MATHEMATICS 52
SOLUTION SET 2

14.3.28 The volume is:

$$\begin{aligned} V &= \int_0^2 \int_{y/2}^{(4-y)/2} (8-4x-2y)\,dx\,dy = \int_0^2 \left[8x-2x^2-2xy\right]_{x=y/2}^{(4-y)/2} dy \\ &= \int_0^2 (2y^2-8y+8)\,dy = \left[\frac{2}{3}y^3-4y^2+8y\right]_0^2 = \frac{16}{3} \end{aligned}$$

14.3.30 The region is a triangle whose top has equation $y=\frac{1}{2}(5-x)$ and whose bottom has equation $y=\frac{1}{2}(x-5)$. Hence the volume of the solid is

$$\begin{aligned} V &= \int_{-3}^5 \int_{\frac{(x-5)}{2}}^{\frac{(5-x)}{2}} (25-x^2-y^2)\,dydx = \int_{-3}^5 \left[25y-x^2y-\frac{1}{3}y^3\right]_{\frac{(x-5)}{2}}^{\frac{(5-x)}{2}} dx \\ &= \int_{-3}^5 \left(\frac{1375}{12}-\frac{75}{4}x-\frac{25}{4}x^2+\frac{13}{12}x^3\right) dx \\ &= \left[\frac{1375}{12}x-\frac{75}{4}x^2-\frac{25}{12}x^3+\frac{13}{48}x^4\right]_{-3}^5 \\ &= \frac{1792}{3} \end{aligned}$$

14.3.32 The volume is:

$$\begin{aligned} V &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9-x^2-y^2)\,dy\,dx = \int_{-3}^3 \left[9y-x^2y-\frac{1}{3}y^3\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx \\ &= \int_{-3}^3 \frac{4}{9}(9-x^2)^{3/2}\,dx = \left[\frac{4}{3}\sqrt{9-x^2}\left(\frac{45}{8}x-\frac{1}{4}x^3\right)+\frac{81}{2}\arcsin\left(\frac{x}{3}\right)\right]_{-3}^3 \\ &= \frac{81}{2}\pi \approx 127.234 \end{aligned}$$

Note that sometimes using *Mathematica* to evaluate or simplify integrals might not be a bad idea!

14.3.40

Given: the ellipsoid with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

we assume that a, b, c are all positive. Set z=0 in the equation to find that the ellipsoid intersects the xy-plane in the ellipse with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

that is,

$$y=\frac{b}{a}(a^2-x^2)^{1/2}$$

(we take the positive root because we plan to integrate over the quarter of the ellipse that lies in the first quadrant). Finally, we solve the equation for

$$z = \frac{c}{ab}(a^2b^2 - b^2x^2 - a^2y^2)^{1/2}$$

We integrate to find the volume of the eighth of the ellipsoid that lies in the first octant, then multiply by 8. Hence the volume of the ellipsoid is

$$V = 8\int_0^a \int_0^{(b/a)(a^2-x^2)^{1/2}} \frac{c}{ab}(a^2b^2 - b^2x^2 - a^2y^2)^{1/2}\, dydx$$

Let

$$y = \frac{b}{a}(a^2 - x^2)^{1/2}\sin\theta \Rightarrow dy = \frac{b}{a}(a^2 - x^2)^{1/2}\cos\theta d\theta$$

This substitution yields

$$\begin{aligned}
V &= 8\int_0^a \int_0^{\pi/2} \frac{c}{ab}[(a^2b^2 - b^2x^2)(1 - \sin^2\theta)]^{1/2}\frac{b}{a}(a^2 - x^2)^{1/2}\cos\theta\, d\theta \\
&= \frac{8bc}{a^2}\int_0^a \int_0^{\pi/2} (a^2 - x^2)\cos^2\theta\, d\theta\, dx = \frac{8bc}{a^2}\int_0^a \int_0^{\pi/2} (a^2 - x^2)\frac{1 + \cos 2\theta}{2}\, d\theta\, dx \\
&= \frac{8bc}{a^2}\int_0^a (a^2 - x^2)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} dx = \frac{8bc}{a^2}\int_0^a \frac{\pi}{4}(a^2 - x^2)\, dx \\
&= \frac{2\pi bc}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{2\pi bc}{a^2}\cdot\frac{2}{3}a^3 = \frac{4}{3}\pi abc
\end{aligned}$$

