Here are the solutions to the sixth homework. Please do not print unless necessary.
HW6: 7.3 # 4, 12, 16, 20, 34, 43, 45, 47
7.4 # 2, 8, 11

1. (7.3 # 4) Solve

\[(y^2 + xy^2)y' = 1.\]

**Solution:** We may write this equation as

\[y^2 y' = \frac{1}{1 + x},\]

so that upon separating the variables we find

\[\int y^2 dy = \int \frac{dx}{1 + x},\]

so that

\[\frac{y^3}{3} = \ln|1 + x| + C.\]

Solving for \(y\) we get

\[y = 3\sqrt{3\ln|1 + x| + C}.\]

2. (7.3 # 12) Solve the IVP

\[\frac{dy}{dx} = \frac{\ln x}{xy}, y(1) = 2.\]

**Solution:** Separating variables and integrating gives

\[\int ydy = \int \frac{\ln x \, dx}{x}.\]

The second integral may be evaluated by setting \(u = \ln x\), then \(du = dx/x\). So:

\[\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C.\]

Solving for \(y\) we obtain

\[y = \pm\sqrt{\ln^2(x) + C}.\]

To match the initial condition:

\[2 = y(1) = \pm\sqrt{\ln^2(1) + C}.\]

We must take + and \(C = 4\). So the solution is

\[y = \sqrt{\ln^2(x) + 4}.\]

3. (7.3 # 16) Solve the IVP

\[\frac{dP}{dt} = \sqrt{Pt}, P(1) = 2.\]

**Solution:** To separate the variables note that \(\sqrt{Pt} = \sqrt{P}\sqrt{t}\) so long as \(P\) and \(t\) are positive. Now separating the variables and integrating we get:

\[\int \frac{dP}{\sqrt{P}} = \int \sqrt{t} \, dt,\]

so that

\[2\sqrt{P} = \frac{2}{3} t^{3/2} + C.\]

Note at this stage that \(P(1) = 2\) implies that \(C = 2\sqrt{2} - \frac{2}{3}\). Solving for \(P\) we get:

\[\sqrt{P} = \frac{t^{3/2}}{3} + \sqrt{2} - \frac{1}{3}\]

and so

\[P = \left(\frac{t^{3/2}}{3} + \sqrt{2} - \frac{1}{3}\right)^2.\]
4. (7.3 #20) Find a function \( f \) that satisfies \( f'(x) = f(x)(1 - f(x)) \) and \( f(0) = \frac{1}{2} \).

**Solution:** First set \( y = f(x) \). The IVP becomes

\[
y' = y(1 - y), \quad y(0) = 1/2.
\]

Separating the variables we get

\[
\int \frac{dy}{y(1-y)} = \int dx.
\]

Using partial fractions one can find:

\[
\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} \quad \rightarrow \quad A = 1, \quad B = 1,
\]

so that

\[
\int \frac{dy}{y(1-y)} = \int \frac{1}{y} + \frac{1}{1-y} \, dy = \ln |y| - \ln |1-y| = \ln \left| \frac{y}{1-y} \right|,
\]

by the log laws. Applying this in our situation we find

\[
\ln \left| \frac{y}{1-y} \right| = x + c.
\]

Plug in \( y(0) = 1/2 \) to find that

\[
0 = \ln 1 = \ln \left| \frac{1/2}{1-1/2} \right| = 0 + C,
\]

so that \( C = 0 \). So we’ve seen

\[
\ln \left| \frac{y}{1-y} \right| = x,
\]

solving for \( y \) gives:

\[
\left| \frac{y}{1-y} \right| = e^x.
\]

\[
y = e^x, \quad e^x, \quad e^x - ye^x \quad \rightarrow \quad y = \frac{e^x}{1 + e^x}.
\]

5. (7.3# 34) Solve the integral equation:

\[
y(x) = 2 + \int_1^x \frac{dt}{ty(t)}, \quad x > 0.
\]

**Solution:** First, convert the integral equation to an IVP. Notice that \( y(1) = 2 \) since the integral disappears over an interval of length zero. Next, differentiating the integral and applying the fundamental theorem of calculus gives

\[
\frac{dy}{dx} = 0 + \frac{1}{xy},
\]

a separable DE. Separating variables we find

\[
\int y \, dy = \int \frac{dx}{x}.
\]

So that

\[
\frac{y^2}{2} = \ln(x) + C.
\]

Now

\[
y = \pm \sqrt{2\ln(x) + C}.
\]

To match the intial condition, we must take + and plug in \( y(1) = 2 \),

\[
2 = y(1) = \sqrt{2\ln 1 + C} \quad \rightarrow \quad C = 4.
\]

So the solution is \( y = \sqrt{2\ln(x) + 4} \).
6. (7.3# 43) Model for glucose concentration

\[ \frac{dC}{dt} = r - kC, \]

where \( r, k \) are positive constants.

