① (a) By Cox, $p \mid x^2+3y^2 \Leftrightarrow \left(\frac{-3}{p}\right)=1$

Now $\left(\frac{-3}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) \underset{\text{Reciprocity Law}}{=} \left(\frac{-1}{p}\right)\left(\frac{p}{3}\right)(-1)^{2\cdot\frac{(p-1)}{4}} = \left(\frac{-1}{p}\right)(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)$

$\underset{\text{b/c } \left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}}{=} \left(\frac{p}{3}\right)$; therefore

$\left(\frac{-3}{p}\right)=1 \Leftrightarrow \left(\frac{p}{3}\right)=1 \Leftrightarrow p\equiv 1 \pmod 3$, as desired. □

(b) Arguing just like above, we see that

$p \mid x^2+2y^2 \Leftrightarrow \left(\frac{-2}{p}\right)=1 \Leftrightarrow \left(\frac{2}{p}\right)=(-1)^{\frac{p-1}{2}}$

Now, $\left(\frac{2}{p}\right)=\begin{cases} 1 & p\equiv 1,7\ (8) \\ -1 & p\equiv 3,5\ (8)\end{cases}$; therefore

$\left(\frac{2}{p}\right)=(-1)^{\frac{p-1}{2}} \Leftrightarrow p\equiv 1,3 \pmod 8$, hence

$p \mid x^2+2y^2 \Leftrightarrow p\equiv 1,3 \pmod 8$, as desired. □

② $\left(\frac{-21}{p}\right)=1 \Leftrightarrow \left(\frac{-1}{p}\right)\cdot\left(\frac{3}{p}\right)\left(\frac{7}{p}\right)=1 \Leftrightarrow (-1)^{\frac{p-1}{2}}\cdot\left(\frac{p}{3}\right)\cdot(-1)^{\frac{p-1}{2}}\cdot\left(\frac{p}{7}\right)(-1)^{3\cdot\frac{(p-1)}{2}}=1 \Leftrightarrow \left(\frac{p}{3}\right)\left(\frac{p}{7}\right)=(-1)^{\frac{p-1}{2}}$

So, below are the whole set of R.L. options:

(i) $\left.\begin{array}{l} p\equiv 1 \pmod 4 \\ p\equiv 1 \pmod 3\end{array}\right\} \Rightarrow \begin{array}{lll} p\equiv 1\ (7) & \longleftrightarrow & p\equiv 1\ (84) \\ p\equiv 2\ (7) & \longleftrightarrow & p\equiv 37\ (84) \\ p\equiv 4\ (7) & \longleftrightarrow & p\equiv 25\ (84)\end{array}$

(ii) $\left.\begin{array}{l} p\equiv 1\ (4) \\ p\equiv 2\ (3)\end{array}\right\} \Rightarrow \begin{array}{lll} p\equiv 3\ (7) & \longleftrightarrow & p\equiv 17\ (84) \\ p\equiv 5\ (7) & \longleftrightarrow & p\equiv 5\ (84) \\ p\equiv 6\ (7) & \longleftrightarrow & p\equiv 41\ (84)\end{array}$

(iii) $\left.\begin{array}{l} p\equiv 3\ (4) \\ p\equiv 1\ (3)\end{array}\right\} \Rightarrow \begin{array}{lll} p\equiv 3\ (7) & \longleftrightarrow & p\equiv 31\ (84) \\ p\equiv 5\ (7) & \longleftrightarrow & p\equiv 19\ (84) \\ p\equiv 6\ (7) & \longleftrightarrow & p\equiv 55\ (84)\end{array}$

(iv) $\left.\begin{array}{l} p\equiv 3\ (4) \\ p\equiv 2\ (3)\end{array}\right\} \Rightarrow \begin{array}{lll} p\equiv 1\ (7) & \longleftrightarrow & p\equiv 71\ (84) \\ p\equiv 2\ (7) & \longleftrightarrow & p\equiv 23\ (84) \\ p\equiv 4\ (7) & \longleftrightarrow & p\equiv 11\ (84)\end{array}$

Therefore $\left(\frac{-21}{p}\right)=1 \Leftrightarrow$

$p\equiv 1, 5, 11, 19, 23, 25, 17, 31, 37, 55, 41, 71 \pmod{84}$