Here are the solutions to the fifth homework. Please do not print unless necessary.
HW5: 7.1# 2, 3, 10, 13
7.2# 3-6, 9, 13, 14, 18, 22

1. (7.1 #2) Verify $y = -t \cos t - t$ is a solution to the IVP:

$$t \frac{dy}{dt} = y + t^2 \sin(t). y(\pi) = 0$$

**Solution:** First let’s check the initial condition:

$$y(\pi) = -\pi \cos(\pi) - \pi = \pi - \pi = 0.$$

So the initial conditions works. Now notice that $y' = -\cos t + t \sin t - 1$ by the product rule. So, plugging this into the left hand side of the differential equation we get:

$$t \frac{dy}{dt} = t(- \cos t + t \sin t - 1)$$

$$= -t \cos t + t^2 \sin t - t$$

$$= (-t \cos t - t) + t^2 \sin t$$

$$= y + t^2 \sin(t)$$

as we needed to check.

2. (7.1 # 3) For what values of $r$ does $y = e^{rt}$ satisfy $2y'' + y' - y = 0$? If $r_1$ and $r_2$ are these values, show that every member of the family $y = ae^{r_1} + be^{r_2}$ are solutions, where $a$ and $b$ are constants.

**Solution:** For the first part, take $y = e^{rt}$ and differentiate twice: $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Plug this into the DE:

$$2y'' + y' - y = 0 \rightarrow 2r^2e^{rt} + re^{rt} - e^{rt} = 0$$

$$e^{rt}(2r^2 + r - 1) = 0.$$

Solving for $r$ we find that $r = -1$ or $r = 1/2$. For the second part, suppose we write $y = ae^{-t} + be^{t/2}$ for constants $a$ and $b$. Then $y' = -ae^{-t} + \frac{b}{2}e^{t/2}$ and $y'' = ae^{-t} + \frac{b}{4}e^{t/2}$. Plugging this back into the differential equation we have

$$2y'' + y' - y = 2 \left( ae^{-t} + \frac{b}{2}e^{t/2} \right) + \left( -ae^{-t} + \frac{b}{2}e^{t/2} \right) - \left( ae^{-t} + be^{t/2} \right)$$

$$= (2a - a - a)e^{-t} + \left( \frac{b}{2} + \frac{b}{2} - b \right)e^{t/2}$$

$$= 0$$

as required.

3. (7.1 #10) A function $y(t)$ satisfies the DE

$$y' = y^4 - 6y^3 + 5y^2.$$

What are the constant solutions of the DE? For what values of $y$ is the DE increasing? decreasing?

**Solution:** This equation is autonomous. So first let’s find the constant solutions where $y' = 0$. This means $0 = y^4 - 6y^3 + 5y^2 = y^2(y^2 - 6y + 5) = y^2(y - 1)(y - 5)$. So $y = 0, y = 1, y = 5$ are the constant solutions. Checking the sign of $y'$ on each interval we find that $y$ is increasing for $y \in (-\infty, 0) \cup (0, 1) \cup (5, \infty)$ and decreasing for $y \in (1, 5)$.

4. (7.1 #13) Match the differential equations with the solutions graphed in the text.

**Solution:** Here we’ll only provide the answers. The idea was to look at the slope of the tangent lines at various points on the graphs to make conclusions. Equation (a) matches III, equation (b) matches I, equation (c) matches IV and equation (d) matches II.

5. (7.2 #3-6) Match the differential equation to its direction field and give reasons for your answer.

- 3 matches III. Note that $y' = 2 - y$ is autonomous, i.e. that $t$ does not appear on the right hand side of the equation. Thus the direction field will be determined along any vertical line and constant in the $t$ direction. Only III satisfies this.
4 matches I. Note that $y' = x(2 - y)$ will be zero along $y = 2$, so only I and III are possible matches. By elimination it must be I.

5 matches IV. Notice that at $(0, 0), y = -1$. The only direction field with negative slope at the origin is IV.

6 matches II. Of course we can see this by elimination, but note also that $y' = 0$ whenever $\sin(x) \sin(y) = 0$. One place this happens is when $x$ or $y$ is $\pi$ – and you can see these horizontal tangent lines form a 'box' around the rest of the direction field.

Other justifications possible.

6. (7.2 #9, 13, 14) For all of these problems you are asked to sketch a direction field. The best way is to use the link I provided on my website. Please play around with this and various combinations of equations to get intuition. None of these problems were graded this week.

7. (7.2 # 18) Make a rough sketch of a direction field for $y' = f(y)$ for the $f(y)$ plotted in the book. How does the limiting behaviour depend on the value of $y(0)$.

Here is what the direction field should look like:
This direction field is for solutions of the form $y = y(t)$. The SLOPE of the tangent lines of $y$ are given by $f$. This is an important thing to think about. So for example, notice from the text that $f(y)$ has zeros at $y = -2, -1, 1, 2$. These correspond to constant solutions to our DE, and we see them as horizontal ticks on the direction field.

The curve that is plotted is the solution with $y(0) = 0.5$. Notice that as $t \to \infty$ that the curve is asymptotics to the line $y = 1$. This is the key point. Choose any value on the vertical line above $t = 0$. This corresponds to picking a value for $y(0)$. There is a solution curve through your point that follows the direction field. Where the solution goes depends on the nearby constant solutions. If you pick $y(0) = -2, -1, 1, 2$, then the solution is constant and stays at one of $y = -2, -1, 1, 2$. If you pick $-1 < y(0) < 1$, then all of these solutions increase and are asymptotics to $y = 1$. If you pick $1 < y(0) < 2$ then solutions decrease and are asymptotics to $y = 2$. If you pick $y(0) > 2$ solutions grow without bound. If you pick $-2 < y(0) < -1$, solutions decrease and are asymptotics to $y = -2$. Finally if $y(0) < -2$, solutions increase and are asymptotic to $y = -2$.

We got all of this information because the equation is autonomous. This is a very important example understand.

8. (7.2 # 22) This problem was done in problem 1 of Lab 6. Please see the solutions for Lab 6.