MATH 52 FINAL EXAM SOLUTIONS (AUTUMN 2003)

1. Evaluate the integral by reversing the order of integration

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy.$$

Solution.

$$\begin{aligned}
\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy &= \int_0^3 \int_0^{x/3} e^{x^2}\, dxdy \\
&= \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\, dx \\
&= \frac{1}{6} e^{x^2}\Big|_0^3 = \boxed{\frac{e^9 - 1}{6}}
\end{aligned}$$

2. Rewrite the following integral in rectangular coordinates.

$$\int_0^{\pi/4} \int_0^{\frac{2}{\cos\theta + \sin\theta}} r^2 \cos\theta \; dr \; d\theta$$

Do **not** evaluate the integral.

Solution. The equation $r = 2/(\cos\theta + \sin\theta)$ becomes $x + y = 2$ in rectangular coordinates. Since $r\, drd\theta$ becomes dA, the integrand $r\cos\theta$ becomes x in rectangular coordinates, so the integral equals

$$\boxed{\int_0^1 \int_y^{2-y} x\, dxdy} \quad \text{or} \quad \boxed{\int_0^1 \int_0^x x\; dydx + \int_1^2 \int_0^{2-x} x\; dydx}$$

3. (a) Consider the change of variables defined by

$$u = \frac{y}{x}, \qquad v = xy.$$

Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ in terms of u and v.

Solution. First we find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$. It is equal to the determinant

$$\begin{bmatrix} -y/x^2 & y \\ 1/x & x \end{bmatrix} = -2y/x = -2u$$

Hence the Jacobian $\boxed{\dfrac{\partial(x, y)}{\partial(u, v)} = -\dfrac{1}{2u}}$.

(b) Use your answer to part (a) to find the area of the region in the first quadrant bounded by the lines $y = x$ and $y = 2x$ and the curves $xy = 2$ and $xy = 3$.

Solution. In the uv-coordinates the region becomes $1 \le u \le 2$ and $2 \le v \le 3$. Hence its area is:

$$\int_{u=1}^{2} \int_{v=2}^{3} \frac{1}{2u}\, dv\, du = \int_{u=1}^{2} \frac{1}{2u}\, du = \left[\frac{1}{2}\ln u\right]_{u=1}^{2} = \boxed{\frac{1}{2}\ln 2}$$

4. Find the area of the part of hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution. $z = f(x, y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2} r\, dr d\theta = \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2} r\, dr$$
$$= 2\pi \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_1^2 = \boxed{\frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})}$$

5. (a) Let E be the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$. Set up, but **do not evaluate** a triple integral in cylindrical coordinates which computes the volume of E.

Solution. The paraboloids intersect when $x^2 + y^2 = 36 - 3x^2 - 3y^2$. Hence $D = \{(x, y) \mid x^2 + y^2 \le 9\}$, and in cylindrical coordinates $E = \{(r, \theta, z) \mid 0 \le r \le 3, 0 \le \theta \le 2\pi, r^2 \le z \le 36 - 3r^2\}$. Finally,

$$\boxed{V = \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} r\, dz dr d\theta}$$

(b) Set up, but **do not evaluate** a triple integral in spherical coordinates which computes the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Solution. In spherical coordinates, the sphere $x^2 + y^2 + z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2 + y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\{(\rho, \theta, \phi) \mid 0 \le \rho \le 2, 0 \le \theta \le 2\pi, \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\}$ and

$$\boxed{V = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi\, d\rho d\theta d\phi}$$

6. Let C be the circle in the plane $x + y + z = 0$, centered at $(0, 0, 0)$, with radius 1. Suppose C is parametrized positively with respect to the upward unit normal of the plane. Let $\mathbf{F}(x, y, z) = (z, x, y)$. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Hint: Use Stokes' Theorem.

Solution. The vector $\mathbf{n} = \frac{1}{\sqrt{3}}(1, 1, 1)$ is the upward unit normal to the plane, and

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)$$

so by Stokes' Theorem, if we denote by D the disk in the plane enclosed by C,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \sqrt{3}\,dS = \boxed{\pi\sqrt{3}}$$

7. Prove that the volume of a solid T with boundary surface S is equal to $\iint_S z\,dxdy$.

   Solution. By the Divergence Theorem applied with $\mathbf{F}(x,y,z) = (0,0,z)$,

$$\begin{aligned}\operatorname{vol}(T) &= \iiint_T 1\,dV \\ &= \iiint_T \operatorname{div}\mathbf{F}\,dV \\ &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS \\ &= \iint_S 0\,dydz + 0\,dzdx + z\,dxdy \\ &= \iint_S z\,dxdy\end{aligned}$$

8. Let $\mathbf{F}(x,y,z) = (x,y,z)$ and let S be sphere of radius 2 centered at $(1,2,7)$. Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

   Solution. Let B denote the solid ball bounded by the sphere S. Since $\operatorname{div}\mathbf{F} = 3$, the Divergence Theorem implies

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B 3\,dV = 3\operatorname{vol}(B) = \boxed{32\pi}$$

9. (a) Evaluate the line integral $\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy$, where C is the positively oriented boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

   Solution. Using Green's Theorem

$$\begin{aligned}\int_C (x^3 - y^3)\,dx + (x^3 + y^3)\,dy &= \iint_{1\le x^2+y^2\le 9} \left[\frac{\partial}{\partial x}(x^3 + y^3) - \frac{\partial}{\partial y}(x^3 - y^3)\right] dA \\ &= \iint_{1\le x^2+y^2\le 9} (3x^2 + 3y^2)\,dA \\ &= 3\int_0^{2\pi}\int_1^3 r^3\,drd\theta \\ &= 6\pi\left(\frac{81}{4} - \frac{1}{4}\right) = \boxed{120\pi}\end{aligned}$$

   (b) Let C be the curve parametrized by $\mathbf{r}(t) = (3\cos t, 4\sin t)$ for $0 \le t \le \pi$. Find $\displaystyle\int_C (e^x + y^2\cos x)dx + (e^{2y} + 2y\sin x)dy$. Hint: Complete C to a closed curve and use Green's theorem.

