RESOLVENTS AND MARTIN BOUNDARIES OF PRODUCT SPACES

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Abstract. In this paper we examine the Laplacian on the product of two asymptotically hyperbolic spaces from the point of view of geometric scattering theory. In particular, we describe the asymptotic behavior of the resolvent applied to Schwartz functions and that of the resolvent kernel itself. We use these results to find the Martin boundary of the product space. This behaves (nearly) as expected when the factors have no L^2 eigenvalues, but it experiences a substantial collapse in the presence of such eigenvalues.

1. Introduction

Geometric scattering theory, as espoused in [25], is the study of the Laplacian and other natural elliptic operators, using geometrically-informed and fully microlocal methods, on various classes of non-compact Riemannian manifolds X with controlled asymptotic geometry. A basic goal of this subject is the definition of a class of pseudodifferential operators on X large enough to contain the resolvent of the Laplacian. The Schwartz kernels of these operators are characterized by the fact that they are particularly simple distributions on a suitable compactification of $X^2 = X \times X$, and the proper choice of this compactification (as well as a related compactification of X) is an important preliminary step in this construction. Notable successes in this program, detailed below, include treatments of scattering theory on conformally compact (asymptotically hyperbolic) and asymptotically Euclidean spaces. In this paper we examine the case where X is a product of conformally compact spaces from this point of view. This is the simplest case of a geometry modelled on a higher rank symmetric space, but many of the analytic difficulties of the more general situation already appear here. This work is intended as a guide for what to expect in further developments in this area. We emphasize, in particular, that while many of the arguments and the results here use the product structure explicitly, the methods are designed to, and in fact do, extend to more general higher rank situations, cf. [21].

Before describing the results here in more detail, we recall some of the mathematical context for our work. The geometry of symmetric spaces of rank greater than one is complicated by the fact that the curvature is zero in certain directions at infinity and negative in others. One of the challenges and attractions of studying these spaces through the lens of geometric scattering theory is to isolate the specific ways in which the flat and negatively curved directions interact with one another and affect the analysis. This relies on an initial knowledge of geometric scattering

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on spaces which are asymptotically Euclidean or asymptotically hyperbolic (conformally compact).

Recall that $(\overline{M}, g)$ is conformally compact if $\overline{M}$ is a smooth compact manifold with boundary, such that for some defining function x for $\partial\overline{M}$, g takes the form $(dx^2+h)/x^2$, where h is a nonnegative smooth symmetric 2-tensor which restricts to a nondegenerate metric on $\partial\overline{M}$. (Note, however, that only the conformal class of h on $\partial\overline{M}$ is well-defined from g.) The prototypes are hyperbolic space and its convex cocompact quotients. There is an extensive literature on scattering on this class of manifolds which contains geometric constructions of the resolvent, scattering operator and generalized eigenfunctions, as well as trace formulæ and asymptotics for the counting function for resonances. For brevity, we list only [20], [18], [19], [34] as references, and cf. [27], [13], [16] for alternate points of view. A parallel theory for complex hyperbolic manifolds and their perturbations is initiated in [6], and while not appearing explicitly, this theory extends to manifolds with the asymptotic structure of quaternionic hyperbolic spaces and the Cayley plane.

Alongside this is the study of the asymptotically Euclidean spaces initiated in [24], [26], where the metrics are called scattering metrics. This theory extends the vast classical literature on two-body Schrödinger operators and has also led to a new and detailed understanding of quantum N-body scattering, see e.g. [31], [10], [32] and [33]. The geometric setting used in these last works on the quantum N-body problem has much in common with the structure and role of flats in non-compact symmetric spaces.

The fundamental groundwork for scattering on higher rank noncompact symmetric spaces (and to some extent their quotients too) is due to Harish-Chandra, cf. [12], but there have also been recent important refinements, [3], [4]. Closely related is the theory of Martin compactifications for these spaces in [9]. However, these works all rely heavily on the special algebraic structure, so there are no obvious ways to extend these results to even relatively modest geometric perturbations of these spaces.

We now come to the contents of this paper. Let $\overline{X} = \overline{M}_1 \times \overline{M}_2$, where both factors $(\overline{M}_j, g_j)$ are conformally compact; the prototype is the product of hyperbolic spaces, but observe that $\overline{X}$ is asymptotically like this model only near the corner $\partial\overline{M}_1 \times \partial\overline{M}_2$; elsewhere, especially near $M_1 \times \partial\overline{M}_2$ and $\partial\overline{M}_1 \times M_2$, it is a severe perturbation, both metrically and topologically, of this model product space. Endowing X with the product metric $g = g_1 \otimes I + I \otimes g_2$, then our basic questions involve the resolvent $R_X(\mu) = (H-\mu)^{-1}$, where $H = -\Delta_g$ is the Laplacian for g on X (with sign chosen so that its spectrum is positive). We also write $R_X(\mu)$ as $R_g(\mu)$ or simply $R(\mu)$. This is a holomorphic family of elliptic pseudodifferential operators of order -2 for μ in the resolvent set $\mathbb{C} \setminus \mathrm{spec}(\Delta_g) \supset \mathbb{C} \setminus [0,\infty)$. The precise extent of the spectrum is straightforward to determine from the spectra on either factor. The first substantial question is to understand the behaviour of $R(\mu)$ as μ approaches the continuous spectrum of H. Existence of a limit, in an appropriate sense, is known as the limiting absorption principle, and this is implied by a stronger property (which holds more rarely) that $R(\mu)$ continues meromorphically beyond the spectrum. When it exists, this continuation usually lives on some Riemann surface covering the complex plane and ramified at the thresholds of the spectrum. The Laplacian on a conformally compact manifold admits a meromorphic continuation [20], and our first main result here is that $R_X(\mu)$ also continues meromorphically.

The compactification of $X \times X$, on which the Schwartz kernel of $R_X(\mu)$ lives as a polyhomogeneous (or more generally, a Legendrian) distribution, is obtained by resolving $\overline{X} \times \overline{X}$ with a sequence of real blow-ups. A careful description of this space is central in understanding the finer properties of the resolvent, and conversely is key in its initial geometric construction, but requires substantial notation. Thus for the purposes of this introduction, we describe instead a simpler compactification $\widetilde{X}$ of X itself, and state some representative results concerning the asymptotics of $R_X(\mu)f$, where f is a Schwartz function on X. These asymptotics, which are in any case the main technical steps in this paper, are phrased in terms of the geometry of $\widetilde{X}$, and illustrate the relationship between the structure of the compactification and the structure of the resolvent.

Let (M_j, g_j), $j = 1, 2$, be conformally compact, as above,with $\dim M_j = k_j + 1$. Writing $g_j = x_j^{-2}\overline{g}_j$, then a brief calculation [18] shows that as $p \to q \in \partial M_j$, all sectional curvatures of g_j at p converge to $-|dx_j|^2_{\overline{g}_j}(q)$. (This function only depends on g_j when $x_j = 0$.) We always assume that our spaces are asymptotically hyperbolic in the sense that this asymptotic curvature function is identically -1. The usual compactification $\overline{M}_j$ of M_j has its standard $\mathcal{C}^\infty$ structure, with smooth boundary defining function x_j. (For example, on the Poincaré ball model of hyperbolic space, $1 - |z|^2$ is a suitable choice for this defining function.) We define a new $\mathcal{C}^\infty$ structure on $\overline{M}_j$ by adjoining $\rho_j = -1/\log x_j$ as a new boundary defining function. We write this new $\mathcal{C}^\infty$ manifold with boundary as $(\overline{M}_j)_{\log}$ and call it the logarithmic blow up of $\overline{M}_j$. The compactification $\widetilde{X}$ of X is defined by

$$\widetilde{X} = [(\overline{M}_1)_{\log} \times (\overline{M}_2)_{\log}; \partial(\overline{M}_1)_{\log} \times \partial(\overline{M}_2)_{\log}]. \tag{1.1}$$

By definition, this space is the resolution of $\overline{X} = \overline{M}_1 \times \overline{M}_2$ obtained by logarithmically blowing up each factor and then performing the standard blow-up of the corner of the product. Note that $\rho_j = -1/\log x_j$ is smooth on $(\overline{M}_j)_{\log}$ and hence the function $\rho = (\rho_1^{-2} + \rho_2^{-2})^{-1/2}$ is smooth and vanishes simply at the three boundary hypersurfaces of $\widetilde{X}$.

