# SELECT SOLUTIONS TO ASSIGNMENT 6

**14.9.Solution** : Not given.

**15.1.Solution** : For $H \leq G$, $K \leq G$, we are asked to prove $HK \leq G \Leftrightarrow HK = KH$. Assume first that $KH = HK$. Since $H$ and $K$ are subgroups, they both contain the identity element $e$, so $e \in HK \Rightarrow HK \neq \emptyset$. Let $a, b \in HK$. Then we can write $a = h_1k_1$ and $b = h_2k_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$. Let $k_3 = k_1k_2^{-1} \in K$ and $h_3 = h_2^{-1}$. Thus $ab^{-1} = h_1k_3h_3$. Since $HK = KH$, we can write $k_3h_3 = h_4k_4$ for some $h_4 \in H, k_4 \in K$. Thus $ab^{-1} = h_1h_4k_4$, and since $h_1h_4 \in H, k_4 \in K$, we obtain $ab^{-1} \in HK$ as required. So $HK \leq G$.

Now assume $HK \leq G$. Since $K \leq HK$ and $H \leq HK$, by the closure property of subgroups, $KH \subset HK$. To show the reverse containment let $hk \in HK$. Since $HK$ is assumed to be a subgroup, write $hk = a^{-1}$ for some $a \in HK$. Then if we write $a = h_1k_1$, we have $hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$, so $HK \subset KH \Rightarrow HK = KH$ as required.

**15.2.Solution** : Not given.

**15.5.Solution** : We let $G = D_8$, $H = \langle s, r^2\rangle$, and $J = \langle s\rangle$. Then certainly $\langle s\rangle \trianglelefteq \langle s, r^2\rangle \trianglelefteq D_8$ since each subgroup is of index two in the next (and an index two subgroup is always normal). However, $\langle s\rangle$ is not normal in $D_8$ since $rsr^{-1} = sr^2 \notin \langle s\rangle$

**15.6.Solution** : We want to show that

$$xy = yx \quad \text{for all } x \in H \text{ and } y \in J$$

This is equivalent to showing that $xyx^{-1}y^{-1} = e$ for all such $x$ and $y$. But $H$ and $J$ are normal subgroups. So $xyx^{-1} \in J$, and hence $xyx^{-1}y^{-1} \in J$ by the closure property of subgroups. Similarly $y(x^{-1})y^{-1} \in H$ (since $H$ normal), and hence $xyx^{-1}y^{-1} \in H$ (by closure), so the element $xyx^{-1}y^{-1}$ is in both $H$ and $J$. But $H \cap J = \{e\}$ by assumption - so we are done.

**15.7.Solution** : We want to show that the subgroup $K$ of $G \times H$ is abelian. Direct products are abelian if and only if their components in each group are abelian. So we must show that for any $(g_1, h_1)$ and $(g_2, h_2)$ in $K$, we have $g_1g_2 = g_2g_1$ and $h_1h_2 = h_2h_1$. Again, notice that if we could show $(g_1g_2g_1^{-1}g_2^{-1}, e)$ is in $K$, then by assumption, we would have $g_1g_2 = g_2g_1$.

Given $(g_2, h_2) \in K$, since K is a subgroup, $(g_2^{-1}, h_2^{-1}) \in K$. Furthermore, since $K$ is normal, if we conjugate by any element of $G \times H$ we will stay in K. This implies

$$(g_1, e)(g_2, h_2)(g_1, e)^{-1} = (g_1 g_2 g_1^{-1}, h_2) \in K$$

Multiplying

$$(g_1 g_2 g_1^{-1}, h_2) * (g_2^{-1}, h_2^{-1}) = (g_1 g_2 g_1^{-1} g_2^{-1}, e) \in K$$

which implies (by assumption) that $g_1 g_2 g_1^{-1} g_2^{-1} = e$. Hence $g_1 g_2 = g_2 g_1$. An identical argument on the second component shows the same is true for any $h_1$ and $h_2$. Hence $K$ is abelian as required.

**15.8.Solution** : First recall that a group $G$ is abelian iff $[G, G] = \{1\}$. We always have $[G, G] \trianglelefteq G$. So since $A_n$ is non-abelian and simple for $n \geq 5$, we must have $[A_n, A_n] = A_n$ for $n \geq 5$. Now $A_4$ is non-abelian, so its commutator subgroup is non-trivial. Also recall $V \trianglelefteq A_4$ and $A_4/V \cong \mathbb{Z}_3$, which is abelian, and so $[A_4, A_4] \leq V$. In fact $[A_4, A_4] = V$, since V this is the only non-trivial normal subgroup of $A_4$.

**15.10.Solution** : $H \trianglelefteq G$, and we have a set $X \subset G$ such that $\langle X \rangle = G$ and $x \in X \Rightarrow x^{-1} \in X$. Also we have a set $Y \subset H$ such that $\langle Y \rangle = H$. The claim is that

$$xyx^{-1} \in H \ \forall x \in X, y \in Y \Rightarrow H \trianglelefteq G$$

So take an arbitrary conjugate $ghg^{-1}$ for some $g \in G, h \in H$. Write $g = x_1....x_n, h = y_1....y_m$ for $x_i \in X$ and $y_j \in Y$. Then $ghg^{-1} = x_1...x_n y_1...y_m x_n^{-1}...x_1^{-1}$. Now insert extra terms to obtain $ghg^{-1} = x_1...x_n y_1 x_n^{-1} x_n y_2 x_n^{-1}....x_n y_m x_n^{-1}...x_1^{-1}$. Then by hypothesis we can combine all the conjugates into a single element $h' \in H$ to get $ghg^{-1} = x_1...x_{n-1} y' x_{n-1}^{-1}...x_1^{-1}$. It is now clear how to finish the proof - rewrite $h'$ as a product of element in $Y$ and repeat the above process. Then use an induction argument to conclude $ghg^{-1} \in H$ as required.

**15.12.Solution** : This problem shows that $A_5$ is simple: 14.5 implies that all 3-cycles are in the same conjugacy class, and so any normal subgroup must contain all 3-cycles if it contains a single one. Moreover, we know that the 3-cycles generate all of $A_n$ for $n \geq 4$. So putting these two facts together, there can be no proper normal subgroup of $A_5$. The trick is to show that any normal subgroup of $A_5$ contains a 3-cycle. But notice any element in $A_5$ is either a product of disjoint transpositions, a 3-cycle, or a 5-cycle. So if a subgroup contains any of these types, then the commutators in the question show that there is a way to produce a 3-cycle and remain in the proposed normal subgroup (conjugate the element, and then multiply by the inverse of the element, and you stay in the normal subgroup).