SOLUTIONS TO ASSIGNMENT 7

16.1.Solution : Let $G := \mathbb{C} - \{0\}$, equipped with normal multiplication (of complex numbers). So every $z \in G$ has a unique expression $z = re^{i\theta}$ for some $r > 0$ and some $\theta \in [0, 2\pi)$.

a) $z \to z*$ is a homomorphism. For $z, w \in G$, we always have $(zw)* = z * w*$. If we write $z = re^{i\theta}$ and $w = Re^{i\phi}$, the $zw = rRe^{i(\theta+\phi)}$, and so $(zw)* = rRe^{-i(\theta+\phi)} = re^{-i\theta}Re^{-i\phi} = z*w*$ as required.

b) This also is a homomorphism. For $(zw)^2 = zwzw = z^2w^2$ since multiplication is commutative.

c) This is not a homomorphism. For let $z = w = 1$. Then $(iz)(iw) = -1 \neq i = i(zw)$.

d) This is a homomorphism. With the notation from a), we have $|zw| = rR = |z||w|$.

16.3.Solution : First we show that the subgroup $G \times \{e\}$ is normal. Let $(x, e) \in G \times \{e\}$ and $(g, h) \in G \times H$.

$$(g, h)^{-1}(x, e)(g, h) = (g^{-1}xg, h^{-1}eh) = (g^{-1}xg, e) \in G \times \{e\}$$

Hence $G \times \{e\} \trianglelefteq G \times H$.

Define a map $\psi : G \times H \to H$ by

$$\psi(g, h) = h.$$

Then ψ is a surjective homomorphism. The kernel of ψ is $G \times \{e\}$. By the First Isomorphism Theorem, $G \times H/G \times \{e\} \cong H$.

16.4.Solution : Let $p_A : G \to G/A$ and $p_B : H \to H/B$ be the natural quotient maps. Then verify that $f : G \times H \to (G/A) \times (H/B)$ given by $f(g, h) := (p_A(g), p_B(h)) = (gA, hB)$ is a surjective homomorphism with kernel $A \times B$. The the First Isomorphism Theorem yields the conclusion.

16.8.Solution : Let H be the subset of elements $\{(g, \phi(g)) : g \in G\}$. Assume $\phi : G \to G'$ is a homomorphism. We show that H is a subgroup of $G \times G'$.

H is nonempty since $(e, e') \in H$. Let $(g, \phi(g))$ and $(h, \phi(h))$ be two elements in H.

$$(g, \phi(g))(h, \phi(h))^{-1} = (gh^{-1}, \phi(g)\phi(h)^{-1}) = (gh^{-1}, \phi(gh^{-1})) \in H$$

Hence H is a subgroup of $G \times G'$.

Conversely, suppose H is a subgroup of $G \times G'$. Then given $(g, \phi(g))$ and $(h, \phi(h))$ in H, we have $(g, \phi(g))(h, \phi(h)) \in H$; but $(g, \phi(g))(h, \phi(h)) = (gh, \phi(g)\phi(h))$, by the definition of H, $\phi(g)\phi(h) = \phi(gh)$, which implies that ϕ is a homomorphism.

16.12.Solution : Here is a statement of the Correspondence Theorem (sometimes called the fourth isomorphism theorem):

Let G be group and let $H \trianglelefteq G$. Write $\pi : G \to G/H$ for the natural map. Then

$$S \mapsto \pi(S) = S/H$$

is a bijection from $Sub(G; H)$, the family of all those subgroups S of G that contain H, and $Sub(G/H)$, the family of all the subgroups of G/H. Furthermore, $T \trianglelefteq S$ if and only if $T/H \trianglelefteq S/H$, in which case $S/T \cong (S/H)/(T/H)$.

This theorem tells us that every subgroup of G/H has the form S/H for a unique subgroup $S \leq G$ containing H; also S/H is normal in G/H if and only if the corresponding S is normal in G.

The question now follows from this theorem - once you recall that a group is simple iff it has only the trivial normal subgroups.