Set $H_j = -\Delta_{g_j}$, so that $H = H_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes H_2$, and denote by ϕ_{ji} the L^2 eigenfunctions of H_j with eigenvalue $\lambda_{ji} < k_j^2/4$. The asymptotics of $R(\mu)f$, for μ in the spectrum of H is given by the

Theorem 1.1. *(See Theorem 8.1.) Suppose* $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\mathbb{R} \ni \mu > k^2/4 = (k_1^2 + k_2^2)/4$. *Then on* $\widetilde{X}$*:*

$$\begin{aligned} R(\mu - i0)f &= x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho) g \\ &+ \sum_{i=1}^{N_1} x_2^{k_2/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (\phi_{1i} \otimes g_{1i}) + \sum_{i=1}^{N_2} x_1^{k_1/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (g_{2i} \otimes \phi_{2i}), \end{aligned} \tag{1.2}$$

where g *is polyhomogeneous on* $\widetilde{X}$, *and* g_{1i} *and* g_{2i} *are polyhomogeneous on* $\bar{M}_1$ *and* $\bar{M}_2$, *respectively.*

The leading term of each part in the asymptotics can be described explicitly. For example, since ϕ_{ji} decays like $x_j^{k_j/2 + \sqrt{k_j^2/4 - \lambda_{ji}}}$, the first term dominates the other two in the interior of the front face of $\widetilde{X}$; the second term, involving the eigenfunctions of H_1, is only comparable to it at the lift of $\overline{M}_1 \times \partial\overline{M}_2$, while the third term, involving the eigenfunctions of H_2, is only comparable to it at the lift of $\partial\overline{M}_1 \times \overline{M}_2$.

A similar result holds when μ is outside the continuous spectrum, but with exponents which are no longer purely imaginary; in a sense this is more elementary, but its statement is complicated by two facts: first, we must omit certain terms in the last two sums which correspond to eigenvalues on either factor which give lower order exponential growth rates; second, when μ is real (and less than $k^2/4$), additional singularities in the asymptotics appear along the front face of $\widetilde{X}$. These new singularities can be resolved either by additional blow-ups, or else understood as a type of 'Legendre singularity'. ([11] contains a discussion of Legendrian singularities and their relationship to blowups.) We refer to Theorem 7.12 for the detailed statement of the results in that case.

These results lead to our definition of a 'resolvent compactification' of $X \times X$. The final main theme of this paper concerns two better-known compactification constructions on X which are defined through more elementary geometric or analytic considerations: the geodesic (also known as the conic) and Martin compactifications. The former of these is defined using equivalence classes of geodesic rays, while the latter one is defined using function theory, specifically the space of positive solutions of $(H - \lambda)u = 0$ for λ real and below the bottom of $\operatorname{spec}(H)$. This Martin compactification construction is very general, but has been explicitly 'geometrically' identified in many instances, including for the product of hyperbolic spaces [8] and more recently for general symmetric spaces of noncompact type [9]. We shall prove that in the present setting, our analysis of the resolvent family $R(\mu)$ specializes without difficulty and allows us to give an explicit geometric description of the Martin compactification $\overline{X}_M$ of X. As we shall see this space behaves (nearly) as in the already known case of the product of hyperbolic spaces when H_1 and H_2 have no L^2 eigenvalues, but experiences a substantial collapse when such eigenvalues are present.

Theorem 1.2. *There is a natural continuous surjection $\widetilde{X} \to \overline{X}_M$ with the following properties:*

(i) *If neither H_1 nor H_2 have L^2 eigenvalues, the restriction of this map to a neighborhood of the front face of the blow-up in* (1.1) *is injective. Generically, it is also injective on the 'side faces' of $\widetilde{X}$, but this injectivity depends on specific properties of the spherical functions, i.e. on $\frac{d}{d\tau}|_{\tau=0} R_j(k_j^2/4+\tau^2)$.*

(ii) *If both H_1 and H_2 have L^2 eigenvalues, the Martin boundary $\partial\overline{X}_M$ is of the form $\partial\overline{M}_1 \cup \partial\overline{M}_2 \cup (\partial\overline{M}_1 \times \partial\overline{M}_2 \times I)$, I an open interval. This set is naturally identified with a collapsed version of $\widetilde{X}$, and the map $\partial\widetilde{X} \to \partial\overline{X}_M$ factors through it.*

(iii) *If H_1 has L^2 eigenvalues, but H_2 does not, then $\widetilde{X} \to \overline{X}_M$ is injective near the corner given by the intersection of (the lift of) $\partial\overline{M}_1 \times \overline{M}_2$ with the front face, as in (i) above, but a neighborhood of the lift of $\overline{M}_1 \times \partial\overline{M}_2$ collapses to $\partial\overline{M}_2$ as in (ii).*

There are several features of this work to which we wish to draw particular attention. First, the main tool in deriving the resolvent asymptotics is stationary phase, which is in many respects local. The previous identification of the Martin compactification (at least for the product of hyperbolic spaces) in [8] relies heavily on global heat kernel bounds, which we feel are intrinsically more complicated. Note also that by using stationary phase we are taking advantage of the oscillatory nature of the resolvent, even when studying it for certain values of μ where it has been more traditional to rely on 'positivity methods' such as the maximum principle, the Harnack inequality, etc. Another interesting feature here is the surprisingly complicated way the existence of bound states, i.e. L^2 eigenvalues, for the Laplacian on either factor affects the asymptotics and the structure of the Martin boundary. Finally, we have given a detailed description of the smooth structure of the various compactifications we construct; this aspect is usually neglected in other discussions of compactification theory, but as we show, plays a significant role.

The rest of the paper is organized as follows. In the next section we prove a contour integral representation formula for $R_X(\mu)$ in terms of the resolvents of the two factors $R_{M_j}(\mu_j)$. A related expression, written as an integral over the spectral measure, derived in the context in Euclidean scattering, appears in work of Ben-Artzi and Devinatz [5]. The representation formula here holds in great generality, and is useful in many other situations, cf. [21]. In §3 we specialize to our specific geometric setting and review the detailed structure of the resolvent R_M when M is conformally compact; we also derive some new weighted estimates for resolvents of conformally compact manifolds which are required later. An immediate consequence of the representation formula of §2 is the existence of an appropriate meromorphic continuation for the resolvent $R_X(\lambda)$, and we describe this in §4. In §5, we give a preliminary construction of a (rough) parametrix for $R_X(\lambda)$, which is later refined in §9 to give a precise description of the resolvent, using the results in §7 and §8. §6 contains a discussion of general compactification theory, specialized to this context. The main technical work in this paper appears in §7 and §8, where we describe the asymptotics of $R_X(\mu)f$ when f is Schwartz, first when μ is in the resolvent set and then when μ is in the main sheet of continuous spectrum of H. As already noted, this is applied in §9, where we construct the 'resolvent double space', or resolvent compactification of $X \times X$. Finally, in §10, we use the resolvent asymptotics to determine the Martin compactification of X and prove Theorem 1.2.

