MATHEMATICS 52
SOLUTION SET 1

14.1.4 In this partition, the area ΔA_i of each partition square is 1, so the Riemann sum will be:

$$\sum_{i=1}^{4} f(x_i^*, y_i^*)\Delta A_i = f(\frac{1}{2}, \frac{1}{2}) + f(\frac{3}{2}, \frac{1}{2}) + f(\frac{1}{2}, \frac{3}{2}) + f(\frac{3}{2}, \frac{3}{2}) = 4$$

14.1.8 In this partition, the area ΔA_i of each partition rectangle is $\Delta x \Delta y = \pi/6$, so the Riemann sum will be:

$$\begin{aligned} & \textstyle\sum_{i=1}^{6} f(x_i^*, y_i^*)\Delta A_i \\ = \quad & \tfrac{\pi}{6}\left(f(\tfrac{1}{4}, \tfrac{\pi}{6}) + f(\tfrac{3}{4}, \tfrac{\pi}{6}) + f(\tfrac{1}{4}, \tfrac{\pi}{2}) + f(\tfrac{3}{4}, \tfrac{\pi}{2}) + f(\tfrac{1}{4}, \tfrac{5\pi}{6}) + f(\tfrac{3}{4}, \tfrac{5\pi}{6})\right) = \tfrac{\pi}{2} \end{aligned}$$

14.1.14 We integrate first with respect to x, then with respect to y:

$$\int_{-2}^{1}\int_{2}^{4} x^2y^3\,dy\,dx = \int_{-2}^{1}\left[\frac{1}{4}x^2y^4\right]_2^4 dx = \int_{-2}^{1} 60x^2\,dx = \left[20x^3\right]_{-2}^{1} = 180$$

14.1.20 Here, we integrate first with respect to y, then with respect to x

$$\begin{aligned} \int_0^{\pi/2}\int_0^{\pi/2} \cos x \sin y\,dy\,dx &= \int_0^{\pi/2} \left[-\cos x \sin y\right]_0^{\pi/2} dx = \int_0^{\pi/2} \cos x\,dx \\ &= \left[\sin x\right]_0^{\pi/2} = 1 \end{aligned}$$

14.1.24 Here, we integrate first with respect to x, then with respect to y:

$$\begin{aligned} \int_0^1\int_0^1 e^{x+y}\,dx\,dy &= \int_0^1 \left[e^{x+y}\right]_0^1 dy = \int_0^1 (e^{y+1} - e^y)\,dy = \left[e^{y+1} - e^y\right]_0^1 \\ &= (e^2 - 2e + 1) \approx 2.9525 \end{aligned}$$

14.1.26 We integrate first with respect to x, then with respect to y.

$$\begin{aligned} \int_0^{\pi/2}\int_0^{\pi/2} (y-1)\cos x\,dx\,dy &= \int_0^{\pi/2} \left[(y-1)\sin x\right]_0^{\pi/2} dy = \int_0^{\pi/2} (y-1)\,dy \\ &= \left[\frac{1}{2}y^2 - y\right]_0^{\pi/2} = \frac{1}{8}(\pi^2 - 4\pi) \approx -0.337 \end{aligned}$$

14.1.30 We integrate first with respect to y, then with respect to x:

$$\begin{aligned} \int_1^2\int_1^3 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx &= \int_1^2 \left[\frac{y^2}{2x} + x\ln y\right]_1^3 dx = \int_1^2 \left(\frac{4}{x} + x\ln 3\right) dx \\ &= \left[\frac{1}{2}x^2\ln 3 + 4\ln x\right]_1^2 = 4\ln 2 + \frac{3}{2}\ln 3 \approx 4.4205 \end{aligned}$$

14.1.32 The first evaluation, when we integrate first with respect to x and then with respect to y, yields:

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\sin x\cos y\,dx\,dy &= \int_{-\pi/2}^{\pi/2}\left[-\cos x\cos y\right]_0^{\pi}\,dy\\ &= \int_{-\pi/2}^{\pi/2}2\cos y\,dy = \left[2\sin y\right]_{-\pi/2}^{\pi/2} = 2-(-2) = 4\end{aligned}$$

The second evaluation, when we integrate first with respect to y and then with respect to x, yields the same result:

$$\begin{aligned}\int_0^{\pi}\int_{-\pi/2}^{\pi/2}\sin x\cos y\,dy\,dx &= \int_0^{\pi}\left[\sin x\sin y\right]_{-\pi/2}^{\pi/2}\,dx\\ &= \int_0^{\pi}2\sin x\,dx = \left[-2\cos x\right]_0^{\pi} = 2-(-2) = 4\end{aligned}$$

