MATHEMATICS 52
SOLUTION SET 3

14.5.32 The polar moment of inertia of the lamina is

$$I_0 = \int_0^\pi \int_0^a r^4 \sin\theta \, dr \, d\theta = \int_0^\pi \frac{1}{5} a^5 \sin\theta \, d\theta = \left[-\frac{1}{5} a^5 \cos\theta\right]_0^\pi = \frac{2}{5} a^5$$

14.6.4 The value of the triple integral is

$$\begin{aligned} I &= \int_{z=-2}^{6} \int_{y=0}^{2} \int_{x-1}^{3} (x+y+z) \, dx \, dy \, dz \\ &= \int_{-2}^{6} \int_{0}^{2} \left[\frac{1}{2}x^2 + xy + xz\right]_{-1}^{3} dy \, dz = \int_{-2}^{6} \int_{0}^{2} (4+4y+4z) \, dy \, dz \\ &= \int_{-2}^{6} \left[2y^2 + 4y + 4yz\right]_{y=0}^{2} dz = \int_{-2}^{6} (16+8z) \, dz = \left[16z + 4z^2\right]_{-2}^{6} \\ &= 256 \end{aligned}$$

14.6.10 The value of the triple integral is:

$$\begin{aligned} I &= \int_{-1}^{1} \int_{-2}^{2} \int_{y^2}^{8-y^2} z \, dz \, dy \, dx = \int_{-1}^{1} \int_{-2}^{2} \left[\frac{1}{2} z^2\right]_{y^2}^{8-y^2} dy \, dx \\ &= \int_{-1}^{1} \int_{-2}^{2} (32 - 8y^2) \, dy \, dx = \int_{-1}^{1} \left[32y - \frac{8}{3} y^3\right]_{-2}^{2} dx = \int_{-1}^{1} \frac{256}{3} \, dx \\ &= \frac{512}{3} \end{aligned}$$

14.6.12 The volume is:

$$\begin{aligned} V &= \int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{y} 1 \, dz \, dy \, dx = \int_{-2}^{2} \int_{x^2}^{4} y \, dy \, dx = \int_{-2}^{2} \left[\frac{y^2}{2}\right]_{x^2}^{4} dx \\ &= \int_{-2}^{2} \left(8 - \frac{x^4}{2}\right) dx = \left[8x - \frac{x^5}{10}\right]_{-2}^{2} = \frac{128}{5} \end{aligned}$$

14.6.20 The volume is:

$$\begin{aligned} V &= \int_{0}^{2} \int_{0}^{2-x} \int_{0}^{4-x^2-z^2} 1 \, dy \, dz \, dx = \int_{0}^{2} \int_{0}^{2-x} (4 - x^2 - z^2) \, dz \, dx \\ &= \int_{0}^{2} \left[4z - x^2 z - \frac{1}{3} z^3\right]_{0}^{2-x} dx = \int_{0}^{2} \frac{1}{3}(16 - 12x^2 + 4x^3) \, dx \\ &= \frac{1}{3}\left[16x - 4x^3 + x^4\right]_{0}^{2} = \frac{16}{3} \end{aligned}$$

14.6.26 The moment of inertia of the solid(with density $\delta = 1$) with respect to the z-axis is

$$\begin{aligned} I_z &= \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{y}(x^2+y^2)\,dz\,dy\,dx = \int_{-2}^{2}\int_{x^2}^{4}(x^2y+y^3)\,dy\,dx \\ &= \int_{-2}^{2}\left[\frac{1}{2}x^2y^2+\frac{1}{4}y^4\right]_{x^2}^{4}dx = \int_{-2}^{2}\left(648x^2-\frac{1}{2}x^6-\frac{1}{4}x^8\right)dx \\ &= \left[64x+\frac{8}{3}x^3-\frac{1}{14}x^7-\frac{1}{36}x^9\right]_{-2}^{2} = \frac{15872}{63} \approx 251.9365 \end{aligned}$$

14.6.42 Note first that the two surfaces intersect in a curve that projects vertically onto the ellipse

$$\left(\frac{x-1}{2}\right)^2 + z^2 = 1$$

in the xz-plane. Hence the volume of the solid is

$$\begin{aligned} V &= \int_{-1}^{1}\int_{1-2\sqrt{1-z^2}}^{1+2\sqrt{1-z^2}}\int_{x^2+4z^2}^{2x+3} 1\,dy\,dx\,dz = \int_{-1}^{1}\int_{1-2\sqrt{1-z^2}}^{1+2\sqrt{1-z^2}}(3+2x-x^2-4z^2)\,dx\,dz \\ &= \int_{-1}^{1}\int_{1-2\sqrt{1-z^2}}^{1+2\sqrt{1-z^2}}[-(x-1)^2+4(1-z^2)]\,dx\,dz \\ &\quad \text{Let } z=\sin\theta, \text{ then } dz=\cos\theta d\theta \\ &= \int_{-\pi/2}^{\pi/2}\left[-\frac{1}{3}(x-1)^3+4x\cos^2\theta\right]_{1-2\cos\theta}^{1+2\cos\theta}\cos\theta\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\left[-\frac{16}{3}\cos^4\theta+16\cos^4\theta\right]d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{32}{3}\cos^4\theta\right)d\theta \\ &= \cos^3\theta\sin\theta\Big|_{-\pi/2}^{\pi/2}+\frac{32}{3}\int_{-\pi/2}^{\pi/2}3\cos^2\theta\sin^2\theta\,d\theta = 0+8\int_{-\pi/2}^{\pi/2}\sin^2 2\theta\,d\theta \\ &= 4\int_{-\pi}^{\pi}\sin^2\theta\,d\theta = 4\int_{-\pi}^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta = 4\pi \end{aligned}$$