**Solution:** For part (a) we solve the DE with initial condition \( C(0) = C_0 \). First separate the equation

\[ \int \frac{dC}{r - kC} = \int dt. \]

Let \( u = r - kC \), then \( du = -kdC \) so that \( dC = (-1/k)du \). This means the integral above becomes

\[ -\frac{1}{k} \int \frac{du}{u} = t + B, \]

where \( B \) is a constant of integration. Now

\[ -\frac{1}{k} \ln |r - kC| = t + B, \]

so that

\[ \ln |r - kC| = -kt + B \]
\[ |r - kC| = Be^{-kt} \]
\[ r - kC = Be^{-kt} \]
\[ kC = r + Be^{-kt} \]
\[ C = \frac{r}{k} + Be^{-kt}. \]

In each of the lines above \( B \) is a new (arbitrary) constant from line to line. Solving for the initial condition we find

\[ C_0 = C(0) = \frac{r}{k} + B, \rightarrow B = C_0 - \frac{r}{k}. \]

We conclude

\[ C = \frac{r}{k} + \left( C_0 - \frac{r}{k} \right) e^{-kt}. \]

For part (b) note that if \( C_0 < r/k \) then the second term is negative, and so \( C \) is increasing and (since \( k > 0 \)),

\[ \lim_{t \to \infty} C(t) = \frac{r}{k}. \]

7. (7.3# 45) A tank contains 1000L of brine with 15kg of salt dissolved. Pure water enters the tank at 10L/min and the well mixed solution drains at the same rate. How much salt is in the tank after \( t \) minutes? after 20 minutes?

**Solution:** Let \( t \) be time in minutes and \( y(t) \) be the amount of salt in the tank, measured in kg. Recall the basic fact about mixing problems:

\[ \frac{dy}{dt} = \text{rate in} - \text{rate out}. \]

From the text we see that there is no salt flowing in, so the rate in is zero. As for the rate out, the flow rate is 10L/min and the concentration of salt in the tank at any time is \( y(t)/1000 \). So

\[ \frac{dy}{dt} = -\frac{10y}{1000} \text{kg/min} = -\frac{y}{100}. \]

Separating variables we find

\[ \frac{dy}{y} = -\frac{1}{100} dt, \]

So that

\[ y(t) = Ce^{-t/100}, \]

and since \( y(0) = 15 \),

\[ y(t) = 15e^{-t/100}, \]

answering part (a). As for the second question, after 20 minutes,

\[ y(20) = 15e^{-0.2} = 12.3 \text{kg}. \]
8. (7.3# 47) A vat with 500 gallons of beer contains 4% alcohol by volume. Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

**Solution:** Let \( t \) be time in minutes, and let’s again cast this problem in terms of amount alcohol \( y(t) \) (also measured in gallons). We’re told initially that \( y(0) = 4\% \) of 500 gallons = 20. We also have

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}.
\]

From the text, the rate in is

\[
0.06 \cdot 5 \text{gal/min}
\]

The rate out it

\[
\frac{y}{500} \cdot 5 \text{gal/min}
\]

So that

\[
\frac{dy}{dt} = 0.3 - \frac{y}{100} = \frac{30 - y}{100}.
\]

Solving this equation we find

\[
\frac{dy}{30 - y} = \frac{dt}{100},
\]

which yields

\[-\ln|30 - y| = \frac{t}{100} + C,
\]

and solving for \( y \) gives

\[y(t) = 30 - 10e^{-\frac{t}{100}}.
\]

After one hour, \( t = 60 \), we find

\[y(60) = 30 - 10e^{-0.6} = 24.51
\]

gal of alcohol. Since there is 500 gallons in the tank, we find that is about 4.9% alcohol by volume.

9. (7.4 #2) A cell of E. coli divides into 2 cells every 20 minutes. The initial population is 60 cells.

