A FEW SOLUTIONS

Here are solutions to a couple of problems that I gave in Section. I’ve chosen to write up a solution to the first problem because I know that giving a proof of when a collection of vectors forms a subspace can be tricky sometimes. I chose to write up a solution to the second problem because I think it’s not easy, and therefore a good test of how comfortable you feel with proofs (in fact, I doubt that anyone was able to solve this problem after I gave it in class). I’m doubt that anything like these problems will show up on your test, but I am convinced that if you understand how these problems work you can probably knock down a lot of the more theoretical problems you’ll come across.

**Problem.** Suppose that $W \subseteq \mathbb{R}^r$ is a subspace and that $T : \mathbb{R}^c \rightarrow \mathbb{R}^r$ is a linear transformation. Show that

$$T^{-1}(W) = \{ \overrightarrow{v} \in \mathbb{R}^c : T(\overrightarrow{v}) \in W \}$$

is a subspace of $\mathbb{R}^c$.

**Solution.** We need to check the 3 rules that a subspace must obey. First, we have to show that $\overrightarrow{0} \in \mathbb{R}^c$ is an element of $T^{-1}(W)$; unwinding the definition of $T^{-1}(W)$, this means that we need to show $T(\overrightarrow{0}) \in W$. Remember that we proved that $T(\overrightarrow{0}) = \overrightarrow{0}$ since $T$ is a linear operator (the $\overrightarrow{0}$ on the left hand side of the equation is the zero vector in $\mathbb{R}^c$, while the $\overrightarrow{0}$ on the right hand side of the equation is in $\mathbb{R}^r$). Since $W$ is a subspace, $\overrightarrow{0} \in W$ (this $\overrightarrow{0}$ is in $\mathbb{R}^r$, of course, since that’s where $W$ lives). Putting these together gives $T(\overrightarrow{0}) = \overrightarrow{0} \in W$, and therefore $\overrightarrow{0} \in T^{-1}(W)$. Great!

Now for the second rule of subspace, we have to check that if $\overrightarrow{z}, \overrightarrow{u} \in T^{-1}(W)$, then $\overrightarrow{z} + \overrightarrow{u} \in T^{-1}(W)$. Unwinding the definition of what it means to be in $T^{-1}(W)$, this means that $T(\overrightarrow{z}) \in W$ and $T(\overrightarrow{u}) \in W$, and we have to prove that $T(\overrightarrow{z} + \overrightarrow{u}) \in W$. Since $T$ is linear we have $T(\overrightarrow{z} + \overrightarrow{u}) = T(\overrightarrow{z}) + T(\overrightarrow{u})$, and since $W$ is a subspace, $T(\overrightarrow{z}) + T(\overrightarrow{u}) \in W$ by closure under addition. Therefore we have

$$T(\overrightarrow{z} + \overrightarrow{u}) = T(\overrightarrow{z}) + T(\overrightarrow{u}) \in W,$$

and hence $\overrightarrow{z} + \overrightarrow{u} \in T^{-1}(W)$.

Finally, we have to check that $T^{-1}(W)$ is closed under scalar multiplication. So suppose that the devil gives you $\overrightarrow{u} \in T^{-1}(W)$ and some scalar $c \in \mathbb{R}$, and your job is to show $c \overrightarrow{u} \in T^{-1}(W)$. Again, we’ll begin by writing down what it means for all these elements to be in $T^{-1}(W)$: we are given that $T(\overrightarrow{u}) \in W$, $c \in \mathbb{R}$ some scalar, and we are supposed to prove that $T(c \overrightarrow{u}) \in W$. Since $T$ is linear we know that $cT(\overrightarrow{u}) = T(c \overrightarrow{u})$, and since $W$ is a subspace we know that $cT(\overrightarrow{u}) \in W$ since $W$ is closed under scalars (remember, we already had that $T(\overrightarrow{u}) \in W$). Therefore

$$T(c \overrightarrow{u}) = cT(\overrightarrow{u}) \in W,$$

and hence $c \overrightarrow{u} \in T^{-1}(W)$.

**Problem.** Suppose that $V \subseteq W$, where both are subspace. Show that $\dim(V) \leq \dim(W)$.

**Solution.** Recall that $\dim(V)$ is the number of elements in a basis of $V$, and that $\dim(W)$ is the number of elements in a basis for $W$. So to show the desired inequality, let $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_s$ and $\overrightarrow{w}_1, \ldots, \overrightarrow{w}_t$ be bases for $V$ and $W$, respectively.
and \( W \), respectively. (Note that our notation means that \( s = \dim(V) \) and \( t = \dim(W) \), and we’re now aiming to show that \( s \leq t \).)

Notice first that both \( \overrightarrow{v}_1, \cdots, \overrightarrow{v}_s \) and \( \overrightarrow{w}_1, \cdots, \overrightarrow{w}_t \) are collections of vectors in the subspace \( W \) (we’ve used the fact that \( V \subseteq W \) here). Since \( \overrightarrow{v}_1, \cdots, \overrightarrow{v}_s \) are a basis for \( V \), they form a \textit{linearly independent} collection of vectors in \( V \) (and therefore, also a linearly independent collection of vectors in \( W \)—think about why this is true).

Using the other property of bases, since \( \overrightarrow{w}_1, \cdots, \overrightarrow{w}_t \) is a basis of \( W \), this collection \textit{spans} the space \( W \). Now a proposition in your book says the following: if you have a collection of spanning vectors for a subspace \( U \) and another collection of linearly independent vectors inside \( U \), the number of elements in the linearly independent collection is no bigger than the number of elements in the spanning collection (this is Proposition 12.1 rearranged a bit). In our case, this means that \( s \leq t \), which is what we wanted to prove. \( \square \)