SOLUTIONS TO ASSIGNMENT 5

10.2.Solution : Suppose (for a contradiction) $\mathbb{Z} \times \mathbb{Z}$ is isomorphic to $\mathbb{Z}$. We know that $\mathbb{Z}$ is infinite cyclic (that is, generated by a single element which does not have finite order), and that any isomorphism preserves the order of an element and maps generators to generators. Hence $\mathbb{Z} \times \mathbb{Z}$ contains a single generator of infinite order, and so is also infinite cyclic. We write this generator as $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Now $(1, 0), (0, 1) \in \mathbb{Z} \times \mathbb{Z}$. So since (a, b) is a generator of this group, there must be non-zero integers m, n such that $(1, 0) = m(a, b)$ and $(0, 1) = n(a, b)$. But then $a = b = 0$, and $(0, 0)$ does not generate it $\mathbb{Z} \times \mathbb{Z}$. We have reached a contradiction. Hence $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.

10.4.Solution : Since $V = \mathbb{Z}_2 \times \mathbb{Z}_2$, we see $\mathbb{Z}_3 \times V = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. On the other hand, $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ since $\gcd(2, 3) = 1$, and so $\mathbb{Z}_2 \times \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Now it is clear that $\mathbb{Z}_3 \times V \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ by the obvious isomorphism.

10.5.Solution : Let H be the diagonal $\{(x, x) : x \in G\}$. If $a = (x, x)$ and $b = (y, y)$ are both contained in H, $ab^{-1} = (xy^{-1}, xy^{-1})$ is also contained in H, hence H is a subgroup. The map $\psi : G \to G \times G$ defined as $x \to (x, x)$ is an isomorphism of G to the image H.

10.7.Solution : There are two abelian groups: $\mathbb{Z}_{24}$ and $\mathbb{Z}_{12} \times \mathbb{Z}_2$. They are not isomorphic, since $\mathbb{Z}_{24}$ is cyclic but $\mathbb{Z}_{12} \times \mathbb{Z}_2$ is not cyclic. In fact, every element in $\mathbb{Z}_{12} \times \mathbb{Z}_2$ has order no greater than 12.

In the following nonabelian groups

$$D_4 \times \mathbb{Z}_3, D_{12}, A_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times D_6, S_4,$$

the maximal order of elements is $12, 12, 6, 6, 4$ respectively. Consider $D_4 \times \mathbb{Z}_3$ and D_{12}. In D_4, there are 5 elements which have order 2, and in $\mathbb{Z}_3$ there are 2 elements of order 3, so in $D_4 \times \mathbb{Z}_3$ there are $5 \times 2 = 10$ elements of order 6. But in D_{12}, there are only 2 elements of order 6. Hence they are not isomorphic. For $A_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times D_6$, there are 8 elements of order 6 in $A_4 \times \mathbb{Z}_2$, but there are only 6 elements of order 6 in $\mathbb{Z}_2 \times D_6$. They are also not isomorphic.

So in conclusion, no two of them are isomorphic.

10.11.Solution : First we recall the following theorem: If G is a group with normal subgroups H and K such that $H \cap K = 1$, then $HK \cong H \times K$.

Now let n be a positive odd integer. We write $D_{2n} = \langle r, s : r^{2n} = s^2 = 1, srs = r^{-1}\rangle$ (the usual presentation of D_{2n}). Let $H = \langle s, r^2\rangle$ and let $K = \langle r^n\rangle$. Geometrically, if D_{2n} is the group of symmetries of a regular $2n$-gon, H is the group of symmetries of the regular n-gon inscribed in the $2n$-gon by joining vertex $2i$ to vertex $2i + 2$, for all i mod $2n$ (and if we let $r_1 = r^2$, H has the usual presentation of the dihedral group of order $2n$ with generators r_1 and s). Note that $H \lhd D_{2n}$ since it as index 2. Since $o(r) = 2n, o(r^n) = 2$. Also since $srs = r^{-1}$, we have $sr^ns = r^{-n} = r^n$, that is, s centralizes r^n. Since clearly r centralises r^n, $K \lhd Z(D_{2n})$. Thus $K \lhd D_{2n}$. Finally, K is not a subgroup of H since r^2 has odd order (or because r^n sends vertex i into vertex $i + n$, hence does not preserve the set of even vertices of the $2n$-gon). Thus $H \cap K = 1$, by Lagranges Theorem. Now we apply the about theorem to complete the proof.

12.3.Solution : Let $G = \mathbb{Z}$ and $H = \{0\}$. Then $\{(x, y) : xy \in H\}$ is not an equivalence relation - since for any $x \neq 0$, $x + x \notin H$, i.e. x is not related to itself.

12.8.Solution : When $g \in H$, $gH = H = Hg$. Assume $g \notin H$, H and gH form a partition G. Similarly, H and Hg also form a partition of G, hence $gH = Hg$.

14.2.Solution :We use the usual generators and relations for the dihedral group D_n. If $n = 2k$ then the conjugacy classes in D_n are: $\{1\}, \{r^k\}, \{r^{\pm}\}, \{r^{\pm 2}\}, ..., \{r^{\pm(k-1)}\}, \{sb^{2b} : b = 1, ..., k\}$ and $\{sr^{2b-1} : b = 1, ...k\}$.

For $n = 2k + 1$ the conjugacy classes in D_n are $\{1\}$,$\{r^{\pm 1}\}, \{r^{\pm 2}\}, ..., \{r^{\pm k}\}, \{sb^b : b = 1, ...n\}$.

14.3.Solution : The conjugacy class of an element $x \in G$ is usually written x^G, and is defined by $x^G := \{axa^{-1} : a \in G\}$. Let $\phi : G \to G'$ be an isomorphism. We will prove $\phi(x^G) = \phi(x)^{G'}$. On the one hand $\phi(axa^{-1}) = \phi(a)\phi(x)\phi(a)^{-1} \in \phi(x)^{G'}$. Conversely, since ϕ is an isomorphism, any $b \in G'$ is of the form $\phi(a)$ for a unique $a \in G$. So $b\phi(x)b^{-1} = \phi(axa^{-1}) \in \phi(x^G)$ as required.

14.5.Solution : We recall that given any cycle $C = (a_1a_2...a_n)$, then

$$gCg^{-1} = (g(a_1)g(a_2)...g(a_n)).$$

So to show that a three cycle is conjugate to any other three cycle, we will show that any 3-cycle is conjugate to (123). We know

$$g(123)g^{-1} = (g(1)g(2)g(3))$$

To show $g(123)g^{-1} = (abc)$ for any 3 numbers a, b, c between 1 and 5, pick $g \in A_5$ such that $g(1) = a, g(2) = b$ and $g(3) = c$. To check that we can find such a g in A_5 note that adding the transposition $(g(4)g(5))$ does not affect the permutations $g(123)g^{-1}$ so we can make g odd or even by adding $(g(4)g(5))$ as needed.