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Riemannian Manifolds of Positive Curvature

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Abstract

The study of positive sectional curvature is one of the oldest pursuits in Riemannian geometry, but despite the considerable efforts of many researchers, basic questions remain unanswered. In this lecture we will briefly summarize the state of knowledge in this area and outline the techniques which have had success. These techniques include geodesic and comparison methods, minimal surface methods, and Ricci flow. We will then describe our recent work (see [18], [21], [22]) which uses the Ricci flow to resolve the differentiable sphere theorem; that is, the complete classification of manifolds whose sectional curvatures are 1/4-pinched.

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1. Preliminaries and the Main Theorems

We let M denote a smooth manifold of dimension n. Recall that a Riemannian metric on M is a choice g of inner product on each tangent space which varies smoothly from point to point. Any manifold admits an infinite dimensional family of Riemannian metrics, but the question of whether a manifold admits metrics with desired geometric properties is one of the basic questions of global Riemannian geometry. Surfaces embedded in $\mathbb{R}^3$ provide important examples of two dimensional Riemannian manifolds where the metric g is the restriction of the euclidean inner product to each tangent space. The geometry of surfaces was developed by Gauss in the early nineteenth century. Gauss understood

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aspects of the geometry of surfaces which are intrinsic in that they depend only on the metric g as opposed to those geometric aspects which depend on the way in which the surface is embedded in space. In his famous Theorema Egregium, Gauss identified the function K equal to the product of the principal curvatures, which we call the Gauss curvature of a surface. He showed that K can be expressed in terms of g and its first two derivatives. This means that if we choose local coordinates x^1, x^2 on the surface and express the metric in terms of the coordinate vector fields ∂_1, ∂_2 by $g_{ij} = \langle \partial_i, \partial_j \rangle$ then the function K has an expression involving the g_{ij} and its first and second derivatives. Among other things Gauss showed that $K = 0$ if and only coordinates can be introduced in a neighborhood of any point in which $g_{ij} = \delta_{ij}$; that is, the metric is locally equivalent to the euclidean space $\mathbb{R}^2$.

In 1854 Riemann extended Gauss' theory of the intrinsic geometry of surfaces to higher dimensions. In particular, he found an expression involving the metric and its first two derivatives whose vanishing characterizes those metrics which are locally equivalent to the euclidean metric on $\mathbb{R}^n$. To describe this expression, let M denote a manifold of dimension n, and let g be a Riemannian metric on M. The curvature of (M, g) is described by the Riemann curvature tensor R. This gives, for each point $p \in M$, a multilinear function $R : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$. From its construction, the Riemann curvature tensor satisfies the symmetries

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Z, W, X, Y) \tag{1}$$

and the first Bianchi identity

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0 \tag{2}$$

for all tangent vectors $X, Y, Z, W \in T_pM$. By contracting the Riemann curvature tensor with respect to the metric, we obtain the Ricci and scalar curvature of (M, g):

$$\mathrm{Ric}(X, Y) = \sum_{k=1}^{n} R(X, e_k, Y, e_k)$$

and

$$\mathrm{scal} = \sum_{k=1}^{n} \mathrm{Ric}(e_k, e_k).$$

Here, X, Y are arbitrary vectors in the tangent space T_pM, and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Although the curvature of a higher dimensional Riemannian manifold is a much more complicated object than the Gaussian curvature of a surface, it turns out to be possible to understand R in terms of Gauss curvatures of surfaces embedded in M. To explain this, we consider a two dimensional plane π in the tangent space T_pM, and we consider all geodesics emanating from p that are tangent to the plane π. The union of these geodesics rays forms a two-dimensional surface $\Sigma \subset M$; more formally, the surface Σ is defined as $\Sigma = \exp_p(U \cap \pi)$, where $\exp_p : T_pM \to M$ denotes the exponential map and $U \subset T_pM$ denotes a small ball centered at the origin. With this understood, the sectional curvature $K(\pi)$ is defined to be the Gaussian curvature of the two-dimensional surface Σ at the point p.

The sectional curvatures can be described precisely in terms of R: given any point $p \in M$ and any two dimensional plane $\pi \subset T_pM$, the sectional curvature of π is defined by

$$K(\pi) = \frac{R(X, Y, X, Y)}{|X|^2\,|Y|^2 - \langle X, Y\rangle^2},$$

where $\{X, Y\}$ is a basis of π. Note that this definition is independent of the choice of the basis $\{X, Y\}$, and that if X, Y are chosen to be an orthonormal basis then the denominator is equal to 1. Because of the symmetries of R it turns out that the sectional curvatures algebraically determine all components of R at a given point p.

We say that a Riemannian manifold has *positive curvature* if all sectional curvatures are positive at all points of M. Perhaps the most basic example of a Riemannian manifold of positive curvature is the n-dimensional sphere S^n with its standard metric arising from its embedding as the unit sphere in $\mathbb{R}^{n+1}$. This manifold has constant sectional curvature 1; that is, $K(\pi) = 1$ for all two-dimensional planes π. Conversely, it was shown by H. Hopf in 1926 that a compact, simply connected Riemannian manifold with constant sectional curvature 1 is necessarily isometric to the sphere S^n, equipped with its standard metric (see [53], [54]). More generally, if (M, g) is a compact Riemannian manifold with constant sectional curvature 1, then (M, g) is isometric to a quotient S^n/Γ, where Γ is a finite group of isometries acting freely. These quotient manifolds are completely classified (see [83]) and they are referred to as *spherical space forms*. The simplest examples of spherical space forms are the sphere S^n and the real projective space $\mathbb{RP}^n$. When n is even, these are the only examples. By contrast, there is an infinite collection of spherical space forms for each odd integer n.

In case of dimension $n = 2$ it is a classical result that the only compact surfaces which can be given metrics of positive curvature are S^2 and $\mathbb{RP}^2$. This follows from the Gauss-Bonnet Theorem which asserts that for any metric the integral of K over the surface is equal to 2π times the Euler characteristic. Thus if K is positive the Euler characteristic must be positive, and from the classification of compact surfaces it follows that M is diffeomorphic to either S^2 or $\mathbb{RP}^2$.

For $n \geq 3$ it is a much more difficult task to classify those compact manifolds which can be given metrics of positive curvature, and we can only give partial answers. There are families of positively curved manifolds which are called *compact rank one symmetric spaces* (CROSS). These include S^n and $\mathbb{RP}^n$ as well as other manifolds which are not spherical space forms. One such family consists of the complex projective spaces $\mathbb{CP}^n$ for $n \geq 2$. The manifold $\mathbb{CP}^n$ is the set of complex lines through the origin in $\mathbb{C}^{n+1}$, or it may be alternatively described as the quotient of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ by the circle action consisting of multiplication by $e^{i\theta}$ for $\theta \in S^1$. The metric on S^{2n+1} induces a natural metric on $\mathbb{CP}^n$. Now since $\mathbb{CP}^n$ has a complex structure which is compatible with this metric, a two plane π in T_pM might be complex, meaning that it is invariant under multiplication by $\sqrt{-1}$; it might be totally real, meaning that multiplication by $\sqrt{-1}$ takes π to an orthogonal two plane; or it might be somewhere between these extremes. It turns out that if we normalize the metric so that all complex two planes have curvature 1, then the algebra of the curvature tensor implies that the totally real two planes must have curvature 1/4, and all two planes have curvature in the interval $[1/4, 1]$. Note that the real dimension of $\mathbb{CP}^n$ is $2n$. There is a similar construction with the complex numbers replaced by the quaternions $\mathbb{H}$, and this produces the quaternionic projective spaces $\mathbb{HP}^n$ for $n \geq 2$. These are CROSS manifolds of dimension $4n$, and they also have natural metrics with sectional curvatures in the interval $[1/4, 1]$. There is one remaining CROSS manifold of dimension 16 called the Cayley projective plane (see [4] for a detailed description). It also has a natural metric with sectional curvatures in the interval $[1/4, 1]$.