14.4.2 The circle with polar equation $r = 3\sin\theta$ has area

$$\begin{aligned}
A &= \int_0^\pi \int_0^{3\sin\theta} r\, dr\, d\theta = \int_0^\pi \left[\frac{1}{2}r^2\right]_0^{3\sin\theta} d\theta = \int_0^\pi \frac{9}{2}\sin^2\theta\, d\theta \\
&= \int_0^\pi \frac{9}{4}(1 - \cos 2\theta)\, d\theta = \frac{9}{8}\left[2\theta - \sin 2\theta\right]_0^\pi = \frac{9}{4}\pi
\end{aligned}$$

14.4.4 The area bounded by the right-hand loop of the four-leaved rose with polar equation $r = 2\cos 2\theta$ is:

$$\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4}\int_0^{2\cos 2\theta} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{2\cos 2\theta} d\theta = \int_{-\pi/4}^{\pi/4} 2\cos^2 2\theta\, d\theta \\
&= \frac{1}{4}\left[4\theta + \sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{2} \approx 1.571
\end{aligned}$$

14.4.10 Note that as r is always positive, the domain of integration is generated as θ varies from $-\pi/2$ to $\pi/2$, so that the integral is:

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\, dr\, d\theta &= \int_{-\pi/2}^{\pi/2}\left[\frac{1}{4}r^4\right]_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta \\
&= \frac{1}{8}\left[12\theta + 8\sin 2\theta + \sin 4\theta\right]_{-\pi/2}^{\pi/2} = \frac{3}{2}\pi \approx 4.712
\end{aligned}$$

The integral of $\cos^4\theta$ can also be read off from the table of integrals at the end of the textbook.

$$\int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta = 8\int_0^{\pi/2}\cos^4\theta\, d\theta = 8\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2} = \frac{3}{2}\pi$$

14.4.14 Conversion of the integral to polar coordinates yields

$$K = \int_0^{\pi/2} \int_0^1 \frac{r}{\sqrt{4-r^2}} \, dr \, d\theta = \frac{\pi}{2} \cdot \left[-\sqrt{4-r^2} \right]_0^1 = \frac{\pi}{2} \cdot (2 - \sqrt{3}) \approx 0.4209$$

14.4.18 Following the general strategy for finding bounds in polar coordinates, we see (with a little trigonometry) that a ray at angle θ enters the quarter-circle at distance $r_1 = 1/\cos\theta$ from the origin and leaves it at distance $r_2 = 2\cos\theta$ from the origin. Once $\theta > \pi/4$, the ray will not even touch our region of integration. The integral in polar coordinates will hence be:

$$\begin{aligned}
\int_0^{\pi/4} \int_{1/\cos\theta}^{2\cos\theta} \frac{1}{r} r \, dr \, d\theta &= \int_0^{\pi/4} \left(2\cos\theta - \frac{1}{\cos\theta} \right) d\theta \\
&= \left[2\sin\theta - \ln\left(\frac{1}{\cos\theta} + \tan\theta \right) \right]_0^{\pi/4} = \sqrt{2} - \ln(1 + \sqrt{2}) \\
&\approx 0.533
\end{aligned}$$

14.4.20 The volume is

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 (2 + r\cos\theta) \cdot r \, dr \, d\theta = \int_0^{2\pi} \left[r^2 + \frac{1}{3} r^3 \cos\theta \right]_0^2 d\theta \\
&= \int_0^{2\pi} \left(4 + \frac{8}{3}\cos\theta \right) d\theta = \left[4\theta + \frac{8}{3}\sin\theta \right]_0^{2\pi} = 8\pi \approx 25.1327
\end{aligned}$$

