Here are the solutions to the second homework. Please do not print unless necessary.

HW2: 8.2# 12, 29, 31, 32, 51
8.3# 14, 16, 17, 18

1. (8.2 #12) Determine if the following series is convergent or divergent. If convergent, find the sum.

\[ 4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots \]

**Solution:** Notice we can rewrite the series as:

\[ 4 + 3 + \frac{3^2}{4^1} + \frac{3^3}{4^2} + \cdots \]

We’re also told this series is geometric. We can also write \( 3 = \frac{3^1}{4^0} \), so this term fits the pattern. In fact, \( 4 = \frac{3^0}{4^{-1}} \), so even the first term fits the pattern.

Thus one way to write the series is

\[ \sum_{k=-1}^{\infty} \frac{3^{k+1}}{4^k} \]

(There are other correct ways to write this). Let’s reindex this to get

\[ \sum_{n=0}^{\infty} \frac{3^n}{4^{n-1}} = 4 \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n \]

The sum is therefore:

\[ \frac{4}{1 - \frac{3}{4}} = 16. \]

2. (8.2 #29) Determine whether the sequence converges or diverges, if it converges find the sum.

\[ \sum_{n=1}^{\infty} \left( \frac{1}{e^n} + \frac{1}{n(n+1)} \right) \]

**Solution:** To solve this problem we would like to say that

\[ \sum_{n=1}^{\infty} \left( \frac{1}{e^n} + \frac{1}{n(n+1)} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{e^n} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right) \]

but in order to assert this we must know that each series on the right converges separately. The first series is convergent geometric (because \( e < 1 \)), and we can easily compute the sum:

\[ \sum_{n=1}^{\infty} \left( \frac{1}{e^n} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{e} \right)^n = \frac{\frac{1}{e}}{1 - \frac{1}{e}} = \frac{1}{e - 1}. \]

The second series is also convergent. This was the telescoping series done in class. See page 568 of the text for the argument that shows

\[ \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right) = 1. \]

We conclude since both sums are convergent that:

\[ \sum_{n=1}^{\infty} \left( \frac{1}{e^n} + \frac{1}{n(n+1)} \right) = \frac{1}{e - 1} + 1 = \frac{e}{e - 1}. \]

3. (8.2 #31) Determine whether the sequence converges or diverges, if it converges find the sum.

\[ \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \]
Solution: Notice that applying partial fractions to this expression gives
\[
\frac{2}{n^2 - 1} = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.
\]

Let’s write out a few partial sums of the series:
\[
\sum_{n=2}^{\infty} \frac{2}{n^2 - 1} = \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right).
\]
(Note: we are starting with \(n = 2\!\!)

\[
s_1 = 1 - \frac{1}{3} = \frac{2}{3}.
\]
\[
s_2 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{5}{6}.
\]
\[
s_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} = 1 - \frac{1}{6}.
\]
\[
s_4 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} = \frac{1}{2} - \frac{1}{6}.
\]
\[
s_5 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} = \frac{1}{3} - \frac{1}{6}.
\]
\[
s_n = 1 + \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}.
\]

As \(n \to \infty\), \(s_n\) approaches \(\frac{3}{2}\). We conclude
\[
\sum_{n=2}^{\infty} \frac{2}{n^2 - 1} = \frac{3}{2}.
\]

4. (8.1 #32) Determine whether the sequence converges or diverges, if it converges find the sum.

\[
\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3}
\]
converges or diverges. If it converges, find the limit.

Solution: Notice that applying partial fractions to this expression gives
\[
\frac{2}{n^2 + 4n + 3} = \frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}.
\]

Let’s write out a few partial sums of the series:
\[
\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3} = \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+3} \right).
\]

\[
s_1 = \frac{1}{2} - \frac{1}{4}.
\]
\[
s_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} = \frac{5}{6}.
\]
\[
s_3 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} = 1 - \frac{1}{6}.
\]
\[
s_4 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} = \frac{1}{2} - \frac{1}{6}.
\]
\[
s_5 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} = \frac{1}{3} - \frac{1}{6}.
\]
\[
s_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} + \frac{1}{n+2}.
\]

As \(n \to \infty\), \(s_n\) approaches \(\frac{5}{6}\). We conclude
\[
\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3} = \frac{5}{6}.
\]
5. (8.2 #52) When money is spent on goods and services those who receive the money also spend some of it. The people that receive the twice spent money spend some of that and so on. In a hypothetical isolated community suppose the local government spends \( D \) dollars. Suppose each recipient spends 100\(c\)% and saves 100\(s\)% of the money they receive. The numbers \( s \) and \( c \) satisfy \( s + c = 1 \).

