GRAPHS WITH DISJOINT LINKS IN EVERY SPATIAL EMBEDDING

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ABSTRACT

We exhibit a graph, $G_{12}$, that in every spatial embedding has a pair of non-splittable 2 component links sharing no vertices or edges. Surprisingly, $G_{12}$ does not contain two disjoint copies of graphs known to have non-splittable links in every embedding. We exhibit other graphs with this property that cannot be obtained from $G_{12}$ by a finite sequence of $\Delta-Y$ and/or $Y-\Delta$ exchanges. We prove that $G_{12}$ is minor minimal in the sense that every minor of it has a spatial embedding that does not contain a pair of non-splittable 2 component links sharing no vertices or edges.

Keywords: embedded graphs, intrinsically linked, spatial graph, minor minimal, self-linked.

1. Introduction

It has previously been shown by Conway-Gordon [1] and Sachs [6] that $K_6$ is intrinsically linked, that is, every spatial embedding of $K_6$ contains at least one pair of disjoint cycles that form a non-splittable link. Robertson, Seymour and Thomas [4] later proved that a graph is intrinsically linked if and only if it contains as a minor one of the seven graphs in the so called Petersen family (graphs that can be obtained from $K_6$ through $\Delta-Y$ exchanges and $Y-\Delta$ exchanges). The main focus of this paper is on the search for an analogue of this theorem for graphs that have disjoint pairs of non-splittable links in every spatial embedding. A graph has the disjoint linking property if in every spatial embedding there exists a pair of non-splittable links that share no vertices and no edges. Of course, any graph that contains two disjoint sub-graphs that are each intrinsically linked will have this property. As the main result of this paper, we exhibit several graphs that have the disjoint linking property and do not contain disjoint copies of intrinsically linked graphs as sub-graphs.

A vertex splitting of a vertex $v$ in a graph $G$ is achieved by replacing $v$ with two vertices $v'$ and $v''$, adding the edge $v'v''$ and connecting a subset of the edges that were incident to $v$ to $v'$ and the rest of the edges that were incident to $v$ to $v''$. A graph $G$ is considered to be an expansion of a graph $H$ if $G$ can be obtained by vertex splittings of $H$. A contraction of an edge is the inverse of an expansion. If we contract the graph $G$ along the edge $e$, we denote the resulting graph $G\setminus e$. Let $G$ be a graph and let $v$ be a vertex of $G$ with only three incident edges. Let $H$ be obtained from $G$ by deleting $v$ and adding an edge between each pair of vertices that were connected to vertex $V$ initially. We say that $H$ is obtained from $G$ by a $Y-\Delta$ exchange and that $G$ is obtained from $H$ by a $\Delta-Y$ exchange. In [3], it was shown that vertex splittings and also $\Delta-Y$ exchanges preserve intrinsic linking. By similar reasoning, both operations preserve the disjoint linking property.

Throughout the paper we will take an embedded graph to mean a graph embedded in 3-space, where all of our embeddings are tame. A link is a finite set of disjoint cycles embedded in 3-space. A link is said to be non-splittable if there exists no sphere in 3-space disjoint from the link that separates components of the link.
We will use the terms non-splittably linked cycles and linked cycles interchangeably; both will represent cycles in a spatially embedded graph that form a non-splittable link. A graph $H$ is said to be a minor of a graph $G$ if $H$ can be obtained from $G$ by deleting and/or contracting a finite number of edges, and possibly by removing isolated vertices. A graph $G$ is minor minimal with respect to a given property iff no minor of $G$ has the property. By the important result of Robertson and Seymour [5], there are only finitely many minor minimal graphs with respect to the disjoint linking property. The main result of this paper suggests that this set may be very large.

2. Lemmas involving $K_{3,3,2}$ and $K_{3,1,1,1,1}$

Here we prove that the graphs $K_{3,3,2}$ and $K_{3,1,1,1,1}$ contain a pair of non-splittably linked three-cycles in every spatial embedding. We will later use these graphs to build other graphs with the disjoint linking property.

Let $A_1$ and $A_2$ be disjoint graphs in space, such that the projection of $A_1 \cup A_2$ is regular. Define $\omega(A_1, A_2)$ to be the number of times $(\mod 2)$ that $A_1$ crosses over $A_2$ in the projection. If $A_1$ and $A_2$ are both cycles, then $\omega(A_1, A_2)$ is equal to the mod 2 linking number of $A_1$ and $A_2$, $lk(A_1, A_2)$.