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2. Resolvent formula

Let H_1, H_2 be self-adjoint operators on Hilbert spaces V_1 and V_2 which are bounded below; the precise structure of their spectra will be unimportant for the present. We denote $\inf \operatorname{spec}(H_j) = \lambda_0(H_j)$. Now let

$$H = H_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes H_2$$

be the self-adjoint operator on the completed tensor product space $V_1 \hat{\otimes} V_2$. Then $\inf \operatorname{spec}(H) \equiv \lambda_0(H) = \lambda_0(H_1) + \lambda_0(H_2)$. Let

$$R_j(\mu_j) = (H_j - \mu_j)^{-1}, \quad j = 1, 2, \qquad \text{and} \quad R(\mu) = (H - \mu)^{-1}$$

be the resolvents of the H_j and H, respectively. This setting has been investigated by Ben-Artzi and Devinatz in [5], with the main goal of determining (in an abstract setting) when the limiting absorption principle, i.e. the existence of the boundary values $R(\mu \pm i0)$, $\mu \in [\lambda_0(H), \infty)$ on suitable weighted spaces, holds assuming that the limiting absorption principle holds for each of the R_j individually.

One of our first goals is to show that if both families $R_j(\mu_j)$ admit meromorphic continuations, then $R(\mu)$ does as well. Later we wish to obtain precise asymptotics for the Schwartz kernel of $R(\mu)$ for μ in the resolvent set and in the continuous spectrum. In this section we derive a representation of $R(\mu)$ as a contour integral which will be useful for both of these purposes.

So, fix μ in the resolvent set $\mathbb{C}\backslash[\lambda_0(H_1)+\lambda_0(H_2), \infty)$, and let γ be a parametrized curve in the complex plane which is disjoint from $[\lambda_0(H_1), \infty)$ and such that $\mu - \gamma$ is disjoint from $[\lambda_0(H_2), \infty)$, or in other words, such that γ does not intersect $[\lambda_0(H_1), \infty) \cup (\mu - [\lambda_0(H_2), \infty))$. Suppose also that $\gamma(t) = c_\pm t$ for $\pm t \geq T > 0$ with $\operatorname{Im} c_\pm > 0$. This is illustrated in Figure 1. The precise values of $c_\pm$ are unimportant for our purposes, and indeed there is considerably more leeway than this in choosing γ, but the linear separation of γ from the spectra, following from $\operatorname{Im} c_\pm > 0$, is important. (Actually, one can also allow slightly sublinear separation, but this is of no interest here.) Then we claim that

$$R(\mu) = \frac{1}{2\pi i}\int_\gamma R_1(\mu_1) \otimes R_2(\mu - \mu_1)\, d\mu_1. \tag{2.1}$$

To prove this formula, first observe that since

$$\|R_j(\mu_j)\| \leq |\operatorname{Im} \mu_j|^{-1},$$

the norm of the integrand in (2.1) (as a bounded operator on L^2) is estimated by $C(1 + |t|)^{-2}$, and hence the integral converges. Next, note that it suffices to show that both sides of (2.1) produce the same result when restricted to the range of $\chi_I(H_1) \otimes \mathrm{Id}$ where I is any compact interval and χ_I its characteristic function, because the union of these ranges is dense.

Fixing the interval I, the integrand $R_1(\mu_1) \otimes R_2(\mu - \mu_1)$ is holomorphic for $\mu_1 \notin I \cup (\mu - \operatorname{spec}(H_2))$. Therefore we may deform the contour to one which is the union of two curves, $\hat\gamma$ and $\tilde\gamma$, where $\hat\gamma$ agrees with γ for $t \geq T' > 0$ and intersects the real axis precisely once, somewhere in $(\sup I, \infty)$, while $\tilde\gamma$ surrounds I once. Thus, on the range of $\chi_I(H_1) \times \mathrm{Id}$,

$$\begin{aligned}&\frac{1}{2\pi i}\int_\gamma R_1(\mu_1) \otimes R_2(\mu - \mu_1)\, d\mu_1 \\ &\quad= \frac{1}{2\pi i}\int_{\hat\gamma} R_1(\mu_1) \otimes R_2(\mu - \mu_1)\, d\mu_1 + \frac{1}{2\pi i}\int_{\tilde\gamma} R_1(\mu_1) \otimes R_2(\mu - \mu_1)\, d\mu_1.\end{aligned} \tag{2.2}$$

But now, if $\hat\gamma$ is moved to infinity, the first integral on the right tends to 0, as follows by directly estimating the integral. On the other hand, letting $\tilde\gamma$ tend to I and applying Stone's theorem, the second integral on the right is the same as

$$\int_I (dE_1(\mu_1) \otimes R_2(\mu - \mu_1)),$$

where dE_j denotes the spectral measure of H_j. This is identical to $R(\mu)$ on the range of $\chi_I(H_1)\otimes\mathrm{Id}$ since applying $H - \mu$ to it gives (with a slight abuse of notation)

$$\begin{aligned}&\int_I ((H_1 - \mu_1) \otimes \mathrm{Id} + \mathrm{Id} \otimes (H_2 - (\mu - \mu_1)))\ (dE_1(\mu_1) \otimes R_2(\mu - \mu_1)) \\ &\quad= \int_I (((H_1 - \mu_1)\, dE_1(\mu_1)) \otimes R_2(\mu - \mu_1)) + \int_I (dE_1(\mu_1) \otimes \mathrm{Id})\end{aligned}$$

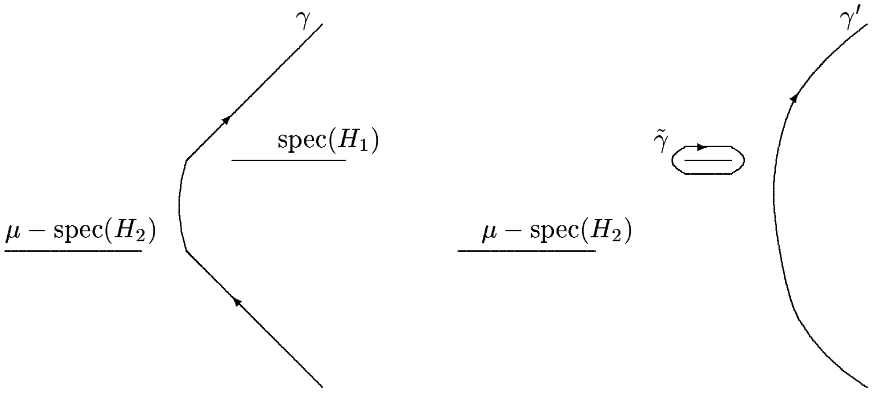

FIGURE 1. Contours of integration in the μ_1 plane.

$$= \int_I (dE_1(\mu_1) \otimes \mathrm{Id}) = \chi_I(H_1) \otimes \mathrm{Id},$$

i.e. the identity on this subspace. Thus (2.1) is established.

3. RESOLVENTS OF CONFORMALLY COMPACT MANIFOLDS

In this section we briefly collect some facts about the resolvent family $R_M(\lambda)$, which we abbreviate simply as $R(\lambda)$ for the duration of this section, when (M, g) is a conformally compact manifold. These are all discussed and proved in [20], to which we refer for all details.