In addition to manifolds which are locally CROSS (covered by a CROSS manifold), there are a few other constructions which have yielded metrics of positive curvature on other manifolds. First, the compact homogeneous manifolds of positive curvature have been classified by Berger [10], Aloff and Wallach [2], Wallach [81], and Bérard-Bergery [6]. Secondly, there are biquotient constructions by Eschenburg [31] and Bazaikin [5]. The combination of these constructions give non-CROSS examples in dimensions 6, 7, 12, 13, and 24. In dimensions 7 and 13 they produce infinitely many distinct examples. The study of manifolds of positive curvature with symmetry is being actively pursued by a number of authors. We refer the reader to Grove [41] for a recent survey of this topic.

There are very few general obstructions known to the existence of metrics of positive curvature on compact manifolds of dimension 4 or more. In the next section we will summarize what is known and give an overview of the methods which have been effective. As a step toward the classification problem Hopf conjectured that a compact, simply connected Riemannian manifold whose sectional curvatures are close to 1 should be homeomorphic to a sphere (see Marcel Berger's account in [13], page 545). This idea is formalized by the notion of curvature pinching, which goes back to H. Hopf and H.E. Rauch:

Definition 1.1. A Riemannian manifold (M, g) is said to be weakly δ-pinched in the global sense if the sectional curvature of (M, g) satisfies $\delta \leq K \leq 1$. If the strict inequality holds, we say that (M, g) is strictly δ-pinched in the global sense.

For our purposes, it will be convenient to consider the weaker notion of pointwise pinching. This means that we only compare sectional curvatures corresponding to different two dimensional planes based at the same point $p \in M$:

Definition 1.2. We say that (M, g) is weakly δ-pinched in the pointwise sense if $0 \leq \delta\, K(\pi_1) \leq K(\pi_2)$ for all points $p \in M$ and all two dimensional planes $\pi_1, \pi_2 \subset T_pM$. If the strict inequality holds, we say that (M, g) is strictly δ-pinched in the pointwise sense.

Hopf's pinching problem was first taken up by H.E. Rauch after he visited Hopf in Zürich during the late 1940s ([13], page 545). In a seminal paper [72], Rauch showed that a compact, simply connected Riemannian manifold which is strictly δ-pinched in the global sense is homeomorphic to S^n ($\delta \approx 0.75$). Furthermore, Rauch posed the question of what the optimal pinching constant δ should be. This question was settled around 1960 by the celebrated Topological Sphere Theorem of M. Berger and W. Klingenberg:

Theorem 1.3 (M. Berger [8], W. Klingenberg [59]). *Let (M, g) be a compact, simply connected Riemannian manifold which is strictly $1/4$-pinched in the global sense. Then M is homeomorphic to S^n.*

The classical proof of the Topological Sphere Theorem relies on comparison geometry techniques which were refined during the 1950's (see e.g. [28], Chapter 6). There are several ways in which one might hope to improve Theorem 1.3. A natural question to ask is whether the global pinching condition in Theorem 1.3 can be replaced by a pointwise one. Furthermore, one would like to extend the classification in Theorem 1.3 to include manifolds that are not necessarily simply connected. By applying Theorem 1.3 to the universal cover, one can conclude that any compact Riemannian manifold which is strictly 1/4-pinched in the global sense is homeomorphic to a quotient of a sphere by a finite group, but this leaves open the question of whether the group is conjugate to one which acts by standard isometries, the condition required to show that the manifold is homeomorphic to a spherical space form. We point out that exotic $\mathbb{Z}_2$-actions on the standard sphere S^4 have been constructed in [26] and [33].

Another fundamental question is whether a Riemannian manifold satisfying the assumptions of Theorem 1.3 is diffeomorphic, instead of just homeomorphic, to S^n. This is a highly non-trivial matter as the smooth structure on S^n is not unique in general. In other words, there exist examples of so-called exotic spheres which are homeomorphic, but not diffeomorphic, to S^n. Hence, we may rephrase the problem as follows:

Conjecture 1.4. *An exotic sphere cannot admit a metric with $1/4$-pinched sectional curvature.*

The first examples of exotic spheres were constructed in a famous paper by J. Milnor [65] in 1957. M. Kervaire and J. Milnor proved that there exist exactly 28 different smooth structures on S^7 (cf. [58]). It was shown by

E. Brieskorn that the exotic 7-spheres have a natural interpretation in terms of certain affine varieties (cf. [24], [25], [51]). To describe this result, let Σ_k denote the intersection of the affine variety

$$\{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\}$$

with the unit sphere in $\mathbb{C}^5$. Brieskorn proved that, for each $k \in \{1, \ldots, 28\}$, Σ_k is a smooth manifold which is homeomorphic to S^7. Moreover, the manifolds Σ_k, $k \in \{1, \ldots, 28\}$, realize all the smooth structures on S^7.

In 1974, D. Gromoll and W. Meyer [38] described an example of an exotic seven-sphere that admits a metric of nonnegative sectional curvature. It was shown by F. Wilhelm [82] that the Gromoll-Meyer sphere admits a metric which has strictly positive sectional curvature outside a set of measure zero (see also [32]). P. Petersen and F. Wilhelm have recently proposed a construction of a metric of strictly positive sectional curvature on the Gromoll-Meyer sphere, which is currently in the process of verification.

For each $n \geq 5$, the collection of all smooth structures on S^n has the structure of a finite group Θ_n, called the Kervaire-Milnor group. If $n \equiv 1, 2 \mod 8$, there is a natural invariant $\alpha : \Theta_n \to \mathbb{Z}_2$. This invariant is described in detail in [57]. In particular, half of all smooth structures on S^n have non-zero α-invariant. Using the Atiyah-Singer index theorem, N. Hitchin [52] showed that an exotic sphere with non-zero α-invariant cannot admit a metric of positive scalar curvature. On the other hand, it follows from a theorem of S. Stolz [79] that every exotic sphere with vanishing α-invariant does admit a metric of positive scalar curvature.

Conjecture 1.4 is known as the Differentiable Pinching Problem. This problem has been studied by a large number of authors since the 1960s, and various partial results have been obtained. D. Gromoll [37] and E. Calabi (unpublished) showed that a simply connected Riemannian manifold which is $\delta(n)$-pinched in the global sense is diffeomorphic to S^n. The pinching constant $\delta(n)$ depends only on the dimension, and converges to 1 as $n \to \infty$. In 1971, M. Sugimoto, K. Shiohama, and H. Karcher [80] proved an analogous theorem with a pinching constant δ independent of n ($\delta = 0.87$). The pinching constant was subsequently improved by E. Ruh [73] ($\delta = 0.80$) and by K. Grove, H. Karcher, and E. Ruh [43] ($\delta = 0.76$).

