$$\Rightarrow \left| \begin{array}{ll} 3p^2+2pq+5q^2=3 & \text{(multiply by 12 both sides)} \\ 3r^2+2rs+5s^2=5 & \text{(multiply by 12 both sides)} \end{array} \right\} \begin{array}{ll} (6p+2q)^2+56q^2=36 & (1) \\ (6r+2s)^2+56s^2=60 & (2) \end{array}$$

$(1) \Rightarrow q=0, \quad p=\mp 1$

$(2) \Rightarrow s=0$ or ∓ 1. If $s^2=0$, then $(6r+2s)^2=60$, which is impossible since $6r+2s \in \mathbb{Z}$. Therefore $s=\mp 1$, and $(6r+2s)^2=4$

But this implies $r=0$.

So $s=\mp 1, \quad p=\mp 1, \quad q=r=0$.

But $1=3pr+qr+ps+5qs = ps$, so $ps=1$, therefore

$$\det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = ps-qr=1,$$ which proves the claim.

2.2 (c) $f \sim g \Leftrightarrow g \sim f$ (proof of reflexivity for proper equiv. in (2.2)(a) works for improper equiv. as well)

So ~~it~~ it suffices to prove the following: with $\det\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\mp 1$

— Assume $f(x,y)=g(px+qy,\ rx+sy)$; and $f(x_0,y_0)=m$ for some $(x_0,y_0)\in \mathbb{Z}\times\mathbb{Z}$. Then g represents m as well.

Proof: Just let $x=px_0+qy_0$, $y=rx_0+sy_0$, then $g(x,y)=m$.

— If f,g are as above and f represents m properly, then so does g.

Proof: We only need to prove $\gcd(x_0,y_0)=1 \Rightarrow \gcd(px_0+qy_0,\ rx_0+sy_0)=1$.

But say $d \mid px_0+qy_0$, and $d \mid rx_0+sy_0 \Rightarrow d \mid s(px_0+qy_0)-q(rx_0+sy_0)$

$$\Downarrow \qquad\qquad\qquad\qquad \shortparallel$$

$$d \mid r(px_0+qy_0)-p(rx_0+sy_0)=\mp y_0 \qquad (ps-qr)x_0=\mp 1\, x_0$$

$d \mid x_0$ and $d \mid y_0 \Rightarrow \underline{d=1, \text{ as desired}}$

□

2.2 (d) Assume that $f(x,y)=ax^2+bxy+cy^2 \sim g$, and g is primitive. And assume that $\gcd(a,b,c)=d>1$. Since $f\sim g$, by 2.2 (c) f and g represent same integers, hence g (b/c f does so) represents only integers div. by d. This contradicts Cox / Lemma 2.15 (which you still have not covered)