Convergence of the Q-curvature flow on S^4, Advances in Mathematics 205, 1–32 (2006)

In this paper, we prove the convergence of the Q-curvature flow in the critical case of S^4. More precisely, given any initial metric which is conformal to the standard metric on S^4, we show that the Q-curvature flow exists for all time and converges to the round metric as $t \to \infty$.

The Q-curvature flow can be viewed as the gradient flow. To explain this, let g_0 be the standard metric on S^4. We consider the functional

$$\begin{aligned} E_{g_0}[w] = 2 \int_{S^4} (\Delta_{g_0} w)^2 \, dV_{g_0} + 4 \int_{S^4} |dw|_{g_0}^2 \, dV_{g_0} \\ - 16\pi^2 \log \left(\frac{3}{8\pi^2} \int_{S^4} e^{4(w-\overline{w})} \, dV_{g_0} \right), \end{aligned}$$

where $\overline{w}$ denotes the mean value of w:

$$\overline{w} = \frac{3}{8\pi^2} \int_{S^4} w \, dV_{g_0}.$$

It is known that the functional $E_{g_0}[w]$ is bounded from below. However, the functional $E_{g_0}[w]$ is invariant under the conformal diffeomorphism group of S^4, which is non-compact. To overcome this obstacle, we modify the flow by conformal diffeomorphisms.