(a) Let \( S_n \) be the total spending generated after \( n \) transactions. Find an equation for \( S_n \).

(b) Show \( \lim_{n \to \infty} S_n = kD \), where \( k = 1/s \). What is this number if \( c = 0.8 \)?

**Solution:**

First we tackle part (a). Let \( s_n \) represent the amount of TOTAL spending done by all individuals at the \( n \)th step. At step 1, the government infuses \( D \) dollars into the economy by giving it to a recipient. So at the first step,

\[ s_1 = D. \]

But now the recipient will spend \( c \) percent of this, i.e. \( cD \) dollars will now be given to a new recipient, and so \( s_2 = D + cD \).

The next recipient gets \( cD \) dollars and will spend \( c \cdot cD = c^2D \). So the now \( s_3 = D + cD + c^2D \).

The third recipient gets \( c^2D \) dollars and spends \( c \cdot c^2D = c^3D \) of these. So the total spending after 3 steps is \( s_4 = D + cD + c^2D + c^3D \).

The \( n \) recipient will get \( c^{n-1}D \) dollars and will spend \( c^nD \) of it. The total spending is then:

\[ s_n = D + cD + c^2D + \cdots + c^nD = \sum_{k=0}^{n} Dc^k. \]

For part (b) we want to take a limit of \( s_n \) as \( n \to \infty \). But \( s_n \) is just the partial sum of

\[ \sum_{k=0}^{\infty} Dc^k, \]

which is a geometric series! Since \( 0 < c < 1 \), we take \( r = c \) and \( a = D \) to find the limit is just the sum

\[ \lim_{n \to \infty} s_n = \sum_{k=0}^{\infty} Dc^k = \frac{D}{1 - c}. \]

Since \( s = 1 - c \), this limit is just

\[ \sum_{k=0}^{\infty} Dc^k = \frac{D}{1 - c} = \frac{D}{s}, \]

as we needed to show. If \( c = 0.8 \), then \( s = 0.2 \), so \( \frac{D}{s} = 5D \).

6. (8.3 #14) Determine whether or not the series is convergent or divergent.

\[ 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots \]

**Solution:**

This series is

\[ \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \]

Any of the following justifications is correct:

- This is just a \( p \)-series with \( p = 3/2 \). Since \( p > 1 \), the series converges.
- The series comes from the continuous, positive, decreasing function \( f(x) = x^{-3/2} \) by setting \( a_n = f(n) \). So the series will converge if and only if

\[ \int_{1}^{\infty} \frac{dx}{x^{3/2}} \]

converges. But this is a \( p \)-integral which converges, as you may cite or verify by explicit computation.
7. (8.3 #16). Determine whether or not the series is convergent or divergent.

\[
\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}.
\]

**Solution:** We compare to \( \sum \frac{1}{n} \) using the LCT. Set \( a_n = \frac{n^2}{n^3 + 1} \) and \( b_n = \frac{1}{n} \). Then

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^3}{n^3 + 1}
\]

after simplification. But this limit is just 1. So the LCT applies and both series behave in the same manner. Since \( \sum \frac{1}{n} \) is the divergent harmonic series \( (p = 1) \), \( \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} \) diverges too.

8. (8.3 #17). Determine whether or not the series is convergent or divergent.

\[
\sum_{n=1}^{\infty} \frac{1}{n \ln n}.
\]

**Solution:** Whoops. This was actually on lab 2. Please see the Lab 2 solutions. The integral test is the way to go.

9. (8.3 #18). Determine whether or not the series is convergent or divergent.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + 9}.
\]

**Solution:** The proof here is by the LCT. You should also check that there is an easier proof using the comparison test, and a slightly harder proof using the integral test. We compare to \( \sum \frac{1}{n^2} \). Set \( a_n = \frac{1}{n^2 + 9} \) and \( b_n = \frac{1}{n^2} \). Then

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^2 + 9}
\]

after simplification. But this limit is just 1. So the LCT applies and both series behave in the same manner. Since \( \sum \frac{1}{n^2} \) is a convergent \( p \)-series, \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 9} \) converges.