$=\int_0^{2\pi}\frac{81}{4}(\cos^3\theta + 2\sin^3\theta - 3)\,d\theta$$

$$=\left[\frac{27}{16}(2\cos 3\theta - 18\cos\theta + \sin 3\theta + 9\sin\theta - 36\theta)\right]_0^{2\pi} = -\frac{243\pi}{2}$$

15.5.28 The surface S may be parametrized by spherical coordinates (ϕ,θ) where $0\le\phi\le\pi/4$, $0\le\theta\le 2\pi$. As before, we find that $dS = a^2\sin\phi\,dA$. We know that the coordinates $(\bar{x},\bar{y},\bar{z})$ of the centroid satisfy $\bar{x}=\bar{y}=0$. Assuming S has constant density 1, we get that the mass and the z-coordinate of the centroid are:

$$m = \iint_S 1\,dS = \int_0^{2\pi}\int_0^{\pi/4} a^2 \sin\phi\,d\phi\,d\theta = 2\pi\Big[-\cos\phi\Big]_0^{\pi/4} = (2-\sqrt{2})\pi a^2$$

$$\begin{aligned} I_z &= \frac{1}{m}\iint_S z\,dS = \frac{1}{m}\int_0^{2\pi}\int_0^{\pi/4} a^3 \sin\phi\cos\phi\,d\phi\,d\theta \\ &= \frac{2\pi}{m}\left[\frac{1}{2}a^3\sin^2\phi\right]_0^{\pi/4} = \frac{2\pi a^3}{4m} = \frac{a}{2(2-\sqrt{2})} \end{aligned}$$

15.5.40 The integrals are set up as follows (for calculations, you might want to use a computer system such as Mathematica):

$$\begin{aligned} |N| &= \sqrt{\left[\frac{\partial(y,z)}{\partial(t,\theta)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(t,\theta)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(t,\theta)}\right]^2} \\ &= \sqrt{16 + \frac{3}{4}t^2 + 8t\cos\left(\frac{\theta}{2}\right) + \frac{1}{2}t^2\cos\theta} \end{aligned}$$

$$A = \int_0^{2\pi}\int_{-1}^{1} |N|\,dt\,d\theta \approx 50.399$$

$$I_z = \int_0^{2\pi}\int_{-1}^{1} (x^2+y^2)\cdot|N|\,dt\,d\theta \approx 831.470$$

15.6.2 Here we have

$$\mathbf{F}(x,y,z) = (x^2+y^2+z^2)^{1/2}\langle x,y,z\rangle \quad \text{and} \quad \mathbf{n} = \frac{1}{3}\langle x,y,z\rangle$$

is a unit vector normal to the surface S. Because $\mathbf{F}\cdot\mathbf{n} = \frac{1}{3}(x^2+y^2+z^2)^{3/2}$, $\mathbf{F}\cdot\mathbf{n}$ takes on the constant value 9 on S. Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 9\cdot area(S) = 9\cdot 4\pi\cdot 3^2 = 324\pi \approx 1017.876$$

Next, let B denote the solid ball bounded by S. Then

$$\begin{aligned} \nabla\cdot\mathbf{F} &= \frac{x^2}{\sqrt{x^2+y^2+z^2}} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \frac{z^2}{\sqrt{x^2+y^2+z^2}} + 3\sqrt{x^2+y^2+z^2} \\ &= 4\sqrt{x^2+y^2+z^2} \end{aligned}$$

and thus

$$\begin{aligned} \iiint_B \nabla\cdot\mathbf{F}\,dV &= \iiint_B 4\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{3} 4\rho^3\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi\int_0^{\pi} 81\sin\phi\,d\phi = 2\pi\Big[-81\cos\phi\Big]_0^{\pi} = 324\pi \end{aligned}$$

15.6.8 Denote by T the solid paraboloid bounded by the given surface S. The divergence theorem applied to the pair S, T gives:

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_T \nabla\cdot\mathbf{F}\,dV = \iiint_T 4(x^2+y^2)\,dV \\
&= \int_0^{2\pi}\int_0^5\int_0^{25-r^2} 4r^3\,dz\,dr\,d\theta = 2\pi\int_0^5 (100r^3-4r^5)\,dr \\
&= \frac{31250}{3}\pi \approx 32724.923
\end{aligned}$$