14.7.4 The moment of inertia of a solid sphere of density δ, radius a, and center $(0,0,0)$ with respect to the z-axis is

$$\begin{aligned} I_z &= \int_{0}^{2\pi}\int_{0}^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}\delta r^3\,dz\,dr\,d\theta = 2\pi\delta\int_{0}^{a}2r^3(a^2-r^2)^{1/2}\,dr \\ &= 2\pi\delta\cdot\left[-\frac{2}{15}(2a^4+a^2r^2-3r^4)(a^2-r^2)^{1/2}\right]_{0}^{a} = \frac{8}{15}\pi\delta a^5 = \frac{2}{5}ma^2 \end{aligned}$$

where $m = \frac{4}{3}\pi\delta a^3$ is the mass of the sphere.

14.7.12 The paraboloids meet in the circle $x^2 + y^2 = 4$, $z = 4$. Therefore the volume between them is

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^2 (12r - 3r^3)\,dr = 2\pi\cdot\left[6r^2 - \frac{3}{4}r^4\right]_0^2 \\ &= 2\pi\cdot 12 = 24\pi \end{aligned}$$

14.7.16 Choose a coordinate system in which the cylinder's axis is the z-axis, and its lower base lies in the (r,θ)-plane, so that the points of the cylinder are described by:

$$0 \le r \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le h$$

The volume of the cylinder is $V = a^2\pi h$ and it has constant density δ, so its mass is $m = \delta a^2\pi h$, and the moment of inertia around the z-axis is:

$$\begin{aligned} I_z &= \int_0^{2\pi}\int_0^a\int_0^h \delta r^3\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \delta r^3 h\,dr\,d\theta \\ &= 2\pi\delta\left[\frac{1}{4}r^4 h\right]_0^a = \frac{1}{2}\pi\delta a^4 h = \frac{1}{2}a^2\cdot\pi\delta a^2 h = \frac{1}{2}ma^2 \end{aligned}$$

14.7.28 Note that both the shape and the density of our spherical shell are symmetric around any diameter. So the moment of inertia will be the same for all diameters, and we can restrict our attention to the diameter lying along the z-axis. The distance from this axis for a point with spherical coordinates (ρ,ϕ,θ) is $\rho\sin\phi$, so our moment of inertia will be:

$$\begin{aligned} I_z &= \int_0^{2\pi}\int_0^{\pi}\int_a^{2a} \delta(\rho,\phi,\theta)(\rho\sin\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_a^{2a} \rho^6\sin^3\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\frac{127}{7}a^7\sin^3\phi\,d\phi \\ &= \frac{127}{42}\pi a^7\left[\cos 3\phi - 9\cos\phi\right]_0^{\pi} = \frac{1016}{21}\pi a^7 \end{aligned}$$

14.7.29 The surface with spherical-coordinates equation $\rho = 2a\sin\phi$ is generated as follows. Draw the circle in the xz-plane with center $(a,0)$ and radius a. Rotate this circle around the z-axis. This generates the surface with the given equation. It is called a *pinched torus* - a doughnut with an infinitesimal hole. Its volume is

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2a\sin\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\frac{1}{3}(2a\sin\phi)^3\sin\phi\,d\rho \\ &= \frac{16}{3}\pi a^3\cdot 2\int_0^{\pi/2}\sin^4\phi\,d\rho = \frac{32}{3}\pi a^3\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2} = 2\pi^2 a^3 \end{aligned}$$

We evaluated the last integral with the aid of Formula(113) from the long table of integrals(see the endpapers). The volume of the pinched torus is also easy to evaluate using the first theorem of Pappus.(Section 14.5)

14.8.2 The surface area element is $dS = \sqrt{1 + 2^2 + 2^2}\, dA = 3\, dA$, so the area we are looking for is:

$$A = \int_0^1 \int_{x^2}^{\sqrt{x}} 3\, dy\, dx = \int_0^1 (3\sqrt{x} - 3x^2)\, dx = \left[2x^{3/2} - x^3\right]_0^1 = 1$$

14.8.8 The surface area element is $dS = \sqrt{2^2 + 3^2 + 1^2}\, dA = \sqrt{14}\, dA$, and we integrate in cylindrical coordinates to get the area of the ellipse:

$$\int_0^{2\pi} \int_0^{\sqrt{2}} r\sqrt{14}\, dr\, d\theta = 2\pi \left[\frac{1}{2} r^2 \sqrt{14}\right]_0^{\sqrt{2}} = 2\pi\sqrt{14} \approx 23.509$$

14.8.10 The surface area element is $dS = \sqrt{1 + 4x^2 + 4y^2}\, dA = \sqrt{1 + 4r^2}\, dA$, so the area is:

$$\int_0^{2\pi} \int_0^2 r\sqrt{1 + 4r^2}\, dr\, d\theta = 2\pi \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^2 = \frac{1}{6}\pi(17\sqrt{17} - 1) \approx 36.177$$