**Lemma 1.** Every embedding of $K_{3,3,2}$ contains at least one non-splittable link composed of two three-cycles.

**Proof Sketch:** One can use a proof similar to the proof given in [1] or [6] that shows $K_4$ is intrinsically linked. One can find a particular embedding of the graph $K_{3,3,2}$ and calculate the sum of the linking numbers between every pair of disjoint three-cycles in that graph such that the mod 2 reduction of this sum is one. Thus, at least one pair of disjoint three-cycles must have odd linking number. This implies that there must be at least one pair of non-splittably linked three-cycles in the embedded graph. One can then show that through arbitrary crossing change and through ambient isotopy, this mod 2 reduction of the sum of the linking number between every pair of disjoint three-cycles in $K_{3,3,2}$ is invariant. Therefore in every embedding of $K_{3,3,2}$ there exists at least one pair of disjoint three-cycles that are non-splittably linked.

The following Lemma shows that no minor of $K_{3,3,2}$ has linked three-cycles in every spatial embedding.

**Lemma 2.** For any edge $e$ of $K_{3,3,2}$, there exists a spatial embedding of $K_{3,3,2}$ such that $K_{3,3,2} - e$ and $K_{3,3,2}$\{e do not contain non-splittably linked three-cycles.

**Proof.** Use the three vertex sets $V_1 = \{A, B\}$, $V_2 = \{v_1, v_2, v_3\}$ and $V_3 = \{v_4, v_5, v_6\}$ to form the graph $K_{3,3,2}$, where $V_1, V_2$ and $V_3$ form the three partitions of vertices. There is an edge between vertices of different partitions, but no edges between vertices of the same partition.

Recall that $K_{3,3,2}$ has two equivalence classes of edges, where two edges are considered equivalent if there is a graph automorphism taking one to the other.
One class consists of those edges incident to the set $V_1$. The other class consists of all edges between vertices in $V_2$ and $V_3$. Now there exists an embedding of $K_{3,3,2}$ such that there is only one pair of non-splittably linked three-cycles: $(A, v_3, v_5)$ and $(B, v_3, v_6)$, see Figure 1. Edges from each equivalence class occur in both cycles. Thus, after the removal of any edge, it is possible to embed $K_{3,3,2}$ in a way that has no non-splittably linked three-cycles.

If one contracts an edge that is incident to either $A$ or $B$, then the resulting graph is $K_{3,1,1,1,1}$ with an edge removed. We will show later that such a graph has a spatial embedding without non-splittably linked three-cycles. Now we consider the effect of contracting an edge connecting a vertex in $V_2$ to a vertex in $V_3$. Without loss of generality, we may take the contracted edge to be $(v_1, v_2)$, which is contracted to the vertex $v$. In this case, the resulting subgraph induced by the vertices $\{v, v_1, v_4, v_5, v_6\}$ is $K_5$ with two non-adjacent edges removed. This graph can be embedded in the plane $z = 0$. Place the vertex $A$ at $z = 1$. One can connect $A$ to the vertices in $z = 0$ via straight lines. Similarly, one can place the vertex $B$ in $z = -1$, and connect it to the vertices in $z = 0$ via straight lines. Such an embedded graph clearly has no non-splittably linked three-cycles. Thus contracting an edge of $K_{3,3,2}$ results in a graph that does not have a pair of non-splittably linked three-cycles in every spatial embedding.

![Figure 1: An embedding of $K_{3,3,2}$ with exactly one pair of non-splittably linked three-cycles.](image)

Now we discuss $K_{3,1,1,1,1}$:

**Lemma 3.** In every spatial embedding of $K_{3,1,1,1,1}$ there exists a pair of disjoint three-cycles that are non-splittably linked. In addition, no minor of $K_{3,1,1,1,1}$ has a non-splittably linked pair of three-cycles in every embedding.
Proof. We could use a proof similar to the one given for $K_{3,3,3,3}$, but we present another interesting proof instead. Let $K_{3,1,1,1,1}$ be the graph composed of five sets of vertices $\{A, B, C\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, and $\{v_4\}$, with edges connecting vertices in distinct sets, but no edges connecting vertices in a common set. Consider an arbitrary three-cycle for which there is at least one other disjoint cycle in the graph, $S$. If $S$ does not contain a vertex from $\{A, B, C\}$, then the subgraph induced by the set of vertices disjoint from $S$ is isomorphic to $K_{3,1,1}$, and so does not admit any cycles. We may assume, then, that $S$ contains exactly one vertex from $\{A, B, C\}$ (it cannot have more than one since these vertices are not connected). Since all such cycles are equivalent under graph automorphism, without loss of generality we may take $S$ to be the three-cycle $(A, v_1, v_2)$. There are two distinct three-cycles on these vertices in $K_{3,1,1,1,1,1}$ that are disjoint from $S$, namely $(B, v_3, v_4)$ and $(C, v_3, v_4)$. Now, consider an arbitrary embedding of $K_{3,1,1,1,1,1}$.

Assume that $S$ is not non-splittably linked to either of these three-cycles. This implies that $\omega(S, (B, v_3, v_4)) = \omega(S, (C, v_3, v_4)) = 0 \mod 2$. Again, we consider the crossings edge by edge, and have the following two equations:

$$\omega(S, (B, v_3)) + \omega(S, (v_3, v_4)) + \omega(S, (B, v_4)) = 0 \mod 2$$

$$\omega(S, (C, v_3)) + \omega(S, (v_3, v_4)) + \omega(S, (C, v_4)) = 0 \mod 2$$

We add the two equations above to obtain:

$$\omega(S, (B, v_3)) + \omega(S, (B, v_4)) + \omega(S, (C, v_3)) + \omega(S, (C, v_4)) = 0 \mod 2$$

This equation implies that $S$ has zero mod 2 linking number with the four-cycle $(B, v_3, C, v_4)$, but this is the only four-cycle in the graph that is disjoint with $S$.

The above argument shows that if a given three-cycle is not non-splittably linked with any other three-cycles, then it is not non-splittably linked with any four-cycles. But $K_{3,1,1,1,1,1}$ is a supergraph of $K_{3,3,1,1}$, and we have by [6] that some three-cycle in $K_{3,3,1,1}$ is non-splittably linked with a four-cycle in $K_{3,3,1,1}$. Thus, in our embedding of $K_{3,1,1,1,1,1}$, some three-cycle must be non-splittably linked to a disjoint three-cycle.

Now we move on to the second part of the Lemma. Any graph obtained by a single edge contraction or a single vertex removal on $K_{3,1,1,1,1,1}$ will yield a subgraph of $K_{2,1,1,1,1,1}$. Further edge contractions and vertex removals will yield graphs with five or fewer vertices. The only intrinsically linked graph with six or fewer vertices is the complete graph $K_6$; thus minors of $K_{3,1,1,1,1,1}$ obtained by edge contraction or vertex removal do not have a non-splittably linked pair of three-cycles in every embedding. In the embedding of $K_{3,1,1,1,1,1}$ illustrated in Figure 2, we see that the single pair of non-splittably linked three-cycles uses edges from each of the two distinct edge equivalence classes. Removal of any edge results in a graph that can be spatially embedded with no non-splittably linked pair of three-cycles. Thus, no minor of $K_{3,1,1,1,1,1}$ has a pair of non-splittably linked three-cycles in every spatial embedding.

$\square$

3. The graph $G_{12}$ has the disjoint linking property
Figure 2: (Left) An embedding of $K_{3,1,1,1,1}$ with a single pair of non-splittably linked three-cycles. (Right) An embedded $K_{3,3,1}$ subgraph with a three-cycle non-splittably linked with a four-cycle.

Any graph composed of two disjoint copies of Petersen graphs is minor minimal with respect to the disjoint linking property. Based on such graphs alone, there are at least 28 distinct minor minimal graphs with the disjoint linking property. The discussion below shows that this number is even greater.

