Homework 2, Math 263A: Lie Groups and Lie Algebras
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1. **An alternate description of irreducible representations of SU(2):** Let V_m be the space of homogeneous polynomials in two complex variables of total degree m, an $m+1$-dim'l complex vector space. Define a representation Π_m of $SU(2)$ on V_m by the formula
$$\Pi_m(g)f(z_1, z_2) = f(g^{-1}(z_1, z_2)^T).$$
Let χ_m be the character of this representation and T the maximal torus of $SU(2)$ consisting of elements
$$t_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$
(a) Compute $\Pi_m(t_\theta)$ on each monomial $z_1^k z_2^{m-k}$.

(b) Compute $\pi_m\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ (π is the induced Lie algebra rep'n.) and conclude that Π_m is irreducible (using facts from our discussion about representations of $su(2)$).

(c) Give an explicit finite geometric series for $\chi_m(t_\theta)$ and express the sum as a quotient.

(d) Decompose $\chi_m\chi_n$ as a sum of chi_Ns and give a formula for the multiplicity of Π_N in $\Pi_m \otimes \Pi_n$.

2. Work out the representation of $sl(3, \mathbb{C})$ with highest weight (2,0) acting on a subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$. Determine all the weights of this representation and their multiplicity (i.e. the dimension of the associated weight space). Verify the dimension formula in this case: the irreducible representation of highest weight (m_1, m_2) has dimension
$$\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$$
(a consequence of the Weyl character formula to be shown next week).

3. What changes in the previous exercise for the highest weight representation (1,1)?

4. Let $\mathfrak{h}_\mathbb{C}$ be the complexification of the Lie algebra of the Heisenberg group, i.e. the space of all complex 3×3 upper triangular matrices with 0 on the diagonal. Show that $\mathfrak{h}_\mathbb{C}$ does not have a Cartan subalgebra.

5. Suppose that $\mathfrak{g}$ is a complex Lie algebra with the complete reducibility property. Show that $\mathfrak{g}$ is semisimple.

6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with compact real form $\mathfrak{k} \cong Lie(K)$, for K a compact Lie group. Let $\mathfrak{t} = Lie(T)$ be a maximal commutative subalgebra associated to maximal torus T.

   (a) Let $Z(T)$ be the centralizer of T in K. Show that $Z(T)$ is isomorphic to $Z(\mathfrak{t}) = \{A \in K | Ad_A(H) = H \text{ for all } H \in \mathfrak{t}\}$.

   (b) Let $N(T)$ be the normalizer of T in K. Show that $N(T)$ is isomorphic to $N(\mathfrak{t}) = \{A \in K | Ad_A(H) \subset \mathfrak{t} \text{ for all } H \in \mathfrak{t}\}$.

   Hence the Lie algebra and Lie group definition of the Weyl group coincide.