(a) Find the relative growth rate The exponential model is \( \frac{dP}{dt} = kP \), \( P(0) = P_0 \) with solutions \( P(t) = P_0e^{kt} \). From the text given 2 cells we will have 4 cells 20 minutes later. Let’s let \( t \) be time in minutes and \( P \) be the cell count. Now \( P_0 = 60 \), so \( P(20) = 120 \). We find:

\[120 = P(20) = 60e^{k \cdot 20} \rightarrow k = \frac{1}{20} \ln(2).
\]

So the relative growth rate, \( k = \frac{1}{20} \ln 2 \). We have learned

\[P(t) = 60e^{\frac{t}{20} \ln 2}, \ t \text{ in minutes}
\]

(b) Find an expression for the number of cells after \( t \) hours We are now switching units from (a). 60 minutes to an hour so,

\[P(T) = 60e^{\frac{3T}{20} \ln 2} = 60e^{3T \ln 2}, \ T \text{ in hours}
\]

(c) Find the number of cells after 8 hours

\[P(8) = 60e^{3 \cdot 8 \ln 2} = 1,006,632,960.
\]

(d) Find the growth rate after 8 hours

\[\frac{dP}{dt} \bigg|_{T=8} = kP(8) = (3 \ln(2)) 1006632960 = 2,093,234,394 \text{ cells/hr}
\]

(e) When will the population reach 20,000 cells? Solve for \( T \) in

\[20,000 = 60e^{3T \ln 2},
\]

i.e.

\[3 \ln(2) \cdot T = \ln\left(\frac{20000}{60}\right) \rightarrow T = 2.79 \text{hrs}.
\]
10. (7.4 #8) Strontium-90 has a half life of 28 days.

(a) A sample has an initial mass of 50 mg. Find the formula for the mass after \( t \) days. For radioactive decay, \( \frac{dm}{dt} = km \), \( m(0) = m_0 \) and \( m(t) = m_0 e^{kt} \), where \( m \) is the mass of Strontium-90 in the sample. We are told that \( m_0 = 50 \) and that \( m(28) = 25 \). So

\[
25 = m(0) = 50e^{28k} \rightarrow k = \frac{1}{28} \ln \left( \frac{1}{2} \right) = -\frac{1}{28} \ln(2).
\]

We conclude

\[
m(t) = 50e^{-\frac{1}{28}(\ln 2)t} = 502^{-\frac{t}{28}}.
\]

Of course it is not necessary to simplify the expression as above, but this may make other computations easier.

(b) What is the mass after 40 days.

\[
m(40) = 502^{-\frac{40}{28}} = 502^{-1.429} = 18.569 mg.
\]

(c) How long does it take for the sample to decay to 2 mg? Solve for \( t \) in

\[
2 = m(t) = 502^{-\frac{t}{28}}
\]

\[
\frac{2}{50} = 2^{-\frac{t}{28}}
\]

\[
\ln(2/50) = \ln(2^{-\frac{t}{28}}) = -\frac{t}{28} \ln(2)
\]

\[
t = -\frac{28 \ln(2/50)}{\ln(2)} = \text{about} \ 130 \text{ days}
\]

11. (7.4 #11) Carbon-14 has a half life of 5730 years. If a parchment has 74% as much radioactivity as plant material today, estimate the age of the parchment.

**Solution:** Let \( m(t) \) be the amount of Carbon-14 in the parchment, we know that \( m(t) = m_0 e^{kt} \) from the model in class. First, let’s work out \( k \). Given any initial amount of of Carbon-14, we know \( m(5730) = \frac{m_0}{2} \). So

\[
\frac{m_0}{2} = m_0 e^{5730k} \rightarrow k = \frac{1}{5730} \ln \left( \frac{1}{2} \right) = -\frac{1}{5730} \ln(2).
\]

So we now have

\[
m(t) = m_0 e^{-\frac{t}{5730} \ln(2)}.
\]

Now, given the situation at hand, let us suppose the parchment was created at time \( t = 0 \). We now that right now, \( 0.74m_0 \) of Carbon-14 remains. Let’s solve for \( t \):

\[
0.74m_0 = m_0 e^{-\frac{t}{5730} \ln(2)}
\]

\[
\ln(0.74) = -\frac{t}{5730} \ln(2)
\]

\[
t = -5730 \frac{\ln 0.74}{\ln 2} = 2489.
\]

So the parchment is about 2489 years old.