14.4.28 When we solve the equations of the two paraboloids simultaneously, we get that they intersect along the circle $x^2 + y^2 = 1$, $z = 1$ so that the unit disk centered at the origin lying in xy-plane is a suitable region of integration. This region should be easy enough to parametrize in polar coordinates! The volume we are looking for is obtained by integrating $z_{top} - z_{bot} = (x^2 + y^2) - (2x^2 + 2y^2 - 1) = 1 - r^2$ over this region:

$$V = \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 d\theta = \frac{\pi}{2} \approx 1.571$$

14.4.32 The volume is:

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 2r\sqrt{18 - 2r^2} \, dr \, d\theta = 2\pi \cdot \left[-\frac{1}{3}(18 - 2r^2)^{3/2} \right]_0^2 \\
&= \frac{4}{3}\pi(-5\sqrt{10} + 27\sqrt{2}) \approx 93.713
\end{aligned}$$

14.4.34 The way to understand the given double integral with infinite limits (in regular coordinates) is:

$$\int_0^\infty \int_0^\infty f(x, y) \, dx \, dy = \lim_{b \to \infty} \int_0^b \int_0^b f(x, y) \, dx \, dy$$

It is obtained as a limit of an integral over a square which increases so that it ultimately fills up the first quadrant.

We first convert the given integral to polar form and integrate over the quarter-circle of radius a centered at the origin. This quarter circle has polar description

$0 \leq \theta \leq \pi/2$, $0 \leq r \leq a$. When we let $a \to \infty$, the circle will fill up the first quadrant, just like the square. The integral I_a over the quarter-circle of radius a is:

$$I_a = \int_0^{\pi/2} \int_0^a \frac{r}{(1+r^2)^2}\, dr\, d\theta = \frac{\pi}{2}\left[-\frac{1}{2(1+r^2)}\right]_0^a = \frac{\pi}{4}\left(1 - \frac{1}{1+a^2}\right)$$

so that:

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \lim_{a\to\infty} I_a = \frac{\pi}{4}$$

14.4.38 The circular disk $x^2 + y^2 \leq 4$ in the xy-plane is a suitable domain for the volume integral. The volume of the solid is therefore

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (3 + r\cos\theta + r\sin\theta) r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{3}{2}r^2 + \frac{1}{3}r^3(\cos\theta + \sin\theta)\right]_0^2 d\theta \\ &= \int_0^{2\pi} \left(6 + \frac{8}{3}[\cos\theta + \sin\theta]\right) d\theta = \left[6\theta - \frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta\right]_0^{2\pi} = 12\pi \approx 37.699 \end{aligned}$$

14.5.6 Here we have

$$m = \int_0^1 \int_y^{2-y} 1\, dx\, dy = \int_0^1 (2-2y)\, dy = \left[2y - y^2\right]_0^1 = 1$$

$$M_x = \int_0^1 \int_y^{2-y} y\, dx\, dy = \int_0^1 (2y - 2y^2)\, dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1 = \frac{1}{3}$$

By symmetry, $\bar{x} = 1$. Therefore the centroid is located at $(1, \frac{1}{3})$

14.5.14 The mass and moments of the lemma are

$$\begin{aligned} m &= \int_{-3}^3 \int_0^{9-y^2} x^2\, dx\, dy = \int_{-3}^3 \frac{1}{3}(9-y^2)^3\, dy = \left[243y - 27y^3 + \frac{9}{5}y^5 - \frac{1}{21}y^7\right]_{-3}^3 \\ &= \frac{23328}{35} \\ M_y &= \int_{-3}^3 \int_0^{9-y^2} x^3\, dx\, dy = \int_{-3}^3 \frac{1}{4}(9-y^2)^4\, dy \\ &= \left[\frac{6561}{4}y - 243y^3 + \frac{243}{10}y^5 - \frac{9}{7}y^7 + \frac{1}{36}y^9\right]_{-3}^3 = \frac{139968}{35} \\ M_x &= \int_{-3}^3 \int_0^{9-y^2} x^2 y\, dx\, dy = \int_{-3}^3 \frac{1}{3}y(9-y^2)^3\, dy \\ &= \left[\frac{243}{2}y^2 - \frac{81}{4}y^4 + \frac{3}{2}y^6 - \frac{1}{24}y^8\right]_{-3}^3 = 0 \end{aligned}$$