Consider the graph $G_{12}$ defined by:

$G_{12} = (V(G_{12}), E(G_{12}), \Phi_{12})$, where $V(G_{12}) = \{A, B, v_1, v_2, \ldots, v_6, a, b, c, d\}$ is the vertex set of $G_{12}$, $E(G_{12}) = \{e_1, e_2, \ldots, e_{31}\}$ is the edge set of $G_{12}$, and $\Phi_{12} : E(G_{12}) \rightarrow V(G_{12}) \times V(G_{12})$ is the mapping function associating edges with unique pairs of vertices in $G_{12}$. The graph is constructed as follows: Using vertices $A, B,$ and $v_1$ through $v_6$ with edges $e_1$ through $e_{31}$, let $\Phi_{12}$ map edges to vertices in a way that creates the complete tripartite graph $K_{3,3,3}$, where we partition the vertices into the three sets $V_1 = \{A, B\}, V_2 = \{v_1, v_3, v_5\}$, and $V_3 = \{v_2, v_4, v_6\}$. Let edges $e_1$ through $e_{31}$ be mapped onto vertices $a, b, c, d$ in such a way that creates the complete graph $K_4$. Now let the remaining edges $e_{32}, e_{33}, \ldots, e_{31}$ be mapped in such a way that connects each one of the six vertices $v_1, v_2, \ldots, v_6$ to all of the four vertices $a, b, c, d$.

We now state the following surprising theorem:

**Theorem 1.** The graph $G_{12}$ has disjoint linking property, does not contain two disjoint copies of intrinsically linked graphs, and is minor minimal with respect to the disjoint linking property.

**Proof.** Consider an arbitrary embedding of $G_{12}$. By Lemma 1, the subgraph induced by the vertices $\{A, B, V_1, \ldots, v_6\}$ in $G_{12}$, which forms a $K_{3,3,3}$, contains a pair of non-splittably linked three-cycles. These linked three-cycles will use six of
the vertices of $G_{12}$. Removing these six vertices involved in the link and all incident edges will leave a subgraph of $G_{12}$ which forms a copy of $K_8$, which will have a pair of linked three-cycles in the embedding. It follows that the graph $G_{12}$ has a pair of disjoint non-splittingly linked pairs of three-cycles in every spatial embedding. We claim that this graph does not contain two disjoint copies of intrinsically linked graphs.

Every intrinsically linked graph other than $K_8$ has at least seven vertices. So in order for the twelve-vertex graph $G_{12}$ to contain two disjoint copies of intrinsically linked graphs, both intrinsically linked graphs must be $K_8$. Clearly, there do not exist two disjoint copies of $K_8$ in $G_{12}$.

Now we show that the graph $G_{12}$ is minor minimal with respect to the vertex disjoint linking property. First consider contracting any edge of graph $G_{12}$. The graph $G_{12}\setminus e$ contains only eleven vertices. Since a pair of disjoint links must use at least twelve vertices, this new graph cannot possess the disjoint linking property.

Consider the effect of deleting an edge. The graph $G_{12}$ has exactly four equivalence classes of edges, where two edges are considered equivalent when there exists an automorphism of $G_{12}$ that maps one edge to the other. The first class is the set of edges incident to either of the two vertices in the vertex set $V_1$. We denote this set of edges $E_1$. The second class is composed of the set of edges that connect vertices in the set $V_2$ to vertices in the set $V_3$. We denote this second equivalence class $E_2$. The third class, $E_3$, is composed of edges connecting the vertices in $V_2$ and $V_3$ to the vertices in $V_4$. The final class, $E_4$, is composed of the edges connecting vertices in $V_4$.

We now show that after the deletion of an edge from each of the four classes, it is possible to embed the graph in such a way that it does not have disjoint links. We use the following embedding of $G_{12}$. First embed the subgraph induced by the vertices $\{A, B, V_1, v_2, \ldots, v_6\}$, in such a way that is equivalent to the embedding in Figure 1. In particular, place vertex $A$ in the plane $z = 1$, vertex $B$ in the plane $z = 2$, and the remaining vertices in the plane $z = 0$, see Figure 3. Place all edges connecting $\{v_1, v_2, \ldots, v_6\}$ in the plane $z = 0$, except edges $(v_1, v_4)$ and $(v_3, v_6)$ are allowed to bump up a little, but they stay below $z = 1/2$. Make all edges incident to $A$ or to $B$ stay in $z \geq 0$. Place the vertices $\{a, b, c, d\}$ below the plane $z = 0$ and connect them to the vertices $v_1$ and $v_4$ in such a way that the $K_6$ induced by $\{a, b, c, d, v_1, v_4\}$ has only one pair of linked cycles, namely $(a, v_1, c)$ and $(v_3, b, d)$ (see Figure 4). Finally, place the remaining edges in $E_4$ in such a way that they are unknotted, and exist in $z < 0$, except at one endpoint.