In 1975, H. Im Hof and E. Ruh proved the following theorem, which extends earlier work of Grove, Karcher, and Ruh [42], [43]:

Theorem 1.5 (H. Im Hof, E. Ruh [56]). *There exists a decreasing sequence of real numbers* $\delta(n)$ *with* $\lim_{n\to\infty} \delta(n) = 0.68$ *such that the following statement holds: if* M *is a compact Riemannian manifold of dimension* n *which is* $\delta(n)$*-pinched in the global sense, then* M *is diffeomorphic to a spherical space form.*

E. Ruh [74] has obtained a differentiable version of the sphere theorem under a pointwise pinching condition, albeit with a pinching constant converging to 1 as $n \to \infty$. In 2007, the authors proved the Differentiable Sphere Theorem with the optimal pinching constant ($\delta = 1/4$), thereby confirming Conjecture 1.4.

Theorem 1.6 (S. Brendle, R. Schoen [22]). *Let* (M, g) *be a compact Riemannian manifold which is strictly* $1/4$*-pinched in the pointwise sense. Then* M *is diffeomorphic to a spherical space form. In particular, no exotic sphere admits a metric with strictly* $1/4$*-pinched sectional curvature.*

Note that Theorem 1.6 only requires a pointwise pinching condition. (In fact, we will see in the coming sections that a much weaker curvature condition suffices.) The Differentiable Sphere Theorem, proved in [22], asserts that any compact Riemannian manifold (M, g) which is strictly 1/4-pinched in the pointwise sense admits another Riemannian metric which has constant sectional curvature 1. In particular, this implies that M is diffeomorphic to a spherical space form. In dimension 2, the Differentiable Sphere Theorem reduces to the statement that a compact surface of positive Gaussian curvature is diffeomorphic to S^2 or $\mathbb{RP}^2$. (In dimension 2, there is only one sectional curvature at each point; hence, every two-dimensional surface of positive curvature is 1/4-pinched in the pointwise sense.)

For weakly 1/4-pinched manifolds we have the following result.

Theorem 1.7 (S. Brendle, R. Schoen [21]). *Let* (M, g) *be a compact Riemannian manifold which is weakly* $1/4$*-pinched in the pointwise sense. Then either* M *is diffeomorphic to a spherical space form or* M *is isometric to a locally CROSS manifold.*

2. Methods of Studying Positive Curvature

The proof of Theorem 1.3 relies on comparison methods which involve the study of geodesics and the influence of positive curvature which causes focusing of nearby geodesics. This made possible delicate theorems which compare triangle measurements in a variable curvature manifold to related measurements in the standard sphere. These methods also employ the variational theory of geodesics and the study of their Morse index; that is, the number of negative eigenvalues of the index form

$$I(V, V) = \int_\gamma (|D_{\gamma'} V|^2 - R(\gamma', V, \gamma', V))\, ds$$

where γ is a geodesic and V, a normal vector field along γ which is required to vanish at the endpoints if γ is not closed. The comparison methods were employed in more powerful ways later in the work of Grove and Shiohama [44] on their diameter sphere theorem. This led soon after to the following result of Gromov.

Theorem 2.1 (M. Gromov [39]). *There is a constant C depending only on the dimension n such that the Betti numbers of any compact n-manifold of nonnegative curvature are bounded by C.*

New methods were introduced into the study of positive curvature by Micallef and Moore [64]. Through the study of the variational theory of minimal two spheres immersed in M they were able to weaken the curvature assumptions in the Topological Sphere Theorem. To that end, Micallef and Moore introduced a novel curvature condition, which they called positive isotropic curvature.

Definition 2.2. Let (M, g) be a Riemannian manifold of dimension $n \geq 4$. We say that (M, g) has nonnegative isotropic curvature if

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4) + R(e_2, e_3, e_2, e_3) \\ &+ R(e_2, e_4, e_2, e_4) - 2\, R(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

for all points $p \in M$ and all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$. Moreover, if the strict inequality holds, we say that (M, g) has positive isotropic curvature.

For each point $p \in M$, we denote by $T_p^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ the complexified tangent space to M at p. The Riemannian metric g extends to a complex bilinear form $g : T_p^{\mathbb{C}}M \times T_p^{\mathbb{C}}M \to \mathbb{C}$. Similarly, the Riemann curvature tensor extends to a complex multilinear form $R : T_p^{\mathbb{C}}M \times T_p^{\mathbb{C}}M \times T_p^{\mathbb{C}}M \times T_p^{\mathbb{C}}M \to \mathbb{C}$.

Proposition 2.3. *The manifold (M, g) has nonnegative isotropic curvature if and only if $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$ for all points $p \in M$ and all vectors $\zeta, \eta \in T_p^{\mathbb{C}}M$ satisfying $g(\zeta, \zeta) = g(\zeta, \eta) = g(\eta, \eta) = 0$.*

The key idea of Micallef and Moore is to study harmonic (or equivalently minimal) two-spheres instead of geodesics. More precisely, for each map $f : S^2 \to M$, the energy of f is defined by

$$\mathscr{E}(f) = \frac{1}{2} \int_{S^2} \Big(\Big| \frac{\partial f}{\partial x} \Big|^2 + \Big| \frac{\partial f}{\partial y} \Big|^2 \Big)\, dx\, dy,$$

where (x, y) are the coordinates on S^2 obtained by stereographic projection. A map $f : S^2 \to M$ is called harmonic if it is a critical point of the functional $\mathscr{E}(f)$. This is equivalent to saying that

$$D_{\frac{\partial}{\partial x}} \frac{\partial f}{\partial x} + D_{\frac{\partial}{\partial y}} \frac{\partial f}{\partial y} = 0$$

at each point on S^2. In the special case when (M, g) has positive isotropic curvature, Micallef and Moore obtained a lower bound for the Morse index of harmonic two-spheres.

Proposition 2.4 (M. Micallef, J.D. Moore [64]). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$ with positive isotropic curvature, and let $f : S^2 \to M$ be a nonconstant harmonic map. Then f has Morse index at least $[\frac{n-2}{2}]$.*

The key idea in the proof is to consider complex variations W of the map and to observe that the index form can be written

$$I(W, \overline{W}) = 4 \int_{S^2} g\big(D_{\frac{\partial}{\partial \bar{z}}} W, D_{\frac{\partial}{\partial z}} \overline{W}\big) \, dx \, dy - 4 \int_{S^2} R\Big(\frac{\partial f}{\partial z}, W, \frac{\partial f}{\partial \bar{z}}, \overline{W}\Big) \, dx \, dy.$$

It is then shown, by use of the Riemann-Roch theorem, that the dimension of the space of holomorphic and isotropic variations is at least $[\frac{n-2}{2}]$, and this leads to the Morse index bound.

Combining Proposition 2.4 with the variational theory for harmonic maps (see e.g. [75], Chapter VII), Micallef and Moore were able to draw the following conclusion.

Theorem 2.5 (M. Micallef, J.D. Moore [64]). *Let (M, g) be a compact, simply connected Riemannian manifold of dimension $n \geq 4$ with positive isotropic curvature. Then M is homeomorphic to S^n.*

Sketch of the proof of Theorem 2.5. The idea is to study the homotopy groups of M. If $\pi_k(M) \neq 0$ for some $k \in \{2, \ldots, [\frac{n}{2}]\}$, then the variational theory for harmonic maps implies that there exists a nonconstant harmonic map $f : S^2 \to M$ with Morse index at most $k - 2$. On the other hand, any nonconstant harmonic map from S^2 into M has Morse index at least $[\frac{n-2}{2}]$ by Proposition 2.4. This is a contradiction.

