MATHEMATICS 52
SOLUTION SET 4

14.9.6 Given

$$u = \frac{2x}{x^2+y^2}, \quad v = -\frac{2y}{x^2+y^2}$$

we note first that

$$u^2+v^2 = \frac{4x^2+4y^2}{(x^2+y^2)^2} = \frac{4}{x^2+y^2}$$

so that

$$x^2+y^2 = \frac{4}{u^2+v^2}$$

and substituting this into expressions for u, v in terms of x, y, we get:

$$x = \frac{1}{2}u(x^2+y^2) = \frac{2u}{u^2+v^2} \text{ and } y = -\frac{1}{2}v(x^2+y^2) = -\frac{2v}{u^2+v^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2(v^2-u^2)}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2} \\ \frac{4uv}{(u^2+v^2)^2} & \frac{2(v^2-u^2)}{(u^2+v^2)^2} \end{vmatrix} = \frac{4}{(u^2+v^2)^2}$$

14.9.10 If $y = ux^2$ and $x = vy^2$, then

$$y = uv^2y^4, \ y^3 = \frac{1}{uv^2}, \ y = \frac{1}{u^{1/3}v^{2/3}}$$

Then it follows that

$$x = vy^2 = \frac{v}{u^{2/3}v^{4/3}} = \frac{1}{u^{2/3}v^{1/3}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{2}{3u^{5/3}v^{1/3}} & -\frac{1}{3u^{2/3}v^{4/3}} \\ -\frac{1}{3u^{4/3}v^{2/3}} & -\frac{2}{3u^{1/3}v^{5/3}} \end{vmatrix} = \frac{1}{3u^2v^2}$$

Now, note that that $y = x^2$ corresponds to $u = 1$, $y = 2x^2$ corresponds to $u = 2$, $x = y^2$ to $v = 1$, and $x = 4y^2$ to $v = 4$. Thus we can calculate the required area by integrating over a rectangle in the uv-plane:

$$A = \iint_R 1\,dx\,dy = \int_1^4 \int_1^2 \frac{1}{3u^2v^2}\,du\,dv = \int_1^4 \left[-\frac{1}{3uv^2}\right]_1^2 dv = \int_1^4 \frac{1}{6v^2}\,dv = \frac{1}{8}$$

14.9.12 The transformation

$$u = \frac{2x}{x^2+y^2}, \quad v = \frac{2y}{x^2+y^2}$$

yields

$$u^2+v^2=\frac{4x^2+4y^2}{(x^2+y^2)^2}=\frac{4}{x^2+y^2},\qquad x^2+y^2=\frac{4}{u^2+v^2}$$
$$x=\frac{1}{2}u(x^2+y^2)=\frac{2u}{u^2+v^2},\qquad y=\frac{1}{2}v(x^2+y^2)=\frac{2v}{u^2+v^2}$$

The circle $x^2+y^2=2x$ is thereby transformed into

$$\frac{4}{u^2+v^2}=\frac{4u}{u^2+v^2}\ :\ u=1$$

Similarly, the other three circles are transformed into $u=\frac{1}{3}$, $v=1$, and $v=\frac{1}{4}$. The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{2(v^2-u^2)}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2}\\ -\frac{4uv}{(u^2+v^2)^2} & \frac{2(u^2-v^2)}{(u^2+v^2)^2}\end{vmatrix}=-\frac{4}{(u^2+v^2)^2}$$

Note also that

$$(x^2+y^2)^2=\frac{16}{(u^2+v^2)^2}\ \text{ so that }\ \frac{1}{(x^2+y^2)^2}=\frac{(u^2+v^2)^2}{16}$$

Therefore

$$\begin{aligned}\iint_R\frac{1}{(x^2+y^2)^2}\,dx\,dy &= \int_1^{1/4}\int_1^{1/3}\frac{(u^2+v^2)^2}{16}\cdot\frac{4}{(u^2+v^2)^2}\,du\,dv\\ &= \int_1^{1/4}\int_1^{1/3}\frac{1}{4}\,du\,dv=\frac{-3}{4}\cdot\frac{-2}{3}\cdot\frac{1}{4}=\frac{1}{8}\end{aligned}$$

14.9.16 Under the given transformation, the plane $z=1$ corresponds to $r=1$, the plane $z=4$ to $r=2$, the paraboloid $z=x^2+y^2$ corresponds to:

$$r^2=\frac{r^2}{t^2}\cos^2\theta+\frac{r^2}{t^2}\sin^2\theta=\frac{r^2}{t^2}$$

and thus to $t=1$. The other paraboloid, $z=4(x^2+y^2)$ corresponds to $t=2$. The parameter θ varies from 0 to 2π. The Jacobian of the given transformation is:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,t)}=\begin{vmatrix}\frac{1}{t}\cos\theta & -\frac{r}{t}\sin\theta & -\frac{r}{t^2}\cos\theta\\ \frac{1}{t}\sin\theta & \frac{r}{t}\cos\theta & -\frac{r}{t^2}\sin\theta\\ 2r & 0 & 0\end{vmatrix}=2r\left(\frac{r^2}{t^3}\sin^2\theta+\frac{r^2}{t^3}\cos^2\theta\right)=\frac{2r^3}{t^3}$$

Hence the volume of the solid is:

$$\begin{aligned}V &= \int_0^{2\pi}\int_1^2\int_1^2\frac{2r^3}{t^3}\,dt\,dr\,d\theta=2\pi\int_1^2\left[-\frac{r^3}{t^2}\right]_{t=1}^2dr=2\pi\int_1^2\frac{3r^3}{4}\,dr=\frac{3\pi}{2}\left[\frac{r^4}{4}\right]_1^2\\ &= \frac{45\pi}{8}\end{aligned}$$

14.9.18 The function which maps (x,y) to $(x(u(x,y),v(x,y)),y(u(x,y),v(x,y)))$ is just the identity map, so applying the chain rule to it will give:

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} x_u u_x + x_v v_x & x_u u_y + x_v v_y \\ y_u u_x + y_v v_x & y_u u_y + y_v v_y \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\end{aligned}$$

14.9.20 Given: The solid ellipsoid R with constant density δ and boundary surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(where a, b, c are positive constants). The transformation

$$x = a\rho\sin\phi\cos\theta,\ y = b\rho\sin\phi\sin\theta,\ z = c\rho\cos\phi$$

has Jacobian

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} a\sin\phi\cos\theta & a\rho\cos\phi\cos\theta & -a\rho\sin\phi\sin\theta \\ b\sin\phi\sin\theta & b\rho\cos\phi\sin\theta & b\rho\sin\phi\cos\theta \\ c\cos\phi & -c\rho\sin\phi & 0 \end{vmatrix}$$

$$= abc\rho^2\cos^2\phi\sin\phi\cos^2\theta + abc\rho^2\sin^3\phi\cos^2\theta + abc\rho^2\cos^2\phi\sin\phi\sin^2\theta + abc\rho^2\sin^3\phi\sin^2\theta$$

$$= abc\rho^2\sin\phi$$

This transformation also transforms the ellipsoidal surface into

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2 = 1$$

and thereby transforms R into the solid ball B of radius 1 and center at origin. Therefore the mass of the ellipsoid is

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \delta abc\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\delta abc\int_0^{\pi}\frac{1}{3}\sin\phi\,d\phi \\ &= 2\pi\delta abc\left[-\frac{1}{3}\cos\phi\right]_0^{\pi} = \frac{4}{3}\pi\delta abc \end{aligned}$$