Recall that in order for there to be a pair of disjoint links in $G_{12}$, all of the linked cycles must be three-cycles, using every vertex in the graph. Since they are not connected by an edge, the vertices $A$ and $B$ must be part of different linked three-cycles. We claim that in this embedding, any linked three-cycle that contains vertex $A$, and that is part of a pair of disjoint links, must be linked to a three-cycle that contains the vertex $B$. Consider four three-cycles that make up a pair
of disjoint links, \{S_1, S_2, S_3, S_4\}, for which \(A\) is a vertex in \(S_1\) and \(B\) is a vertex of \(S_2\). We claim that \(S_1\) must be linked with \(S_2\). By the way \(G_{12}\) is constructed, neither \(S_1\) nor \(S_2\) has vertices from \(V_4\). The cycle \(S_3\) must then have either one or two vertices from \(V_2 \cup V_3\). If \(S_3\) has two vertices, then \(S_3\) has all vertices from \(V_4\), and in this case, given the embedding of \(G_{12}\), neither \(S_1\) nor \(S_2\) is linked with \(S_4\), thus \(S_3\) must be linked with \(S_4\) and \(S_1\) is linked with \(S_2\). In the case that \(S_3\) has exactly one vertex from \(V_2 \cup V_3\), then so does \(S_4\). Again, the way our embedding was constructed (only endpoints of edges edges of \(S_3\) or \(S_4\) rise into \(z = 0\)), then neither \(S_1\) nor \(S_2\) is linked with \(S_4\), and by the same argument, \(S_1\) must be linked to \(S_2\).

Now we argue that for each equivalence class of edge, we can remove an edge from that class and we lose our pair of disjoint links in this particular embedding.

Case 1: Removing an edge from \(E_1\), or from \(E_2\).

The removal of edge \((B, v_6)\) from class \(E_1\), or of edge \((v_2, v_5)\) from class \(E_2\) will re-
move the linked three-cycles in the subgraph induced by the vertices \{A, V, v_1, v_2, ..., v_6\}. Thus, by the remark in the previous paragraph, in the given embedding there will be no linked three-cycle containing vertex A linked with a three-cycle containing vertex B, thus no disjoint pair of links.

Case 2: Removing an edge from \( E_3 \) or from \( E_4 \).

In the given embedding, there is only one pair of linked three-cycles that use the vertices \( A \) and \( B \), namely \( \{A, v_2, v_3\} \) and \( \{B, v_2, v_6\} \). Thus, to have a disjoint pair of links, the other pair of linked three-cycles must come from the subgraph induced by the vertices \{v_1, v_4, a, b, c, d\}. This induced subgraph has only one pair of linked cycles, \( \{v_4, b, d\} \) and \( \{v_1, a, c\} \) (see Figure 4). Removal of edge \( (v_4, b) \), which lies in \( E_3 \) will remove this pair of linked cycles. Similarly, removing edge \( (a, c) \), which is in \( E_4 \), will remove the pair of linked cycles.

We have shown that both edge contractions and edge deletions destroy the disjoint linking property in graph \( G_{12} \), so \( G_{12} \) is minor minimal with respect to the disjoint linking property. \( \square \)

4. Other graphs with the disjoint linking property

Recall that Robertson, Seymour and Thomas [4] proved that any intrinsically linked graph contains a graph from the Petersen family as a minor. Each Petersen graph can be obtained from any other one by a finite sequence of \( \Delta \)-Y and Y-\( \Delta \) exchanges. In the following discussion, we will show that one cannot obtain all minor-minimal graphs with the disjoint linking property from a finite sequence of \( \Delta \)-Y and Y-\( \Delta \) exchanges on \( G_{12} \).

We can use the same type of construction we used to make \( G_{12} \) to make other graphs with the disjoint linking property, that do not contain two disjoint copies of intrinsically linked graphs as subgraphs. To make \( G_{12} \), we started with the graphs \( K_{3,3,2} \) and \( K_5 \). The graph \( K_{3,3,2} \) is useful in this construction because it contains a pair of linked cycles in every embedding, and the subgraph induced by the vertices of those linked cycles is not an intrinsically linked graph. We can use \( K_{3,3,2} \) and graphs from the Petersen family, as well as \( K_{3,1,1,1,1} \) and graphs from the Petersen family, to make other graphs with the disjoint linking property. We will not construct all such graphs in detail, but we will go into detail on two more.