Therefore, we have $\pi_k(M) = 0$ for $k = 2, \ldots, [\frac{n}{2}]$. Since M is assumed to be simply connected, it follows that $H_k(M, \mathbb{Z}) = 0$ for $k = 1, \ldots, [\frac{n}{2}]$. Using Poincaré duality, it follows that $H_k(M, \mathbb{Z}) = 0$ for $k = 1, \ldots, n - 1$. This shows that M is a homotopy sphere. Hence, it follows from results of Freedman [35] and Smale [78] that M is homeomorphic to S^n. $\square$

We note that any manifold (M, g) which is strictly 1/4-pinched in the pointwise sense has positive isotropic curvature. Hence, Theorem 2.5 generalizes the Topological Sphere Theorem of Berger and Klingenberg. The following result provides some information about fundamental groups of manifolds with positive isotropic curvature.

Theorem 2.6 (A. Fraser [34]). *Let M be a compact Riemannian manifold of dimension $n \geq 5$ with positive isotropic curvature. Then the fundamental group of M does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

The proof of Theorem 2.6 relies on a delicate analysis of stable minimal tori. This result was proved in dimension $n \geq 5$ by A. Fraser [34]. In [23], the authors extended Fraser's theorem to the four-dimensional case. The topology of manifolds with positive isotropic curvature is also studied in [36].

Finally we introduce the Ricci flow approach to positive curvature. This technique was introduced in seminal work of R. Hamilton in the 1980s (see e.g. [46], [47]). The fundamental idea is to start with a given Riemannian manifold (M, g_0), and evolve the metric by the evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, \qquad g(0) = g_0.$$

Here, $\mathrm{Ric}_{g(t)}$ denotes the Ricci tensor of the time-dependent metric $g(t)$.

Hamilton [46] proved that the Ricci flow always has a solution on some maximal time interval $[0, T)$, where $T > 0$ (see also [30]). Furthermore, if $T < \infty$, then the Riemann curvature tensor of $(M, g(t))$ must be unbounded, so that $\limsup_{t \to T} \sup_M |R_{g(t)}| = \infty$. This result was later improved by N. Šešum [77] who showed that $\limsup_{t \to T} \sup_M |\mathrm{Ric}_{g(t)}| = \infty$ if $T < \infty$.

As pointed out above, the Ricci flow is a nonlinear heat equation for Riemannian metrics. This becomes apparent when we consider the evolution of the curvature tensor of $g(t)$. The Riemann curvature tensor satisfies an evolution equation of the form

$$\frac{\partial}{\partial t} R = \Delta R + \text{quadratic terms in } R,$$

where Δ denotes the Laplace operator associated with the time-dependent metric $g(t)$. The exact form of the quadratic terms will become important later on.

As an example, suppose that g_0 is the standard metric on S^n with constant sectional curvature 1. In this case, the metrics $g(t) = (1 - 2(n-1)t)\, g_0$ form a solution to the Ricci flow. This solution is defined for all $t \in [0, \frac{1}{2(n-1)})$, and collapses to a point as $t \to \frac{1}{2(n-1)}$.

In dimension 3, Hamilton showed that the Ricci flow deforms any initial metric with positive Ricci curvature to a constant curvature metric.

Theorem 2.7 (R. Hamilton [46]). *Let (M, g_0) be a compact three-manifold with positive Ricci curvature. Moreover, let $g(t)$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric g_0. Then the rescaled metrics $\frac{1}{4(T-t)}\, g(t)$ converge to a metric of constant sectional curvature 1 as $t \to T$.*

The proof of Theorem 2.7 relies on pointwise curvature estimates. These are established using a suitable version of the maximum principle for tensors.

Theorem 2.7 has important topological implications. It implies that any compact three-manifold with positive Ricci curvature is diffeomorphic to a spherical space form. Using the classification of spherical space forms in [83], Hamilton was able to give a complete classification of all compact three-manifolds that admit metrics of positive Ricci curvature.

Hamilton's convergence theorem in dimension 3 has inspired a large body of work over the last 25 years. In particular, two lines of research have been pursued:

First, one would like to study the global behavior of the Ricci flow in dimension 3 for general initial metrics (i.e. without the assumption of positive Ricci curvature). This line of research was pioneered by Hamilton, who developed many crucial technical tools (see e.g. [48], [49]). It culminated in Perelman's proof of the Poincaré and Geometrization conjectures (cf. [68], [69], [70]). A non-technical survey can be found in [14] or [60].

Another natural problem is to extend the convergence theory for the Ricci flow to dimensions greater than 3. In this case, one assumes that the initial metric satisfies a suitable curvature condition. The goal is to show that the evolved metrics converge to a metric of constant sectional curvature up to rescaling. One of the first results in this direction was established by Hamilton [47] in 1986.

Theorem 2.8 (R. Hamilton [47]). *Let* (M, g_0) *be a compact Riemannian manifold of dimension* 4*. Assume that* g_0 *has positive curvature operator; that is,* $\sum_{i,j,k,l} R_{ijkl}\,\varphi^{ij}\,\varphi^{kl} > 0$ *for each point* $p \in M$ *and every non-zero two-form* $\varphi \in \wedge^2 T_pM$*. Moreover, let* $g(t)$*,* $t \in [0, T)$*, denote the unique maximal solution to the Ricci flow with initial metric* g_0*. Then the rescaled metrics* $\frac{1}{6(T-t)}\,g(t)$ *converge to a metric of constant sectional curvature* 1 *as* $t \to T$*.*

Again, Theorem 2.8 has a topological corollary: it implies that any compact four-manifold which admits a metric of positive curvature operator is diffeomorphic to S^4 or $\mathbb{RP}^4$.

H. Chen [29] proved that the conclusion of Theorem 2.8 holds under a slightly weaker curvature assumption. A Riemannian manifold M is said to have two-positive curvature operator if $\sum_{i,j,k,l} R_{ijkl}\,(\varphi^{ij}\,\varphi^{kl} + \psi^{ij}\,\psi^{kl}) > 0$ for all points $p \in M$ and all two-forms $\varphi, \psi \in \wedge^2 T_pM$ satisfying $|\varphi|^2 = |\psi|^2 = 1$ and $\langle \varphi, \psi \rangle = 0$. Furthermore, Chen [29] proved that every four-manifold which is strictly 1/4-pinched in the pointwise sense has two-positive curvature operator. This is a special feature of the four-dimensional case, which fails in dimension $n \geq 5$. As a consequence, Chen was able to show that every compact four-manifold, which is strictly 1/4-pinched in the pointwise sense, is diffeomorphic to S^4 or $\mathbb{RP}^4$. B. Andrews and H. Nguyen [3] have recently obtained an alternative proof of this result.

We note that C. Margerin [62] proved a sharp convergence result for the Ricci flow in dimension 4. Combining this theorem with techniques from conformal geometry, A. Chang, M. Gursky, and P. Yang proved a beautiful conformally invariant sphere theorem in dimension 4:

Theorem 2.9 (A. Chang, M. Gursky, P. Yang [27]). *Let* (M, g) *be a compact four-manifold with positive Yamabe constant. Suppose that* (M, g) *satisfies the* *integral pinching condition*

$$\int_M |W|^2 \, d\text{vol} < 16\pi^2 \, \chi(M),$$

where $|W|^2 = \sum_{i,j,k,l} W_{ijkl} \, W^{ijkl}$ *denotes the square of the norm of the Weyl tensor of* (M, g)*. Then* M *is either diffeomorphic to* S^4 *or* $\mathbb{RP}^4$*.*

The key step in the proof is to construct a conformal metric $\tilde{g} = e^{2w} \, g$ which has positive scalar curvature and satisfies the pointwise inequality

$$\frac{1}{6} \, \text{scal}_{\tilde{g}}^2 - 2 \, |\overset{\circ}{\text{Ric}}_{\tilde{g}}|^2 - |W_{\tilde{g}}|^2 > 0.$$

Having constructed a metric $\tilde{g}$ with these properties, a theorem of C. Margerin [62] implies that the Ricci flow evolves the metric $\tilde{g}$ to a constant curvature metric. This shows that M is diffeomorphic to either S^4 or $\mathbb{RP}^4$.

