2.1: Say $f(x,y) = ax^2 + bxy + cy^2 = m$ for some $x, y \in \mathbb{Z}$.
If $\gcd(x,y) = d$, write $x = dx_0$, $y = dy_0$ (so that $\gcd(x_0, y_0) = 1$).
Then, letting $m' = ax_0^2 + bx_0y_0 + cy_0^2$, we see:

$$m = a(dx_0)^2 + b(dx_0)(dy_0) + c(dy_0)^2 = d^2(ax_0^2 + bx_0y_0 + cy_0^2) = dm'$$

and f represents m' properly (since $(x_0, y_0) = 1$).

2.2 (a) For $f(x,y) = ax^2 + 2bxy + cy^2$, let $A_f = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.
Then note that, $f(x,y) = g(px + qy, rx + sy) \Leftrightarrow A_f = U^T A_g U$

where $U = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

Now f is properly equiv. to $g \Leftrightarrow A_f = U^T A_g U$ with $\det U = 1$

- Reflexivity:
  f is properly equiv. to $g \Leftrightarrow A_f = U^T A_g U \Leftrightarrow A_g = (U^{-1})^T A_f U^{-1}$
  (note that U^{-1} has integer entries since $\det U = 1$) (and note $\det(U^{-1}) = 1$ as well)
  $\Leftrightarrow g$ is properly equiv. to f.
- Symmetry:
  f is properly equiv. to f by letting $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ above.
- transitivity:

$$\left.\begin{array}{l} f \text{ is prop equiv. to } g \Leftrightarrow A_f = U^T A_g U \\ g \text{ is " " " } h \Leftrightarrow A_g = V^T A_h V \end{array}\right\} A_f = (VU)^T A_h (VU)$$

and note that $\det(VU) = \det(V)\det(U) = 1$.

2.2 (b) $f(x,y) = 3x^2 + 2xy + 5y^2$ is not improperly equiv. to itself.
In fact say

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3p^2 + 2pq + 5q^2 & 3pr + qr + ps + 5qs \\ 3pr + qr + ps + 5qs & 3r^2 + 2rs + 5s^2 \end{pmatrix}$$