14.5.28 The mass and moments are

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}(1+\cos\theta)^3\,d\theta \\
&= \int_0^{2\pi}\left(\frac{1}{3}+\cos\theta+\cos^2\theta+\frac{1}{3}\cos^3\theta\right)d\theta \\
&= \frac{1}{36}\left[30\theta+45\sin\theta+9\sin 2\theta+\sin 3\theta\right]_0^{2\pi} = \frac{5}{3}\pi \\
M_y &= \int_0^{2\pi}\int_0^{1+\cos\theta} r^3\cos\theta\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}(\cos\theta)(1+\cos\theta)^4\,d\theta \\
&= \int_0^{2\pi}\left(\frac{1}{4}\cos\theta+\cos^2\theta+\frac{3}{2}\cos^3\theta+\cos^4\theta+\frac{1}{4}\cos^5\theta\right)d\theta \\
&= \frac{1}{960}\left[840\theta+1470\sin\theta+480\sin 2\theta+145\sin 3\theta+30\sin 4\theta+3\sin 5\theta\right]_0^{2\pi} \\
&= \frac{7}{4}\pi \\
M_x &= \int_0^{2\pi}\int_0^{1+\cos\theta} r^3\sin\theta\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{4}(1+\cos\theta)^4\,d(1+\cos\theta) \\
&= \left[-\frac{1}{20}(1+\cos\theta)^5\right]_0^{2\pi} = 0
\end{aligned}$$

14.5.42 Note that the quarter of the circular disk $x^2+y^2 \le a^2$ which lies in the first quadrant is symmetric with respect to the line $x = y$, so that the coordinates $(\bar{x}, \bar{y})$ of the centroid satisfy $\bar{x} = \bar{y}$. The area of this circular quarter is one-fourth of the area of the disk - $A = a^2\pi/4$. When we rotate it around the y-axis, its centroid describes a circle of radius $\bar{x} = \bar{y}$ while the body of rotation is a hemiball of volume $2a^3\pi/3$. The first theorem of Pappus now gives us:

$$\frac{2}{3}a^3\pi = V = d\cdot A = 2\pi\bar{x}\cdot\frac{1}{4}a^2\pi$$

so that $\bar{x} = \bar{y} = 4a/3\pi$.

14.5.52 The area of the region is

$$A = \int_0^h\int_0^{\sqrt{2py}} 1\,dx\,dy = \int_0^h \sqrt{2py}\,dy = \left[\frac{2}{3}y^{3/2}\sqrt{2p}\right]_0^h = \frac{2}{3}h^{3/2}\sqrt{2p} = \frac{2}{3}rh$$

(the last equality follows from the substitution of $\frac{r^2}{2h}$ for p) The moments are

$$\begin{aligned}
M_y &= \int_0^h\int_0^{\sqrt{2py}} x\,dx\,dy = \int_0^h py\,dy = \frac{h^2p}{2} = \frac{r^2h}{4} \\
M_x &= \int_0^h\int_0^{\sqrt{2py}} y\,dx\,dy = \int_0^h y^{3/2}\sqrt{2p}\,dy = \frac{2}{5}h^{5/2}\sqrt{2p} = \frac{2}{5}rh^2
\end{aligned}$$

Therefore the centroid of the region is

$$(\bar{x}, \bar{y}) = (\frac{3}{8}r, \frac{3}{5}h)$$

By the first theorem of Pappus, the volume of the paraboloid generated by rotating this region around the y-axis is therefore

$$V = 2\pi\bar{x} \cdot A = 2\pi \cdot \frac{3}{8}r \cdot \frac{2}{3}rh = \frac{1}{2}\pi r^2 h$$