The first convergence result in arbitrary dimension was proved by G. Huisken [55] in 1985.

Theorem 2.10 (G. Huisken [55]). *Assume that* (M, g_0) *is a compact manifold of dimension* $n \geq 4$*. If* (M, g_0) *is* $\delta(n)$*-pinched in the pointwise sense, then the Ricci flow converges to a metric of constant curvature* 1 *up to rescaling. Here,* $\delta(n) \in (0, 1)$ *is an explicit constant that depends only on* n*.*

We note that C. Margerin [61] and S. Nishikawa [67] have also obtained convergence results for the Ricci flow in arbitrary dimension. By introducing new methods into the study of the curvature ODE, Böhm and Wilking were able to extend Chen's theorem to higher dimensions.

Theorem 2.11 (C. Böhm, B. Wilking [16]). *If* (M, g_0) *is a compact manifold with two-positive curvature operator, then the Ricci flow converges to a metric of constant curvature* 1 *up to rescaling.*

3. Proofs of the Main Theorems

All known convergence theorems for the Ricci flow share some common features. In particular, they all exploit the fact that a certain curvature condition is preserved by the Ricci flow. To begin this section, we describe some general tools for verifying that a given curvature condition is preserved by the Ricci flow. These tools are based on the maximum principle, and were developed by Hamilton [46], [47].

Let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on a manifold M. Moreover, let E denote the pull-back of the tangent bundle TM under the map

$$M \times (0, T) \to M, \quad (p, t) \mapsto p.$$

Clearly, E is a vector bundle over $M \times (0,T)$, and the fiber of E over the point $(p,t) \in M \times (0,T)$ is given by the tangent space T_pM. The sections of the vector bundle E can be viewed as vector fields on M that vary in time. Given any section X of E, we define the covariant time derivative of X by

$$D_{\frac{\partial}{\partial t}} X = \frac{\partial}{\partial t} X - \sum_{k=1}^{n} \mathrm{Ric}(X, e_k)\, e_k,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame with respect to the metric $g(t)$. The covariant time derivative $D_{\frac{\partial}{\partial t}}$ is metric compatible in the sense that

$$\begin{aligned} \frac{\partial}{\partial t}(g(X,Y)) &= g\left(\frac{\partial}{\partial t} X, Y\right) + g\left(X, \frac{\partial}{\partial t} Y\right) - 2\,\mathrm{Ric}(X,Y) \\ &= g\left(D_{\frac{\partial}{\partial t}} X, Y\right) + g\left(X, D_{\frac{\partial}{\partial t}} Y\right) \end{aligned}$$

for all sections X, Y of the bundle E.

The Riemann curvature tensor of $g(t)$ can now be viewed as a section of the bundle $E^* \otimes E^* \otimes E^* \otimes E^*$. Furthermore, the covariant time derivative on E induces a covariant time derivative on the bundle $E^* \otimes E^* \otimes E^* \otimes E^*$. With this understood, the evolution equation of the Riemann curvature tensor can be written in the form

$$D_{\frac{\partial}{\partial t}} R = \Delta R + Q(R),$$

where $Q(R)$ denotes the following quadratic expression in R:

$$\begin{aligned} Q(R)(X,Y,Z,W) &= \sum_{p,q=1}^{n} R(X,Y,e_p,e_q)\, R(Z,W,e_p,e_q) \\ &\quad + 2 \sum_{p,q=1}^{n} R(X,e_p,Z,e_q)\, R(Y,e_p,W,e_q) \\ &\quad - 2 \sum_{p,q=1}^{n} R(X,e_p,W,e_q)\, R(Y,e_p,Z,e_q). \end{aligned} \tag{3}$$

This evolution equation was first derived by Hamilton [47]; see also [20], Section 2.3.

We next describe Hamilton's maximum principle for the Ricci flow. To fix notation, let $\mathscr{C}_B(\mathbb{R}^n)$ denote the space of all algebraic curvature operators on $\mathbb{R}^n$. In other words, $\mathscr{C}_B(\mathbb{R}^n)$ consists of all multilinear forms $R : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the relations

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Z,W,X,Y)$$

and

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

for all vectors $X, Y, Z, W \in \mathbb{R}^n$. Moreover, let F be a subset of $\mathscr{C}_B(\mathbb{R}^n)$ which is invariant under the natural action of $O(n)$. Since F is $O(n)$-invariant, it makes sense to say that the curvature tensor of a Riemannian manifold (M, g) lies in the set F. To explain this, we fix a point $p \in M$. After identifying the tangent space T_pM with $\mathbb{R}^n$, we may view the curvature tensor of (M, g) at p as an element of $\mathscr{C}_B(\mathbb{R}^n)$. Of course, the identification of T_pM with $\mathbb{R}^n$ is not canonical, but this does not cause problems since F is $O(n)$-invariant.

Theorem 3.1 (R. Hamilton [47]). *Let $F \subset \mathscr{C}_B(\mathbb{R}^n)$ be a closed, convex set which is invariant under the natural action of $O(n)$. Moreover, we assume that F is invariant under the ODE $\frac{d}{dt}R = Q(R)$. Finally, let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on a compact manifold M with the property that the curvature tensor of $(M, g(0))$ lies in F for all points $p \in M$. Then the curvature tensor of $(M, g(t))$ lies in F for all points $p \in M$ and all $t \in [0, T)$.*

In the remainder of this section, we discuss some important examples of curvature conditions that are preserved by the Ricci flow. Hamilton [47] proved that nonnegative curvature operator is preserved in all dimensions. Furthermore, Hamilton showed that nonnegative Ricci curvature is preserved by the Ricci flow in dimension 3, and nonnegative isotropic curvature is preserved in dimension 4 (see [46], [50]). It turns out that nonnegative Ricci curvature is not preserved by the Ricci flow in dimension $n \geq 4$ (see [63]). By contrast, nonnegative isotropic curvature is preserved by the Ricci flow in all dimensions.

Theorem 3.2 (S. Brendle, R. Schoen [22]; H. Nguyen [66]). *Let M be a compact manifold of dimension $n \geq 4$, and let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on M. If $(M, g(0))$ has nonnegative isotropic curvature, then $(M, g(t))$ has nonnegative isotropic curvature for all $t \in [0, T)$.*

The proof of Theorem 3.2 requires two ingredients: the first is Hamilton's maximum principle for the Ricci flow (cf. Theorem 3.1); the second one is an algebraic inequality for curvature tensors with nonnegative isotropic curvature. We give a sketch of the proof here. A complete proof can be found in [20], Sections 7.2 and 7.3.