Solution. The curve C begins at $(3,0)$ and ends at $(-3,0)$, so we may form a closed curve by using the line segment L from $(-3,0)$ to $(3,0)$. This is parametrized by $\mathbf{r}(t) = (t,0)$ for $-3 \le t \le 3$. The curve $C+L$ encloses the half-ellipse R, and on this region

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y\cos x - 2y\cos x = 0$$

so applying Green's Theorem yields

$$\oint_{C+L} (e^x + y^2\cos x)dx + (e^{2y} + 2y\sin x)dy = \iint_R 0\,dA = 0.$$

But

$$\oint_{C+L} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy + \int_L P\,dx + Q\,dy$$

so

$$\int_C P\,dx + Q\,dy = -\int_L P\,dx + Q\,dy = -\int_{-3}^{3} e^t\,dt = \boxed{e^{-3} - e^3}$$

10. Show that the integral $\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy$ is independent of path and evaluate the integral for a path from $(-1,0)$ to $(5,1)$.

Solution. Here $\mathbf{F}(x,y) = (2x\sin y)\mathbf{i} + (x^2\cos y - 3y^2)\mathbf{j}$. Then $f(x,y) = x^2\sin y - y^3$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence

$$\begin{aligned}\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= f(5,1) - f(-1,0) \\ &= \boxed{25\sin 1 - 1}\end{aligned}$$

11. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant, and let $\mathbf{F}(x,y,z) = (1,1,1)$. Compute $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ represents the outward unit normal to S. Hint: Use the Divergence Theorem.

Solution. The surface S is not closed. By including the quarter disks from the first quadrant of each coordinate plane (call them D_1, D_2 and D_3) a closed surface is obtained. Call T the solid bounded by this surface. Then by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{D_1} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{D_2} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{D_3} \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_T \operatorname{div}\mathbf{F}\,dV \\ &= 0\end{aligned}$$

On the quarter disk D_1 (in the xy-plane), $\mathbf{n} = (0,0,-1)$, so $\mathbf{F}\cdot\mathbf{n} = -1$, and

$$\iint_{D_1} \mathbf{F}\cdot\mathbf{n}\,dS = -\operatorname{area}(D_1) = -\frac{\pi}{4}$$

The flux through the other quarter disks is the same, so

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \boxed{\frac{3}{4}\pi}$$

12. Let $\mathbf{F}(x, y) = (\sin y + (x+1)^x, x\cos y + x^2)$.

(a) Show that

$$\oint_{C_a} \mathbf{F}\cdot d\mathbf{r} = 0$$

for every a, where C_a is the circle of radius a centered at the origin.

Solution. Since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x$, if we denote by D_a the disk bounded by C_a, then Green's Theorem implies

$$\oint_{C_a} \mathbf{F}\cdot d\mathbf{r} = \iint_{D_a} 2x\, dA = \int_0^{2\pi}\int_0^a 2r^2\cos\theta\, dr d\theta = \frac{2}{3}a^3\int_0^{2\pi}\cos\theta\, d\theta = 0$$

(b) Is $\mathbf{F}$ path independent? Explain.

Solution. Since $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$, $\mathbf{F}$ is not path independent.

13. (a) Evaluate the surface integral $\iint_S xy\, dS$, where S is triangular region with vertices $(1,0,0), (0,2,0)$, and $(0,0,2)$

Solution. S is the region in the plane $2x+y+z=2$ or $z = 2-2x-y$ over $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2-2x\}$. Thus

$$\begin{aligned}\iint_S xy\, dS &= \iint_D xy\sqrt{(-2)^2+(-1)^2+1}\, dA \\ &= \sqrt{6}\int_0^1\int_0^{2-2x} xy\, dydx = \sqrt{6}\int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{y=2-2x} dx \\ &= \frac{\sqrt{6}}{2}\int_0^1 (4x-8x^2+4x^3)\, dx = \frac{\sqrt{6}}{2}(2-\frac{8}{3}+1) = \boxed{\frac{\sqrt{6}}{6}}\end{aligned}$$

(b) Evaluate the surface integral

$$\iint_S (xze^y\mathbf{i} - xze^y\mathbf{j} + z\mathbf{k})\cdot d\mathbf{S},$$

where S is the part of the plane $x+y+z=1$ in the first octant with downward orientation.

Solution. We have $\mathbf{F}(x,y,z) = xze^y\mathbf{i} - xze^y\mathbf{j} + z\mathbf{k}$, $z = g(x,y) = 1-x-y$, and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1-x\}$. Since S has downward orientation, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [-xze^y(-1) - (-xze^y)(-1) + z]\, dA \\ &= -\int_0^1\int_0^{1-x}(1-x-y)\, dydx \\ &= -\int_0^1 (\frac{1}{2}x^2 - x + \frac{1}{2})\, dx = \boxed{-\frac{1}{6}}\end{aligned}$$