Here is the next one. Consider the graph \( H_{12} \) defined by:

\[
H_{12} = (V(H_{12}), E(H_{12}), \Psi_{12}),
\]

where \( V(H_{12}) = \{v_1, v_2, v_3, w_1, w_2, w_3, a, b, c, d, e\} \) is the vertex set of \( H_{12} \), \( E(H_{12}) = \{e_1, e_2, ..., e_{43}\} \) is the edge set of \( H_{12} \), and \( \Psi : E(H_{12}) \to V(H_{12}) \times V(H_{12}) \) is the mapping function associating all edges of \( E(H_{12}) \) to unique pairs of vertices. Use the vertices \( v_1, v_2, v_3 \) and \( w_1, w_2, w_3, w_4 \), with edges \( e_1, e_2, ..., e_{18} \) to construct the graph \( K_{3,1,1,1,1} \) with the vertices partitioned into the sets \( V_1 = \{v_1, v_2, v_3\} \), \( V_2 = \{w_1\}, V_3 = \{w_2\}, V_4 = \{w_3\} \) and \( V_5 = \{w_4\} \). Then use the remaining vertices \( \{a, b, c, d, e\} \) with edges \( e_{19} \) through \( e_{43} \) to construct the complete graph \( K_5 \). Finally, use the edges \( e_{29}, e_{34}, ..., e_{43} \) to connect each vertex in \( V_1 \) to each of the vertices in
\{a, b, c, d, e\}.

**Theorem 2.** The graph \(H_{12}\) has the disjoint linking property, does not contain two disjoint copies of intrinsically linked graphs, and cannot be obtained from \(G_{12}\) by a finite sequence of \(\Delta\)-\(Y\) or \(Y\)-\(\Delta\) exchanges.

**Proof.** Consider an arbitrary embedding of \(H_{12}\). Since the subgraph induced by the vertices in \(V_1 \cup \ldots \cup V_5\) is \(K_{3,1,1,1,1}\), by Lemma 3 there is a pair of linked three-cycles in this embedded \(K_{3,1,1,1,1}\). One vertex of the \(K_{3,1,1,1,1}\) is not part of the linked three-cycles. This vertex, together with the vertices in \(\{a, b, c, d, e\}\) induces \(K_5\), which by [1], [6] contains a pair of linked three-cycles in this embedding. This pair of linked three-cycles is disjoint from the first pair of linked three-cycles. Thus \(H_{12}\) has the disjoint linking property. One can check that \(H_{12}\) does not contain two disjoint copies of \(K_5\), so \(H_{12}\) does not contain two disjoint copies of intrinsically linked graphs.

It now follows that \(H_{12}\) has some minor \(H\) that is minor minimal with respect to the disjoint linking property. Since \(H_{12}\) has 43 edges, \(H\) must have no more that 43 edges. Performing \(\Delta\)-\(Y\) and \(Y\)-\(\Delta\) exchanges does not change the total number of edges in a graph, thus \(H\) cannot be obtained from \(G_{12}\) (which has 51 edges) by a finite sequence of \(\Delta\)-\(Y\) and \(Y\)-\(\Delta\) exchanges.

\[\square\]

An argument similar to the one above was given in [2] to show that the collection of minor minimal intrinsically three-connected graphs (those graphs that contain a non-splitting link of three components in every embedding) cannot be obtained by \(\Delta\)-\(Y\) and \(Y\)-\(\Delta\) exchanges on a certain known minor minimal intrinsically three-connected graph.

Now we discuss one more graph with the disjoint linking property. Consider the graph \(G_{13}\) defined by:

\[G_{13} = \langle V(G_{13}), E(G_{13}), \Phi_{13} \rangle, \]