Proposition 3.3. *Let R be an algebraic curvature tensor on $\mathbb{R}^n$ with nonnegative isotropic curvature. Moreover, suppose that*

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2\,R_{1234} = 0$$

for some orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$. Then

$$Q(R)_{1313} + Q(R)_{1414} + Q(R)_{2323} + Q(R)_{2424} - 2\,Q(R)_{1234} \geq 0.$$

Sketch of the proof of Proposition 3.3. Using the definition of $Q(R)$, we compute

$$\begin{aligned}
&Q(R)_{1313} + Q(R)_{1414} + Q(R)_{2323} + Q(R)_{2424} - 2\,Q(R)_{1234} \\
&= \sum_{p,q=1}^{n} (R_{13pq} - R_{24pq})^2 + \sum_{p,q=1}^{n} (R_{14pq} + R_{23pq})^2 \\
&+ 2 \sum_{p,q=1}^{n} (R_{1p1q} + R_{2p2q})\,(R_{3p3q} + R_{4p4q}) - 2 \sum_{p,q=1}^{n} R_{12pq}\,R_{34pq} \\
&- 2 \sum_{p,q=1}^{n} (R_{1p3q} + R_{2p4q})\,(R_{3p1q} + R_{4p2q}) \\
&- 2 \sum_{p,q=1}^{n} (R_{1p4q} - R_{2p3q})\,(R_{4p1q} - R_{3p2q}).
\end{aligned}$$

Hence, it suffices to prove that

$$\begin{aligned}
&\sum_{p,q=1}^{n} (R_{1p1q} + R_{2p2q})\,(R_{3p3q} + R_{4p4q}) - \sum_{p,q=1}^{n} R_{12pq}\,R_{34pq} \\
&\geq \sum_{p,q=1}^{n} (R_{1p3q} + R_{2p4q})\,(R_{3p1q} + R_{4p2q}) \\
&+ \sum_{p,q=1}^{n} (R_{1p4q} - R_{2p3q})\,(R_{4p1q} - R_{3p2q}).
\end{aligned} \tag{4}$$

In order to prove (4), we view the isotropic curvature as a real-valued function defined on the space of all orthonormal four-frames. By assumption, this function attains its global minimum at the point $\{e_1, e_2, e_3, e_4\}$. Hence, the first variation at the point $\{e_1, e_2, e_3, e_4\}$ is zero, and the second variation is nonnegative. In order to take advantage of this information, we consider three different types of variations:

Step 1: We first consider the orthonormal four-frame $\{e_1, \cos(s)\,e_2 - \sin(s)\,e_3, \sin(s)\,e_2 + \cos(s)\,e_3, e_4\}$. Since the first variation of the isotropic curvature is zero, we have $R_{1213} + R_{1242} + R_{3413} + R_{3442} = 0$. An analogous argument gives $R_{1214} + R_{1223} + R_{3414} + R_{3423} = 0$. Using these identities, one can show that

$$\begin{aligned}
&\sum_{p,q=1}^{4} (R_{1p1q} + R_{2p2q})\,(R_{3p3q} + R_{4p4q}) - \sum_{p,q=1}^{4} R_{12pq}\,R_{34pq} \\
&= \sum_{p,q=1}^{4} (R_{1p3q} + R_{2p4q})\,(R_{3p1q} + R_{4p2q}) \\
&+ \sum_{p,q=1}^{4} (R_{1p4q} - R_{2p3q})\,(R_{4p1q} - R_{3p2q}).
\end{aligned} \tag{5}$$

Step 2: We next consider the four-frame $\{\cos(s)\,e_1 + \sin(s)\,e_q, e_2, e_3, e_4\}$, where $q \in \{5, \ldots, n\}$. Since the first variation of the isotropic curvature is equal to zero, it follows that $R_{133q} + R_{144q} + R_{432q} = 0$. Using this and other analogous identities, we obtain

$$\begin{aligned} &\sum_{p=1}^{4} (R_{1p1q} + R_{2p2q})\,(R_{3p3q} + R_{4p4q}) - \sum_{p=1}^{4} R_{12pq}\,R_{34pq} \\ &= \sum_{p=1}^{4} (R_{1p3q} + R_{2p4q})\,(R_{3p1q} + R_{4p2q}) \\ &+ \sum_{p=1}^{4} (R_{1p4q} - R_{2p3q})\,(R_{4p1q} - R_{3p2q}) \end{aligned} \tag{6}$$

for $q = 5, \ldots, n$.

Step 3: To describe the third type of variation, we consider four vectors $w_1, w_2, w_3, w_4 \in \text{span}\{e_5, \ldots, e_n\}$. For each $i \in \{1, 2, 3, 4\}$, we denote by $v_i(s)$ the unique solution of the linear ODE

$$v_i'(s) = \sum_{j=1}^{4} (\langle v_i(s), e_j \rangle\, w_j - \langle v_i(s), w_j \rangle\, e_j)$$

with initial condition $v_i(0) = e_i$. Then $v_i'(0) = w_i$. Moreover, it is easy to see that the vectors $\{v_1(s), v_2(s), v_3(s), v_4(s)\}$ are orthonormal for all $s \in \mathbb{R}$. Since the second variation of the isotropic curvature is nonnegative, we conclude that

$$\begin{aligned} 0 \leq\ & R(w_1, e_3, w_1, e_3) + R(w_1, e_4, w_1, e_4) \\ &+ R(w_2, e_3, w_2, e_3) + R(w_2, e_4, w_2, e_4) \\ &+ R(e_1, w_3, e_1, w_3) + R(e_2, w_3, e_2, w_3) \\ &+ R(e_1, w_4, e_1, w_4) + R(e_2, w_4, e_2, w_4) \\ &- 2\,\big[R(e_3, w_1, e_1, w_3) + R(e_4, w_1, e_2, w_3)\big] \\ &- 2\,\big[R(e_4, w_1, e_1, w_4) - R(e_3, w_1, e_2, w_4)\big] \\ &+ 2\,\big[R(e_4, w_2, e_1, w_3) - R(e_3, w_2, e_2, w_3)\big] \\ &- 2\,\big[R(e_3, w_2, e_1, w_4) + R(e_4, w_2, e_2, w_4)\big] \\ &- 2\,R(w_1, w_2, e_3, e_4) - 2\,R(e_1, e_2, w_3, w_4) \end{aligned} \tag{7}$$

for all vectors $w_1, w_2, w_3, w_4 \in \text{span}\{e_5, \ldots, e_n\}$. We next define linear transformations $A, B, C, D, E, F : \text{span}\{e_5, \ldots, e_n\} \to \text{span}\{e_5, \ldots, e_n\}$ by

$$\begin{aligned} \langle Ae_p, e_q \rangle &= R_{1p1q} + R_{2p2q}, & \langle Be_p, e_q \rangle &= R_{3p3q} + R_{4p4q}, \\ \langle Ce_p, e_q \rangle &= R_{3p1q} + R_{4p2q}, & \langle De_p, e_q \rangle &= R_{4p1q} - R_{3p2q}, \\ \langle Ee_p, e_q \rangle &= R_{12pq}, & \langle Fe_p, e_q \rangle &= R_{34pq} \end{aligned}$$

for $p, q \in \{5, \ldots, n\}$. The inequality (7) implies that the symmetric operator

$$\begin{bmatrix} B & F & -C^* & -D^* \\ -F & B & D^* & -C^* \\ -C & D & A & E \\ -D & -C & -E & A \end{bmatrix}$$

is positive semi-definite. From this, we deduce that $\mathrm{tr}(AB)+\mathrm{tr}(EF) \geq \mathrm{tr}(C^2)+\mathrm{tr}(D^2)$, hence

$$\begin{aligned} &\sum_{p,q=5}^{n} (R_{1p1q} + R_{2p2q})\,(R_{3p3q} + R_{4p4q}) - \sum_{p,q=5}^{n} R_{12pq}\,R_{34pq} \\ &\geq \sum_{p,q=5}^{n} (R_{1p3q} + R_{2p4q})\,(R_{3p1q} + R_{4p2q}) \qquad (8) \\ &+ \sum_{p,q=5}^{n} (R_{1p4q} - R_{2p3q})\,(R_{4p1q} - R_{3p2q}). \end{aligned}$$

Combining (5), (6), and (8), the inequality (4) follows. □

We next describe various curvature conditions that are related to nonnegative isotropic curvature, and are also preserved by the Ricci flow. The following is an immediate consequence of Theorem 3.2.