Recall that M is identified with the interior of a compact smooth manifold with boundary $\overline{M}$. The compactification $\overline{M}$ is geometrically natural: $\overline{M}$ may also be identified with the both the geodesic and Martin compactifications, as we discuss further below. Locally, $R(\lambda)$ is a pseudodifferential operator of order -2, but our focus is on understanding the behavior of its Schwartz kernel $K(z, z') = K(\lambda; z, z')$, $z, z' \in M$, when one or both of the variables tend to infinity in M, i.e. to a point of ∂M. This can occur in various ways, of course, and the most efficient way to encode this information is to consider K as a distribution on the 0-stretched product $\overline{M}_0^2$ introduced in [20] and [18]. This space is obtained from $\overline{M}^2$ by blowing up the boundary of the diagonal $\partial\Delta\iota = \partial\{z = z'\}$; equivalently, $\overline{M}_0^2 = [\overline{M}^2; \partial\Delta\iota]$ is the disjoint union of $\overline{M}^2 \setminus \partial\Delta\iota$ with the interior spherical normal bundle of $\partial\Delta\iota$ at the corner $\partial\overline{M} \times \partial\overline{M}$, this set then being endowed with the minimal $\mathcal{C}^\infty$ structure containing the lifts of smooth functions on $\overline{M}^2$ and polar coordinates around $\partial\Delta\iota$.

$\overline{M}_0^2$ has three different boundary hypersurface faces: the left face which covers (indeed, is identified with) $\partial\overline{M} \times \overline{M}$, the right face which covers $\overline{M} \times \partial\overline{M}$, and the new front face covering $\partial\Delta\iota$. We denote these B_{10}, B_{01} and B_{11}, and their boundary defining functions ρ_{10}, ρ_{01} and ρ_{11}, respectively. Writing $z = (x, y)$ and $z' = (x', y')$ near $\partial\overline{M}$, then

$$\rho_{11} = \sqrt{x^2 + (x')^2 + |y - y'|^2}, \qquad \text{and} \qquad \rho_{10} = x/\rho_{11}, \quad \rho_{01} = x'/\rho_{11}.$$

If $\dim M = n = k + 1$, then write $\lambda = \zeta(k - \zeta)$, where by convention the region $\mathrm{Re}\,\zeta > k/2$ corresponds to the resolvent set $\mathbb{C} \setminus [k^2/4, \infty)$ of Δ_g. Thus for λ in the resolvent set,

$$\zeta = k/2 + i\sqrt{\lambda - k^2/4},$$

where the branch of the square root is chosen so that its imaginary part is negative.

Theorem 3.1. *(Mazzeo and Melrose,* [20, Theorem 7.1]*) For ζ in the half-plane $\operatorname{Re}\zeta > k/2$, the Schwartz kernel K_ζ of $R(\zeta(k-\zeta))$ is a polyhomogeneous distribution on $\overline{M}_0^2$ with the following properties. First, $R(\zeta(k-\zeta))$ has a decomposition*

$$R(\zeta(k-\zeta)) = R'(\zeta(k-\zeta)) + R''(\zeta(k-\zeta)),$$

where R' is an element in the small calculus $\Psi_0^{-2}(\overline{M})$ of 0-pseudodifferential operators on M and R'' is a residual element in the large calculus $\Psi_0^{-\infty,\zeta,\zeta}(\overline{M})$ of 0-pseudodifferential operators. This means that the Schwartz kernel of R' has a standard polyhomogeneous singularity corresponding to pseudodifferential order -2 at the lifted diagonal $\Delta\iota_0$ in $\overline{M}_0^2$ and vanishes to infinite order along B_{10} and B_{01}, while the Schwartz kernel of R'' takes the form

$$\rho_{10}^\zeta \rho_{01}^\zeta F'', \quad F'' \in \mathcal{C}^\infty(\overline{M}_0^2; \pi_R^*\Omega_0);$$

here $\pi_R^*\Omega_0$ is the lift of the 0-density bundle from the right factor (a non-vanishing section of which is given by the Riemannian density dV_g). $R(\zeta(k-\zeta))$ is holomorphic, both as a map into the space of bounded operators on L^2 and also into the space of distributions on $\overline{M}_0^2$, for $\operatorname{Re}\zeta > k/2$, and extends meromorphically, as a function with values in the space of distributions on $\overline{M}_0^2$, to the complex plane when k is even, and to $\mathbb{C} \setminus -\mathbb{N}$ when k is odd, with all poles of finite rank. Because it is constructed using only the symbol calculus, R' is holomorphic in λ. Finally, the restrictions of $\rho_{10}^{-\zeta} K_\zeta$ to the left face B_{10} and of $\rho_{01}^{-\zeta} K_\zeta$ to the right face B_{01} are nonvanishing for all ζ with $\operatorname{Re}\zeta > k/2$.

Remark 3.2. It will be convenient below to write

$$K_\zeta = \rho_{10}^\zeta \rho_{01}^\zeta F$$

where F is smooth on $\overline{M}_0^2$, apart from its conormal singularity at the lifted diagonal $\Delta\iota_0$.

For us, the main import of this theorem is its conclusion that K_ζ has simple polyhomogeneous behavior on the space $\overline{M}_0^2$. One of our ultimate goals here is to find a resolution of the space X^2, where X is the product of two conformally compact manifolds, on which the Schwartz kernel for the resolvent of its Laplacian is also simple.

In the next sections we shall require uniform weighted L^2 estimates for this resolvent $R(\lambda)$ as $\operatorname{Im}\lambda \to \infty$, and we now show how these follow from the parametrix construction in [20]. We set $H = -\Delta_g$ here.

Theorem 3.3. *Fix $s, \epsilon > 0$, and suppose that $\operatorname{Re}\zeta \geq k/2 + s + \epsilon$. Then*

$$R(\zeta(k-\zeta)) : x^s L^2(M, dV_g) \longrightarrow x^s L^2(M, dV_g) \tag{3.1}$$

is bounded and satisfies the estimate

$$\|R(\zeta(k-\zeta))\|_{\mathcal{B}(x^s L^2, x^s L^2)} \leq \frac{C_{s,\epsilon}}{|\operatorname{Im}\zeta(k-\zeta)|} \tag{3.2}$$

uniformly for ζ in this half-plane.

Proof. The proof has a few steps. With $\lambda = \zeta(k-\zeta)$, the construction in [20] gives a parametrix $P(\lambda) \in \Psi_0^{-2,\zeta,\zeta}(\overline{M})$, and residual operators $E(\lambda)$ and $F(\lambda)$, which satisfy

$$P(\lambda)(H-\lambda) = \mathrm{Id} + E(\lambda), \quad (H-\lambda)P(\lambda) = \mathrm{Id} + F(\lambda).$$

These are related to the actual resolvent through the formula

$$R(\lambda) = P(\lambda) - E(\lambda)P(\lambda) + E(\lambda)R(\lambda)F(\lambda). \tag{3.3}$$

This construction will be briefly reviewed in the course of this proof. We shall show that

$$\|P(\lambda)\|_{\mathcal{B}(x^sL^2,x^sL^2)} \le \frac{C}{|\operatorname{Im}\lambda|}, \qquad \|E(\lambda)\|_{\mathcal{B}(L^2,x^sL^2)},\ \|F(\lambda)\|_{\mathcal{B}(x^sL^2,L^2)} \le C,$$

uniformly for $\operatorname{Re}\zeta \ge k/2 + s + \epsilon$. Combining this with the standard estimate

$$\|R(\lambda)\|_{\mathcal{B}(L^2,L^2)} \le \frac{1}{|\operatorname{Im}\lambda|}, \qquad \lambda \in \mathbb{C} \setminus \operatorname{spec}(H), \tag{3.4}$$

we obtain the desired estimate for $R(\lambda)$ in $\mathcal{B}(x^sL^2, x^sL^2)$.