14.1.36 Note that whatever the choice of (x_i^*, y_i^*), $f(x_i^*, y_i^*) = k$, and hence $f(x_i^*, y_i^*)\Delta A_i$ is equal to the product of k and the area$a(R_i)$ of R_i for each i, Hence

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i = \sum_{i=1}^{n} k\cdot a(R) = k\left(\sum_{i=1}^{n} a(R_i)\right) = k\cdot a(R) = k(b-a)(d-c)$$

14.2.2

$$\int_0^2\int_0^{2x}(1+y)\,dy\,dx = \int_0^2\left[y+\frac{1}{2}y^2\right]_{y=0}^{y=2x}\,dx = \int_0^2\left(2x+2x^2\right)\,dx = \left[x^2+\frac{2}{3}x^3\right]_0^2 = \frac{28}{3}$$

14.2.6

$$\begin{aligned}\int_0^1\int_y^{\sqrt{y}}(x+y)\,dx\,dy &= \int_0^1\left[\frac{1}{2}x^2+xy\right]_y^{\sqrt{y}}\,dy = \int_0^1\left(\frac{1}{2}y+y^{3/2}-\frac{3}{2}y^2\right)\,dy\\ &= \left[\frac{1}{4}y^2+\frac{2}{5}y^{5/2}-\frac{1}{2}y^3\right]_0^1 = \frac{3}{20}\end{aligned}$$

14.2.12

$$\begin{aligned}\int_0^{\pi}\int_0^{\sin x}y\,dy\,dx &= \int_0^{\pi}\left[\frac{1}{2}y^2\right]_0^{\sin x}\,dx = \int_0^{\pi}\frac{1}{2}\sin^2 x\,dx = \left[\frac{1}{8}(2x-\sin 2x)\right]_0^{\pi}\\ &= \frac{\pi}{4}\end{aligned}$$

14.2.18 The parabolas $x = 1-y^2$ and $x = y^2-1$ intersect at the points $(0,-1)$ and $(0,1)$, so that:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_{-1}^{1}\int_{y^2-1}^{1-y^2}y\,dx\,dy = \int_{-1}^{1}\left[xy\right]_{y^2-1}^{1-y^2}\,dy\\ &= \int_{-1}^{1}(2y-2y^3)\,dy = \left[y^2-\frac{1}{2}y^4\right]_{-1}^{1} = \frac{1}{2}-\frac{1}{2} = 0\end{aligned}$$

14.2.20 $\cos x$ is nonnegative for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Observing $\sin x$ is an odd fun in x and our domain is symmetric in x, the anwer must be zero. Let's check it:

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} \sin x \, dy \, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [y \sin x]_0^{\cos x} \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x \, dx = \left[\frac{1}{2}\sin^2 x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

14.2.24 The edges of the triangle R have equations $x = 0$, $y = 2x$ and $y = 9 - x$, so that:

$$\begin{aligned} \iint_R f(x,y) \, dA &= \int_0^3 \int_{2x}^{9-x} (9-y) \, dy \, dx = \int_0^3 \left[9y - \frac{1}{2}y^2\right]_{2x}^{9-x} dx \\ &= \int_0^3 \frac{3}{2}(x^2 - 12x + 27) \, dx = \left[\frac{1}{2}(x^3 - 18x^2 + 81x)\right]_0^3 = 54 \end{aligned}$$

14.2.26 Reversal of the order of the integral yields

$$\begin{aligned} \int_0^1 \int_y^{y^{1/4}} (x-1) \, dx \, dy &= \int_0^1 \left[\frac{1}{2}x^2 - x\right]_y^{y^{1/4}} dy = \int_0^1 \frac{1}{2}(2y - y^2 + y^{1/2} - 2y^{1/4}) \, dy \\ &= \left[\frac{1}{2}y^2 - \frac{1}{6}y^3 + \frac{1}{3}y^{3/2} - \frac{4}{5}y^{5/4}\right]_0^1 = -\frac{2}{15} \end{aligned}$$

14.2.32 The domain R of the given integral is bounded above by the line $y = x$, on the right by the line $x = \sqrt{\pi}$, and below by the x-axis. Reversing the order of integration, we get:

$$\begin{aligned} \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin x^2 \, dx \, dy &= \int_0^{\sqrt{\pi}} \int_0^x \sin x^2 \, dy \, dx = \int_0^{\sqrt{\pi}} \left[y \sin x^2\right]_0^x dx \\ &= \int_0^{\sqrt{\pi}} x \sin x^2 \, dx = \left[-\frac{1}{2}\cos x^2\right]_0^{\sqrt{\pi}} = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