Corollary 3.4 (S. Brendle, R. Schoen [22]). *Let M be a compact manifold of dimension $n \geq 4$, and let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on M. Then:*

- *If $(M, g(0)) \times \mathbb{R}$ has nonnegative isotropic curvature, then the product $(M, g(t)) \times \mathbb{R}$ has nonnegative isotropic curvature for all $t \in [0, T)$.*
- *If $(M, g(0)) \times \mathbb{R}^2$ has nonnegative isotropic curvature, then the product $(M, g(t)) \times \mathbb{R}^2$ has nonnegative isotropic curvature for all $t \in [0, T)$.*

Another result in this direction was proved by the first author in [18] (see also [20], Section 7.6). In the following, $S^2(1)$ denotes a two-dimensional sphere of constant curvature 1.

Theorem 3.5 (S. Brendle [18]). *Let M be a compact manifold of dimension $n \geq 4$, and let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on M. If $(M, g(0)) \times S^2(1)$ has nonnegative isotropic curvature, then $(M, g(t)) \times S^2(1)$ has nonnegative isotropic curvature for all $t \in [0, T)$.*

Unlike Corollary 3.4, Theorem 3.5 does not follow directly from Theorem 3.2. This is because the manifolds $(M, g(t)) \times S^2(1)$ do not form a solution to the Ricci flow.

We now discuss the product conditions in more detail. To that end, we assume that (M, g) is a Riemannian manifold of dimension n. We first consider the case $n = 3$. In this case, the following holds:

- The product $(M, g) \times \mathbb{R}$ has nonnegative isotropic curvature if and only if (M, g) has nonnegative Ricci curvature.
- The product $(M, g) \times \mathbb{R}^2$ has nonnegative isotropic curvature if and only if (M, g) has nonnegative sectional curvature.

We now return to the case $n \geq 4$. The following proposition gives a necessary and sufficient condition for the product $(M, g) \times \mathbb{R}$ to have nonnegative isotropic curvature.

Proposition 3.6. *Let (M, g) be a Riemannian manifold of dimension $n \geq 4$. Then the following statements are equivalent:*

(i) The product $(M, g) \times \mathbb{R}$ has nonnegative isotropic curvature.

(ii) We have

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + \lambda^2 \, R(e_1, e_4, e_1, e_4) \\ &+ R(e_2, e_3, e_2, e_3) + \lambda^2 \, R(e_2, e_4, e_2, e_4) \\ &- 2\lambda \, R(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

for all points $p \in M$, all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$, and all $\lambda \in [0, 1]$.

(iii) We have $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$ for all points $p \in M$ and all vectors $\zeta, \eta \in T_p^{\mathbb{C}} M$ satisfying $g(\zeta, \zeta) \, g(\eta, \eta) - g(\zeta, \eta)^2 = 0$.

The proof of Proposition 3.6 is purely algebraic (for details, see [20], Proposition 7.18). We next consider the condition that $(M, g) \times \mathbb{R}^2$ has nonnegative isotropic curvature (cf. [20], Proposition 7.18).

Proposition 3.7. *Let (M, g) be a Riemannian manifold of dimension $n \geq 4$. Then the following statements are equivalent:*

(i) The product $(M, g) \times \mathbb{R}^2$ has nonnegative isotropic curvature.

(ii) We have

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + \lambda^2 \, R(e_1, e_4, e_1, e_4) \\ &+ \mu^2 \, R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 \, R(e_2, e_4, e_2, e_4) \\ &- 2\lambda\mu \, R(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

for all points $p \in M$, all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$, and all $\lambda, \mu \in [0, 1]$.

(iii) We have $R(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0$ for all points $p \in M$ and all vectors $\zeta, \eta \in T_p^{\mathbb{C}}M$.

Theorem 3.2 and Corollary 3.4 provide important examples of preserved curvature conditions. Each of these curvature conditions defines a closed, convex, $O(n)$-invariant cone in $\mathscr{C}_B(\mathbb{R}^n)$, which is preserved by the Hamilton ODE. By adapting a technique of Böhm and Wilking [16], it is possible to construct a family of so-called pinching cones, which are all preserved by the Hamilton ODE. Combining these ideas with general results of R. Hamilton (see [47] or [20], Section 5.4), one can draw the following conclusion.

Theorem 3.8 (S. Brendle, R. Schoen [22]). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ with the property that*

$$\begin{aligned}
&R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\
&+ \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2\mu^2 R(e_2, e_4, e_2, e_4) \\
&- 2\lambda\mu R(e_1, e_2, e_3, e_4) > 0
\end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [0, 1]$. Let $g(t)$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric g_0. Then the rescaled metrics $\frac{1}{2(n-1)(T-t)}\, g(t)$ converge to a metric of constant sectional curvature 1 as $t \to T$.

It turns out that any Riemannian manifold of dimension $n \geq 4$ which is strictly 1/4-pinched in the pointwise sense satisfies the assumption of Theorem 3.8. Hence, we can draw the following conclusion.

Corollary 3.9 (S. Brendle, R. Schoen [22]). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ which is strictly $1/4$-pinched in the pointwise sense. Let $g(t)$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric g_0. Then the rescaled metrics $\frac{1}{2(n-1)(T-t)}\, g(t)$ converge to a metric of constant sectional curvature 1 as $t \to T$.*

The Differentiable Sphere Theorem (Theorem 1.6 above) is an immediate consequence of Corollary 3.9. To conclude this section, we state an improved convergence result for the Ricci flow.

Theorem 3.10 (S. Brendle [18]). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ with the property that*

$$\begin{aligned}
&R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\
&+ R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) \\
&- 2\lambda R(e_1, e_2, e_3, e_4) > 0
\end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ *and all* $\lambda \in [0,1]$. *Let* $g(t)$, $t \in [0,T)$, *denote the unique maximal solution to the Ricci flow with initial metric* g_0. *Then the rescaled metrics* $\frac{1}{2(n-1)(T-t)}\,g(t)$ *converge to a metric of constant sectional curvature* 1 *as* $t \to T$.

Theorem 3.10 extends many known convergence results for the Ricci flow (see also [15]). The main ingredient in the proof is Theorem 3.5. A detailed argument can be found in [20], Section 8.4.

4. Weak Pinching and Further Developments

In this section, we describe various rigidity theorems and results which weaken the 1/4-pinching assumption. For a detailed exposition of the rigidity results, we refer to [20], Chapter 9. We close by outlining some open problems and conjectures in this area.

The first result in this direction was established by M. Berger [9] (see also [28], Theorem 6.6).