The main work involves demonstrating a uniform estimate in weighted spaces for the resolvent of the Laplacian in hyperbolic space, and so we turn to this first. Uniform boundedness of

$$R(\zeta(k-\zeta)) : x^sL^2(\mathbb{H}^{k+1}; dV) \longrightarrow x^sL^2(\mathbb{H}^{k+1}; dV)$$

is equivalent to the uniform boundedness of the conjugated operator $x^{-s}R(\zeta(k-\zeta))x^s$ on $L^2(\mathbb{H}^{k+1}; dV)$. The Schwartz kernel of this conjugated operator is

$$K^s_\zeta(z,z') = K^0_\zeta(z,z')(x'/x)^s = K^0_\zeta(z,z')(\rho_{01}/\rho_{10})^s,$$

where $K^0_\zeta(z,z')$ is the Schwartz kernel of the resolvent of the Laplacian on $\mathbb{H}^{k+1}$. In fact, since $\mathbb{H}^{k+1}$ is rank one symmetric, the resolvent kernel is a point-pair invariant, i.e. depends only on the Riemannian (hyperbolic) distance $\delta(z,z')$ between z and z'. We recall from [20] the explicit formula:

$$K^0_\zeta(\delta) = \begin{cases} c_k \left(\dfrac{1}{\sinh\delta}\dfrac{\partial}{\partial\delta}\right)^{\frac{k-2}{2}} \left(\dfrac{1}{\sinh\delta} e^{-(\zeta-k/2)\delta}\right), & k \text{ even}, \\ c_k \displaystyle\int_0^\infty e^{-(\zeta-k/2)\omega}(\cosh\omega - \cosh\delta)_+^{-k/2}\, d\omega, & k \text{ odd}. \end{cases} \tag{3.5}$$

To analyze the behavior of $K^0_\zeta(\delta(z,z'))$ on $(\mathbb{H}^{k+1})^2_0$, one may use the particularly simple formula in upper half-space coordinates

$$\cosh\delta(z,z') = 1 + \frac{|z-z'|^2}{2xx'}. \tag{3.6}$$

Now separate $(\mathbb{H}^{k+1})^2_0$ into two regions, in each of which K^s_ζ behaves somewhat differently as $\zeta \to \infty$. The first is the neighbourhood of the diagonal where $\delta(z,z') \le 2$, and the other is the complementary region $\delta(z,z') \ge 1$. Accordingly, we write the Schwartz kernel as a sum of two terms using a partition of unity subordinate to this cover, and we shall prove the desired estimates for each piece separately.

The factor $(\rho_{01}/\rho_{10})^s$ is estimated by $e^{s(\delta(p,z)-\delta(p,z'))}$, where p is some fixed point in the interior of $\mathbb{H}^{k+1}$. Hence in the first (near-diagonal) region, it is uniformly bounded, and so the estimate for this piece follows directly from (3.4), as applied to the resolvent on hyperbolic space. On the other hand, in the off-diagonal region, K^0_ζ

decays like $e^{-(\operatorname{Re}\zeta-k/2-\epsilon')\delta(z,z')}$ for any $\epsilon'>0$, and so its product with this factor is bounded by $Ce^{-(\operatorname{Re}\zeta-s-k/2-\epsilon')\delta(z,z')}$, which satisfies a much better estimate than needed provided $\operatorname{Re}\zeta\geq k/2+s+\epsilon$ and $\epsilon'<\epsilon/2$. The desired bound for the operator norm on L^2 now follows using Cauchy-Schwartz. This concludes the proof of (3.2) for the special case of the resolvent $R(\zeta(k-\zeta))$ on $\mathbb{H}^{k+1}$.

To proceed, we recall that $P(\lambda)$ is constructed in stages, and as a sum of two terms, $P(\lambda)=P_1(\lambda)+P_2(\lambda)$. The first, $P_1(\lambda)$, is in the small 0-calculus. It is a slight modification, cf. (3.9) below, of an operator $P_1'(\lambda)$ which is obtained using the symbol calculus with spectral parameter [29] on the conormal bundle of the lifted diagonal $\Delta\iota_0$ in $\overline{M}_0^2$, to solve away the conormal singularity of the identity operator on this submanifold. It can be chosen to to depend holomorphically on $\zeta\in\mathbb{C}$, and so that its Schwartz kernel is supported in a neighbourhood of $\Delta\iota_0$ which does not intersect the side faces of $\overline{M}_0^2$. We have used the symbol calculus with spectral parameter, rather than the ordinary symbol calculus, because one of its properties is that parametrices constructed using it satisfy the estimate (3.2) on L^2. (Indeed, the raison d'être for this refined symbol calculus is to microlocalize the estimate (3.4).) Finally, just as for the resolvent on $\mathbb{H}^{k+1}$ above, the support properties of its Schwartz kernel shows that it satisfies the same estimate on x^sL^2 for any s.

This preliminary parametrix leaves an error term $F_1(\lambda)=I-(H-\lambda)P_1'(\lambda)\in\Psi_0^{-\infty}(\overline{M})$, and the next step in the construction is to find an operator $P_2'(\lambda)$ which solves away the first term in the Taylor expansion of F_1 at the front face B_{11} and some larger number of terms at the side face B_{10}, or in other words, so that for some sufficiently large ℓ,

$$(H-\lambda)P_2'(\lambda)-F_1(\lambda)=F_2(\lambda)\in\Psi_0^{\zeta+\ell,\zeta,1}(\overline{M}). \tag{3.7}$$

The final 1 in the superscript indicates that the Schwartz kernel of F_2 vanishes to first order at B_{11}. The restriction of this equation to B_{11} is the normal equation

$$(N(H)-\lambda)N(P_2'(\lambda))=N(F_1(\lambda)),$$

and the gain here is that the normal operator $N(H)-\lambda$, which is obtained by restricting the action of $H-\lambda$ to this face, is naturally identified with $-\Delta_{\mathbb{H}^{k+1}}-\lambda$. Thus we solve this equation using the hyperbolic space resolvent, and it is at this stage that the support of the Schwartz kernel spreads to the side faces. This solutions is defined only on the front face, so extend it to an element $P_2''(\lambda)\in\Psi_0^{-\infty,\zeta,\zeta}(\overline{M})$. This leaves an error term lying in $\Psi_0^{-\infty,\zeta+1,\zeta,1}(\overline{M})$, and so we must add on an additional term to cancel the first $\ell-1$ terms of this remainder at B_{10}; this is a local, essentially ODE, computation. Let $P_2'(\lambda)$ denote the sum of these two terms. The preceding discussion shows that it satisfies (3.2), and we have

$$(H-\lambda)(P_1'(\lambda)+P_2'(\lambda))=I-F_2(\lambda),\qquad F_2(\lambda)\in\Psi_0^{-\infty,\zeta+1,\zeta,1}(\overline{M}). \tag{3.8}$$