where \(V(G_{13}) = \{A, B, v_1, v_2, \ldots, v_5, a, b, c, d, e\}\) is the vertex set of \(G_{13}\), \(E(G_{13}) = \{e_1, e_2, \ldots, e_{45}\}\) is the edge set of \(G_{13}\), and \(\Phi : E(G_{13}) \rightarrow V(G_{13}) \times V(G_{13})\) is the mapping function associating all edges of \(E(G_{13})\) to unique pairs of vertices. The graph \(G_{13}\) can be constructed as follows: Use vertices \(A, B\) and \(v_1, v_2, \ldots, v_5\) with edges \(e_1, e_2, \ldots, e_{21}\) to construct the complete tripartite graph \(K_{3,3,3}\), as done earlier. Then use the remaining vertices \(\{a, b, c, d, e\}\) with edges \(e_{22}\) through \(e_{45}\) to construct the complete graph on 5 vertices, with one edge removed, say the edge \((d, e)\). Use edges \(\{e_{31}, \ldots, e_{45}\}\) to connect each vertex in \(V_3\) to the vertices \(d\) and \(e\), and the edges \(\{e_{36}, \ldots, e_{45}\}\) to connect each vertex in \(V_3\) to the vertices \(a, b\) and \(c\).

**Theorem 3.** The graph \(G_{13}\) has the disjoint linking property, does not contain two disjoint copies of intrinsically linked graphs, and cannot be obtained from \(G_{12}\) by a finite sequence of \(\Delta\)-\(Y\) or \(Y\)-\(\Delta\) exchanges.

**Proof.** Consider an arbitrary embedding of \(G_{13}\). By Lemma 1, the subgraph induced by the vertices \(\{A, B, V_1, \ldots, v_5\}\) contains a pair of non-splittingly linked three-cycles. If the vertices that compose this link of three cycles and all edges
incident to those vertices are removed, then a subgraph of $G_{13}$ remains, and the subgraph is the seven vertex graph in the Petersen graph obtained from $K_6$ by a single triangle-$Y$ exchange. By [6], this embedded graph contains a three-cycle linked with a four-cycle. Thus this embedding contains a pair of disjoint links. Thus $G_{13}$ has the disjoint linking property.

We claim that this graph does not contain two disjoint copies of intrinsically linked graphs. Since $G_{13}$ contains thirteen vertices, it may contain either two six-vertex intrinsically linked graphs or a six-vertex intrinsically linked graph with a seven-vertex intrinsically linked graph. Because of the particular edge mappings defined for $G_{13}$, there does not exist a complete copy of $K_6$ as a subgraph. Since $K_6$ is the only six-vertex intrinsically linked graph, $G_{13}$ cannot contain two disjoint copies of intrinsically linked graphs.

Finally, since $G_{13}$ contains 45 edges it cannot be obtained from $G_{12}$ via a finite sequence of $\Delta-Y$ and $Y-\Delta$ exchanges.\footnote{It is interesting to note that although we constructed $G_{13}$ out of $K_{3,3,3}$ and the Petersen graph obtained from $K_6$ by a single $Y-\Delta$ exchange, and we constructed $G_{13}$ from $K_{3,3,3}$ and $K_6$, one still cannot obtain $G_{13}$ from $G_{12}$ by any finite sequence of $Y-\Delta$ and $\Delta-Y$ exchanges.

We have constructed more graphs with the disjoint linking property from the intrinsically linked graphs $K_{3,3,3}$ and $K_{3,1,1,1,1}$, using other Petersen graphs. In the interest of space, we will leave out the details here. We would like to point out that the fewest edges used in any such graph we found was 39. One such graph can be constructed using $K_{3,1,1,1,1}$ and $K_{4,4} - \varepsilon$. Start with $K_{3,1,1,1,1}$ and $K_{4,3}$. Connect each of the vertices in the 3-vertex partition in $K_{3,1,1,1,1}$ to any three vertices in the 4-vertex partition of $K_{4,3}$. Take an embedding of the 14-vertex, 39 edged graph. There will be linked three-cycles from the $K_{3,1,1,1,1}$. The remaining vertices induce $K_{4,4} - \varepsilon$, so by [6] there exists another pair of linked (four-)cycles.

Finally, we remark that if one were to attempt to completely discover all minor-minimal graphs with the disjoint linking property, it would be prudent to determine if there are any more graphs like $K_{3,3,3}$ and $K_{3,1,1,1,1}$, that have linked cycles in every embedding, such that the subgraph induced by the vertices of the linked cycles is not intrinsically linked. Note that given an embedding of $K_{3,3,3}$, the subgraph induced by the vertices of two linked three-cycles is $K_{2,2,3}$, and in an embedding of $K_{3,1,1,1,1}$, the subgraph induced by the vertices of two linked three-cycles is $K_{2,1,1,1,1} = K_6 - \varepsilon$.}

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