Theorem 4.1 (M. Berger [9]). *Let* (M, g) *be a compact, simply connected Riemannian manifold which is weakly* $1/4$*-pinched in the global sense. Then* M *is either homeomorphic to* S^n *or isometric to a symmetric space.*

We now describe some rigidity results obtained by means of the Ricci flow. The following result was established by R. Hamilton:

Theorem 4.2 (R. Hamilton [47]). *Let* (M, g_0) *be a compact three-manifold which is locally irreducible and has nonnegative Ricci curvature. Moreover, let* $g(t)$, $t \in [0,T)$, *denote the unique maximal solution to the Ricci flow with initial metric* g_0. *Then the rescaled metrics* $\frac{1}{4(T-t)}\,g(t)$ *converge to a metric of constant sectional curvature* 1 *as* $t \to T$.

In dimension $n \geq 4$, we have the following result:

Theorem 4.3 (S. Brendle, R. Schoen [21]). *Let* M *be a compact manifold of dimension* $n \geq 4$, *and let* $g(t)$, $t \in [0,T]$ *be a solution to the Ricci flow on* M *with nonnegative isotropic curvature. Then, for each* $\tau \in (0,T)$, *the set of all orthonormal four-frames* $\{e_1, e_2, e_3, e_4\}$ *satisfying*

$$\begin{aligned} &R_{g(\tau)}(e_1, e_3, e_1, e_3) + R_{g(\tau)}(e_1, e_4, e_1, e_4) \\ &+ R_{g(\tau)}(e_2, e_3, e_2, e_3) + R_{g(\tau)}(e_2, e_4, e_2, e_4) \\ &- 2\,R_{g(\tau)}(e_1, e_2, e_3, e_4) = 0 \end{aligned}$$

is invariant under parallel transport with respect to the metric $g(\tau)$.

In particular, if the reduced holonomy group of $(M, g(\tau))$ is $\mathrm{SO}(n)$, then $(M, g(\tau))$ has positive isotropic curvature.

Theorem 4.3 is similar in spirit to a result of R. Hamilton [47] concerning solutions to the Ricci flow with nonnegative curvature operator. However, Hamilton's techniques are not applicable in this setting. Instead, the proof of Theorem 4.3 relies on a variant of J.M. Bony's strict maximum principle for degenerate elliptic equations (cf. [17]). This technique was first employed in the context of geometric flows in [21]. It has since found applications to other borderline situations involving Ricci flow (see e.g. [3], [45]).

Theorem 4.3 is particularly effective in combination with M. Berger's classification of holonomy groups (see [7]). For example, the following structure theorem for compact, simply connected manifolds with nonnegative isotropic curvature was established in [19]:

Theorem 4.4 (S. Brendle [19]). *Let (M, g_0) be a compact, simply connected Riemannian manifold of dimension $n \geq 4$ which is irreducible and has nonnegative isotropic curvature. Then one of the following statements holds:*

(i) *M is homeomorphic to S^n.*

(ii) *$n = 2m$ and (M, g_0) is a Kähler manifold.*

(iii) *(M, g_0) is isometric to a symmetric space.*

M. Berger [11] has shown that any quaternionic-Kähler manifold with positive sectional curvature is isometric to $\mathbb{HP}^m$ up to scaling. More recently, H. Seshadri proved that any Kähler manifold which satisfies the assumptions of Theorem 4.4 is biholomorphic to a complex projective space or isometric to a Hermitian symmetric space (see [76], Theorem 1.2). We now state another consequence of Theorem 4.3.

Theorem 4.5. *Let (M, g_0) be a compact, locally irreducible Riemannian manifold of dimension $n \geq 4$ with the property that*

$$\begin{aligned}
&R(e_1, e_3, e_1, e_3) + \lambda^2 \, R(e_1, e_4, e_1, e_4) \\
&+ R(e_2, e_3, e_2, e_3) + \lambda^2 \, R(e_2, e_4, e_2, e_4) \\
&- 2\lambda \, R(e_1, e_2, e_3, e_4) \geq 0
\end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [0, 1]$. Moreover, let $g(t)$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric g_0. Then one of the following statements holds:

(i) *The rescaled metrics $\frac{1}{2(n-1)(T-t)} \, g(t)$ converge to a metric of constant sectional curvature 1 as $t \to T$.*

(ii) *$n = 2m$ and the universal cover of (M, g_0) is a Kähler manifold.*

(iii) *(M, g_0) is locally symmetric.*

In particular, if (M, g_0) is weakly $1/4$-pinched in the pointwise sense, then (M, g_0) satisfies the assumptions of Theorem 4.5. Hence, we have completed the proof of Theorem 1.7.

In the remainder of this section, we describe some results concerning almost $1/4$-pinched manifolds. The first result in this direction was proved by M. Berger in 1983.

Theorem 4.6 (M. Berger [12]). *For every even integer n, there exists a real number $\delta(n) \in (0, 1/4)$ with the following property: if (M, g_0) is a compact, simply connected Riemannian manifold of dimension n which is strictly $\delta(n)$-pinched in the global sense, then M is homeomorphic to S^n or diffeomorphic to a compact symmetric space of rank one.*

The proof of Theorem 4.6 is by contradiction, and relies on a compactness argument in the spirit of Gromov. In particular, the value of the pinching constant $\delta(n)$ is not known in general. U. Abresch and W. Meyer [1] showed that any compact, simply connected, odd-dimensional Riemannian manifold whose sectional curvatures lie in the interval $(\frac{1}{4(1+10^{-6})^2}, 1]$ is homeomorphic to a sphere. Using the classification in Theorem 4.5 and a Cheeger-Gromov-style compactness argument, P. Petersen and T. Tao obtained the following result:

Theorem 4.7 (P. Petersen, T. Tao [71]). *For each integer $n \geq 4$, there exists a real number $\delta(n) \in (0, 1/4)$ with the following property: if (M, g_0) is a compact, simply connected Riemannian manifold of dimension n which is strictly $\delta(n)$-pinched in the global sense, then M is diffeomorphic to a sphere or a compact symmetric space of rank one.*

The conclusion of Theorem 4.7 can be improved slightly when n is odd. In this case, there exists a real number $\delta(n) \in (0, 1/4)$ with the property that every compact n-dimensional manifold (M, g_0) which is strictly $\delta(n)$-pinched in the global sense is diffeomorphic to a spherical space form.

Finally we discuss some open problems in the study of positive curvature. First there are two well-known conjectures of H. Hopf.

Conjecture 4.8 (Hopf). *There is no metric of positive sectional curvature on $S^2 \times S^2$.*

Conjecture 4.9 (Hopf). *If n is even and M^n is a compact manifold with positive sectional curvature, then the Euler characteristic of M is positive.*

Concerning the first problem, we do not know of any compact simply connected manifold which admits a metric of nonnegative sectional curvature but can be shown not to admit a metric of positive sectional curvature. The famous theorem of J.L. Synge (see [28]) which classifies fundamental groups of even dimensional manifolds of positive sectional curvature implies that $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit a metric with positive sectional curvature. There does not seem to be a viable method to approach the second Hopf conjecture at this time.

It would be interesting to understand the fundamental groups of manifolds with positive isotropic curvature (see [34] and [40]).

Conjecture 4.10. *The fundamental group of a compact manifold of positive isotropic curvature contains a free subgroup of finite index.*

One possible approach to this conjecture involves the Ricci flow. For initial metrics with positive isotropic curvature, the Ricci flow will, in general, develop singularities. For $n = 4$, the singularities were analyzed by Hamilton [50], but the case $n \geq 5$ is open.

We expect that there is an almost 1/4-pinching theorem assuming only pointwise pinching. Even the topological version of this is unknown. The proof will require a more sophisticated technique.

Conjecture 4.11. *Theorems 4.6 and 4.7 hold with the assumption of pointwise pinching replacing global pinching.*

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