Now, fix $\ell>s$ and take the first ℓ terms of the Neumann series for $(I-F_2(\lambda))^{-1}$,

$$I+S^{(\ell)}\equiv I+F_2(\lambda)+\ldots+(F_2(\lambda))^{\ell-1}.$$

The composition formula in the 0-calculus [22] shows that $S^{(\ell)}\in\Psi_0^{-\infty,\zeta+\ell,\zeta,\ell}(\overline{M})$. Applying this on the right to both sides of (3.8) gives

$$(H-\lambda)(P_1(\lambda)+P_2(\lambda))=I-F(\lambda),$$

where

$$(3.9) \quad P_j(\lambda) = P_j'(\lambda)(I + S^{(\ell)}(\lambda)), \quad j = 1, 2, \quad \text{and} \qquad F(\lambda) \in \Psi_0^{-\infty,\zeta+\ell,\zeta,\ell}(\overline{M}).$$

Clearly

$$\|F(\lambda))\|_{\mathcal{B}(x^s L^2, L^2)} \leq C$$

uniformly for $\sigma = \zeta(k - \zeta)$, $\mathrm{Re}\,\zeta \geq k/2 + s + \epsilon$. Applying $H - \lambda$ to the right of $P(\lambda) = P_1(\lambda) + P_2(\lambda)$ gives a remainder term $E(\lambda)$ such that $\|E(\lambda)\|_{\mathcal{B}(L^2, x^s L^2)}$ uniformly in this region.

Using these estimates in (3.3) finishes the proof. □

4. Analytic continuation of the resolvent of H

In this section we shall show how to use the integral formula (2.1) to obtain an analytic continuation for the resolvent $R_X(\mu) = (H - \mu)^{-1}$ on $X = M_1 \times M_2$ past the continuous spectrum of $H = -\Delta_g = H_1 + H_2$. For this continuation we can either regard $R_X(\mu)$ as an analytic family of bounded operators between weighted L^2 spaces, or else view its Schwartz kernel as an analytic function with values in an appropriate space of distributions. The key ingredient here is the existence of similar analytic continuations for the resolvents $R_j(\mu) = (H_j - \mu)^{-1}$. Although this continuation result holds in considerably greater generality than just for products of conformally compact spaces, we shall focus exclusively on this case for the sake of being specific.

We first set up some notation. Let $\dim M_j = k_j + 1$. Then it follows from Section 3, cf. also [20] that the spectrum of H_j decomposes into the union of a band of continuous spectrum $[k_j^2/4, +\infty)$, as well as possibly a finite number of L^2 eigenvalues λ_{ji}, $i = 1, \ldots, N_j$, $j = 1, 2$, in $(0, k_j^2/4)$, with the corresponding finite rank eigenprojections Π_{ji}. Next, each $R_j(\mu_j)$ continues meromorphically from the resolvent set to the Riemann surface Σ_j for $\sqrt{\mu_j - k_j^2/4}$, which we think of as two copies of $\mathbb{C}$ attached in the usual way along a cut $\mathcal{C}_j$ extending from $k_j^2/4$ to ∞; the resolvent set is identified with the subset of Σ_j where $\mathrm{Im}\,\sqrt{\mu_j - k_j^2/4} < 0$. Usually $\mathcal{C}_j$ is taken to be the ray along the positive real axis, but it will be convenient to choose it differently later. As already noted, this continuation of R_j is either as a map into an appropriate spaces of distributions, or else for any given $s > 0$, in the region $\mathrm{Im}\,\sqrt{\mu_j - k_j^2/4} < s$, as bounded operators from $x^s L^2$ to $x^{-s} L^2$. (The point here is that as μ_j continues into the nonphysical half-plane, the expansion of the Schwartz kernel of R_j at the left and right faces of $(\overline{M}_j)^2_0$ has terms with increasingly large negative exponents, and so it maps functions which decay at some high rate relative to L^2 to functions which blow up at the negative of this rate.) In any case, its poles are of finite rank, and we denote them by $\tilde{\lambda}_{ji}$, with corresponding finite rank residues $\tilde{\Pi}_{ji}$ (the tildes are meant to distinguish these from the eigendata for H_j). Note, however, that these poles of R_j in the nonphysical part of Σ_j need not be simple.

We assume until near the end of the argument that neither H_j has any L^2 eigenvalues. Define k by

$$\frac{k^2}{4} = \frac{k_1^2}{4} + \frac{k_2^2}{4}.$$

Then $R(\mu)$ is analytic in $\mathbb{C}\setminus[k^2/4,\infty)$, and we wish to show that it continues analytically past the continuous spectrum. We do this by deforming the contour of integration γ in (2.1) in the following manner. First fix μ in the resolvent set, so that $R_1(\mu_1)\otimes R_2(\mu-\mu_1)$ is defined and analytic for $\mu_1\notin\mathcal{C}_1\cup(\mu-\mathcal{C}_2)$, i.e. outside of two horizontal rays, one extending from $k_1^2/4$ to the right and the other from $\mu-k_2^2/4$ to the left. Next, rotate these cuts to rays $\tilde{\mathcal{C}_j}$ by pivoting them by some angle α counterclockwise around their endpoints. Thus we have 'exposed' two sectors of angle α from the nonphysical portion of Σ_1, resp. Σ_2, and at the same time concealed an equal portion of the physical part of the Σ_j. Now it is possible to deform γ to a new contour γ' which lies partly in the newly uncovered sectors in $\mathbb{C}\setminus(\tilde{\mathcal{C}_1}\cup\tilde{\mathcal{C}_2})$, as in Figure 2. Notice that we can always arrange to deform this contour only within a bounded set, so that we do not need to worry about any convergence issues. Finally, μ can then be moved into the nonphysical region.

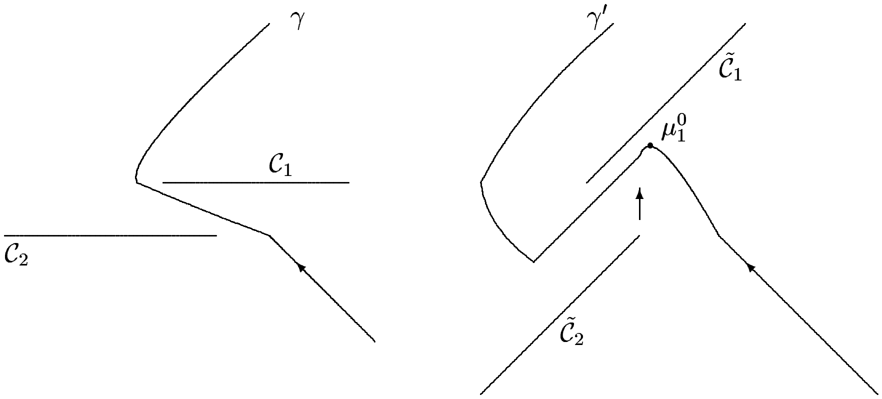

FIGURE 2. Contour of integration to describe the analytic continuation of $R(\mu)$. In the second picture, the cuts $\mathcal{C}_1$ and $\mathcal{C}_2$, corresponding to the spectra of H_1 and H_2, respectively, have been rotated to the cuts $\tilde{\mathcal{C}_1}$ and $\tilde{\mathcal{C}_2}$. Thus near the point μ_1^0 labeled here, $R_1(\mu_1^0)$ is *not* a bounded operator on $L^2(M_1)$. Now μ, and hence $\tilde{\mathcal{C}_2}$ as well, can be moved upwards in the direction of the arrow.