14.2.42 By symmetry around both coordinate axes, the value of the integral is

$$4\int_0^1 \int_0^{1-x} x^2 \, dy \, dx = 4\int_0^1 \left[x^2 y\right]_0^{1-x} dx = 4\int_0^1 (x^2 - x^3) \, dx = 4\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3}$$

14.2.50 The basic idea for solving this problem is to reduce the given situation, in which the region R sits in the plane, to a one-dimensional situation by restricting f to some curve in R. This restriction will then behave as a function of a single variable - After noticing that, the intermediate value theorem from single-variable calculus and Eq. (8) from 14.2 finish the job! Here is one way to write up the proof:

Let $m = f(x_0, y_0)$ and $M = f(x_1, y_1)$ be the minimum and maximum of f over R, respectively, where (x_0, y_0) and (x_1, y_1) are some points in R. By connectedness of R, there is a continuous parametric curve $r : [0,1] \to R$ such that $r(0) = (x_0, y_0)$ and $r(1) = (x_1, y_1)$. In other words, r is a curve lying entirely in the region R which starts off at the point where f is minimal and ends up at the point where f is maximal. Now consider the function $g(t) = f(r(t))a(R)$ where $a(R)$ is the area of R, and $t \in [0,1]$. Note that g is a real-valued function of a single variable t. Since $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, we have that for all $t \in [0,1]$:

$$m \cdot a(R) \leq g(t) \leq M \cdot a(R)$$

Eq. (8) from 14.2. in the textbook tells us that the value of the double integral of f over R lies between the same two values:

$$m \cdot a(R) \leq \iint_R f(x,y)\,dA \leq M \cdot a(R)$$

The intermediate value theorem from single variable calculus, when applied to g, implies that there exists some $\bar{t} \in [0,1]$ such that:

$$g(\bar{t}) = \iint_R f(x,y)\,dA$$

If we let $(\bar{x},\bar{y}) = r(\bar{t})$, we get exactly what we needed to show:

$$\iint_R f(x,y)\,dA = g(\bar{t}) = f(\bar{x},\bar{y})a(R)$$

14.3.4 The graph cross where $2x+3 = 6x-x^2$, that is, where $x=1$ and where $x=3$. The area they enclose is

$$A = \int_1^3 \int_{y=2x+3}^{6x-x^2} 1\,dy\,dx = \int_1^3 (4x-3-x^2)\,dx = \left[2x^2 - 3x - \frac{1}{3}x^3\right]_1^3 = \frac{4}{3}$$

14.3.10 The curves cross where

$$\begin{aligned} x^2 &= \frac{2}{1+x^2} \\ x^4 + x^2 - 2 &= 0 \\ (x^2+2)(x^2-1) &= 0 \end{aligned}$$

That is, they cross only at $x=1$ and $x=-1$. The area of the region they bound is then:

$$\begin{aligned} \int_{-1}^1 \int_{x^2}^{2/(1+x^2)} 1\,dy\,dx &= \int_{-1}^1 \left(\frac{2}{1+x^2} - x^2\right)\,dx = \left[2\arctan x - \frac{1}{3}x^3\right]_{-1}^1 \\ &= \pi - \frac{2}{3} \approx 2.475 \end{aligned}$$

14.3.16 The volume is:

$$\begin{aligned} V &= \int_0^4 \int_0^{\frac{4-x}{2}} (3x+2y)\,dy\,dx = \int_0^4 \left[3xy + y^2\right]_0^{\frac{4-x}{2}}\,dx \\ &= \int_0^4 (4 + 4x - \frac{5}{4}x^2)\,dx = \left[4x + 2x^2 - \frac{5}{12}x^3\right]_0^4 = \frac{64}{3} \end{aligned}$$

14.3.24 The volume of the solid is:

$$\begin{aligned} V &= \int_0^1 \int_{y=x^2}^{\sqrt{x}} (10+y-x^2)\,dy\,dx = \int_0^1 \left[10y - x^2y + \frac{1}{2}y^2\right]_{y=x^2}^{\sqrt{x}}\,dx \\ &= \int_0^1 \left(10x^{\frac{1}{2}} + \frac{1}{2}x - 10x^2 - x^{5/2} + \frac{1}{2}x^4\right)\,dx = \left[\frac{20}{3}x^{\frac{3}{2}} + \frac{1}{4}x^2 - \frac{3}{10}x^3 - \frac{2}{7}x^{\frac{7}{2}} + \frac{1}{10}x^5\right]_0^1 \\ &= \frac{1427}{420} \approx 3.398 \end{aligned}$$