We make this a bit more explicit. Suppose that μ_0 is such that $\mu^0-k_1^2/4\neq\tilde{\lambda}_{1i}$ and $\mu^0-k_2^2/4\neq\tilde{\lambda}_{2i'}$ for any i,i', and also $\mu^0\neq\tilde{\lambda}_{1i}+\tilde{\lambda}_{2i'}$. Rotate the cuts $\mathcal{C}_j$ by an angle $\alpha>\arg(\mu^0-k_j^2/4)$, $j=1,2$. Now deform the contour γ past $\mu^0-k_2^2/4$ to a new contour γ', such that this point is now below γ'. During the deformation, γ may pass through a finite number of poles of R_1, yielding residues $\tilde{\Pi}_{1i}\otimes R_2(\mu-\tilde{\lambda}_{1i})$. So long as μ is (sufficiently) far away from $\mathcal{C}_2$, $\mu-\mu_1$ is in the resolvent set of H_2, hence no poles of R_2 are encountered during this deformation. Thus we have a new representation

$$R(\mu)=\frac{1}{2\pi i}\int_{\gamma'}R_1(\mu_1)\otimes R_2(\mu-\mu_1)\,d\mu_1-\sum_{\text{finite}}\tilde{\Pi}_{1i}\otimes R_2(\mu-\tilde{\lambda}_{1i}).$$

Since μ is in the resolvent set, the left hand side is still a bounded operator on L^2, although on the right hand side, one factor of the integrand $R_1(\mu_1)$ is not bounded on L^2 along the entire contour.

At this point we merely have a new representation of an operator we already know exists, namely $R(\mu)$ for μ in the resolvent set, in terms of operators which are only bounded between weighted spaces. However, fixing γ', we can now let μ vary arbitrarily below this curve, and hence across the spectrum $[k^2/4, \infty)$, as long as the ramification point $k_2^2/4$ of R_2 does not lie on γ'. When $\mu - \tilde{\lambda}_{2i}$ crosses γ', the residue term $-R_1(\mu - \tilde{\lambda}_{2i}) \otimes \tilde{\Pi}_{1i}$ is produced. This yields

$$\begin{aligned} R(\mu) = {} & \frac{1}{2\pi i} \int_{\gamma'} R_1(\mu_1) \otimes R_2(\mu - \mu_1)\, d\mu_1 \\ & - \sum_{\text{finite}} \tilde{\Pi}_{1i} \otimes R_2(\mu - \tilde{\lambda}_{1i}) - \sum_{\text{finite}} R_1(\mu - \tilde{\lambda}_{2i}) \otimes \tilde{\Pi}_{2i}; \end{aligned}$$

Each of the residue terms here continue meromorphically in μ past the spectrum, with ramification points at $\tilde{\lambda}_{1i} + k_2^2/4$ (since R_2 is ramified at $k_2^2/4$) and similarly, with the indices 1 and 2 interchanged. These terms also have poles at $\tilde{\lambda}_{1i} + \tilde{\lambda}_{2j}$, with finite rank residues.

We have proved

Theorem 4.1. *The resolvent $R(\mu)$ for H on $X = M_1 \times M_2$ extends across the cut $[k^2/4, \infty)$ to a meromorphic function (with values in an appropriate space of distributions) on a Riemann surface $\mathcal{T}$ ramified at $\tilde{\lambda}_{1i} + k_2^2/4$ and $\tilde{\lambda}_{2i} + k_1^2/4$ (these points are known as Regge poles), and with finite rank poles at $\tilde{\lambda}_{1i} + \tilde{\lambda}_{2i}$.*

When either H_1 or H_2 has L^2 eigenvalues, then the only difference is that γ' must cross these as well. Thus, the analytic continuation can be written as above, but now we must include sums over the λ_{ji} too, and there are new ramification points at $\lambda_{1i} + k_2^2/4$ and $\lambda_{2i} + k_1^2/4$.

5. The small calculus of product-0 pseudodifferential operators

As explained in the introduction, one of our basic premises is that the proper way to define a calculus of pseudodifferential operators suitable for studying the resolvent is by characterizing the Schwartz kernels of its elements as simple (i.e. polyhomogeneous) distributions on a carefully chosen resolution of $\overline{X} \times \overline{X}$. This resolution is obtained from $\overline{X}^2$ by a sequence of blowups reflecting the various homogeneities of these kernels. We shall define two spaces of operators in this way: the small and large calculi of product-0 pseudodifferential operators. The large calculus is the one which actually contains the resolvent; the resolution of $\overline{X}^2$ needed to define it is the somewhat complicated resolvent double space $\overline{X}^2_{\text{res}}$. We defer discussion of this until §9. In this section, as a first step towards this goal, we shall define a simpler resolution of $\overline{X}^2$, the product-0 double space, on which the small calculus is defined. We construct the basic parametrix for $R(\mu)$ in this calculus using a symbol calculus argument essentially identical to the one for elliptic operators on compact manifolds. Indeed, the point of the small calculus is that it allows us to perform this usual parametrix construction in a neighbourhood of the diagonal of X^2, uniformly to the boundary. However, in our setting, the error term from this parametrix is not a compact operator, and this is what necessitates a further and more refined parametrix construction in the large calculus.

If we were approaching this resolvent construction more systematically, or for example constructing the resolvent on some compact topological or metric perturbation X, then following [20] we would construct successively better parametrices for the resolvent, correcting this initial small calculus parametrix. However, since we are assuming for simplicity that X has a product structure, we actually 'know' the resolvent $R(\mu)$ via its contour integral representation, and so the onus is shifted from constructing $R(\mu)$ to showing that it is lies in the large calculus. Although this obviates the need for the small calculus parametrix, we sketch its construction anyway, in part because it illuminates the precise structure of the singularity of the resolvent kernel along the diagonal of X^2, particularly at the boundary. For, by standard elliptic theory, the Schwartz kernel of $R(\mu)$ is smooth away from the diagonal in the interior of X^2, but this argument breaks down at the boundary, along with the ellipticity of H. In fact, at first glance, our contour integral formula might even seem to allow off-diagonal singularities in $R(\mu)$, e.g. along the partial diagonals of M_j^2 in X^2. These extra singularities are ruled out using standard results about the behaviour of wavefront sets under pushforwards, but the justification of this at the boundary requires extra care.

Recall first that a submanifold Y in a manifold with corners Z is called a p-submanifold if near any $q \in Y$ there is a local coordinate chart $\mathcal{U} \approx [0,1)^\ell \times (-1,1)^{N-\ell}$ adapted to the corner structure of Z, and such that $\mathcal{U} \cap Y = \{0\} \times [0,1)^{\ell'} \times (-1,1)^m$, for some $\ell' \leq \ell$, $m \leq N-\ell$, i.e. some number of the coordinates have been set to zero. It is straightforward [24, Appendix, Section 20] to define the space $I^*(Z;Y)$ of distributions on Z which are one-step polyhomogeneous along Y, and which are extendible across ∂Z.

We now define the product-0 double space of X simply as the product of the 0 double spaces of the factors M_j:

$$\overline{X}^2_{\mathrm{p0}} = (\overline{M}_1^2)_0 \times (\overline{M}_2^2)_0.$$

This has a distinguished submanifold diag, the closure of the lift of the diagonal in X^2, which is the product of the two lifted diagonals diag_j in the 0 double spaces $(\overline{M}_j^2)_0$. Each of these are p-submanifolds, since diag_j intersects the front face boundary hypersurface B^j_{11} of M_j transversely. Thus diag intersects $\partial\overline{X}^2_{\mathrm{p0}}$ only at the two boundary hypersurfaces $B^1_{11} \times (\overline{M}_2^2)_0$ and $(\overline{M}_1^2)_0 \times B^2_{11}$, which we call the front faces of this double space. Let β the denote the blowdown map from $\overline{X}^2_{\mathrm{p0}}$ to $\overline{X}^2$.

Definition 5.1. The small calculus of product-0 pseudodifferential operators on X, denoted $\Psi^*_{\mathrm{p0}}(\overline{X})$, is the space of all pseudodifferential operators A (operating initially on $\mathcal{S}(X)$) which have Schwartz kernels $K_A \in \mathcal{D}'(X \times X)$ which are of the form $\beta_*(\kappa_A)$, where $\kappa_A \in I^*(\overline{X}^2_{\mathrm{p0}}; \mathrm{diag})$, and which vanish to infinite order along boundary hypersurfaces except the front faces. (We remark that the large calculus will be defined similarly; those kernels are defined on the slightly larger resolvent double space obtained by blowing up some of the corners of $\overline{X}^2_{\mathrm{p0}}$, and are allowed to have nontrivial polyhomogeneous expansions along other (non-front) boundary faces.)

The submanifold diag of $\overline{X}^2_{\mathrm{p0}}$ is canonically identified with X, and we define the product-0 tangent and cotangent bundles of X, ${}^{\mathrm{p0}}TX$, ${}^{\mathrm{p0}}T^*X$, as the normal and conormal bundles of diag. These are canonically identified with the ordinary tangent and cotangent bundles over the interior of X, but the identification is not natural at the boundary.

Following the usual practice, there is a symbol mapping ${}^{\mathrm{p0}}\sigma$ on $\Psi^*_{\mathrm{p0}}(X)$ with values in the space of functions on ${}^{\mathrm{p0}}T^*X$ which are homogeneous along the fibres, and hence satisfy the usual symbol estimates. This gives rise to a short exact sequence

$$0 \to \Psi^{*-1}_{\mathrm{p0}}(X) \to \Psi^*_{\mathrm{p0}}(X) \to S^*_{\mathrm{hom}}({}^{\mathrm{p0}}T^*X \setminus o)) \to 0.$$

Ellipticity is defined in the usual way, by nonvanishing of the symbol, and it is straightforward to check that H (and hence $H-\mu$) is elliptic of order 2; implicit here is that H is uniformly elliptic in this calculus along the entire diagonal. The standard iterative scheme now gives

Proposition 5.2. *There exists a parametrix $P(\mu) \in \Psi^{-2}_{\mathrm{p0}}(\overline{X})$ for $(H-\mu)^{-1}$ which has the property that*

$$(H-\mu)P(\mu) = \mathrm{Id} + F(\mu), \ \ P(\mu)(H-\mu) = \mathrm{Id} + E(\mu), \tag{5.1}$$

where $E(\mu), F(\mu) \in \Psi^{-\infty}_{\mathrm{p0}}(\overline{X})$; all operators here may be chosen to depend holomorphically on μ.

We conclude this section with a result about the structure of the difference between the resolvent and this parametrix.

Proposition 5.3. *When $\mu \notin spec(H)$, there is a decomposition of the resolvent*

$$R(\mu) = R'(\mu) + R''(\mu),$$

where $R'(\mu) \in \Psi^{-2}_{\mathrm{p0}}(X)$ depends holomorphically on μ and

$$R''(\mu) = \frac{1}{2\pi i}\int_\gamma \widetilde{R}_{12}(\mu_1, \mu - \mu_1)\, d\mu_1. \tag{5.2}$$

The operator $\widetilde{R}_{12}(\mu_1,\mu_2)$ is meromorphic on $\mathbb{C}^2$ with values in the space of distributions on $\overline{X}^2_{\mathrm{p0}}$. Denoting by $\pi_j : \overline{X}^2_{\mathrm{p0}} \to (\overline{M}_j)^2_0$ the obvious projection and setting $\rho^{(j)}_{10} = \pi_j^*\rho_{10}$, $\rho^{(j)}_{01} = \pi_j^*\rho_{01}$, $j = 1,2$, then $\widetilde{R}_{12}$ has Schwartz kernel of the form

$$\tilde{R}_{12}(\mu_1,\mu_2) = (\rho^{(1)}_{10}\rho^{(1)}_{01})^{k_1/2 + i\sqrt{\mu_1 - k_1^2/4}}(\rho^{(2)}_{10}\rho^{(2)}_{01})^{k_2/2 + i\sqrt{\mu_2 - k_2^2/4}}F_{12}, \tag{5.3}$$

with $F_{12} \in \mathcal{C}^\infty(\overline{X}^2_{\mathrm{p0}}; \pi_R^*(\Omega_0 \otimes \Omega_0))$.

Proof. Using (5.1) and the identities $R(\mu)(H-\mu) = (H-\mu)R(\mu) = I$, we deduce that

$$R(\mu) = P(\mu) - E(\mu)P(\mu) + E(\mu)R(\mu)F(\mu). \tag{5.4}$$

The decomposition we want is obtained by setting $R'(\mu)$ as the sum of the first two terms on the right here and $R''(\mu)$ the final term. Using the composition properties of the small calculus, it is clear that $R'(\mu)$ has the correct structure.

To analyze $R''(\mu)$, we substitute the contour integral (2.1) for $R(\mu)$, which gives

$$R''(\mu) = \frac{1}{2\pi i}\int_\gamma E(\mu)\,(R_1(\mu_1) \otimes R_2(\mu - \mu_1))\,F(\mu)\, d\mu_1. \tag{5.5}$$

Now set

$$\widetilde{R}_{12}(\mu_1,\mu_2) = E(\mu_1+\mu_2)\,(R_1(\mu_1) \otimes R_2(\mu_2))\,F(\mu_1+\mu_2). \tag{5.6}$$

Each of these factors, $E(\mu_1+\mu_2)$, $R_1(\mu_1) \otimes R_2(\mu_2)$ and $F(\mu_1+\mu_2)$, has a Schwartz kernel which is the pushforward of a distribution on $\overline{X}^2_{\mathrm{p0}}$. The kernels of E and F are smooth on this double space and vanish to all orders at all side boundary faces, while from §3, the kernel of $R_1(\mu_1) \otimes R_2(\mu_2)$ is of almost exactly the same form as $\widetilde{R}_{12}(\mu_1, \mu_2)$ in the statement of the proposition, but with F_{12} replaced by a distribution which is a product of two terms, each of which is conormal along one of the two partial diagonals diag_j.

The structure of the composition $E(\mu_1 + \mu_2)\,(R_1(\mu_1) \otimes R_2(\mu_2))$, and the structure of the composition of the resulting operator with $F(\mu_1 + \mu_2)$, may be analyzed using the triple space $\overline{X}^3_{\mathrm{p}0} = (\overline{M}_1)^3_0 \times (\overline{M}_2)^3_0$, in a way completely analogous to how composition in the large 0 calculus is analyzed, e.g. in [22]. Thus, considering the first of these operations, we first lift the Schwartz kernels of $E(\mu_1 + \mu_2)$ and $R_1(\mu_1) \otimes R_2(\mu_2)$ to the p0-double spaces, and then to the triple-space $\overline{X}^3_{\mathrm{p}0}$ via the left-middle and middle-right projections, respectively. This product has conormal singularities along the middle-right partial diagonal and has polyhomogeneous singularities along the various boundary faces. Pushing forward this distribution with the left-right projection $\overline{X}^3_{\mathrm{p}0} \to \overline{X}^2_{\mathrm{p}0}$ gives the Schwartz kernel of the first composition. Its structure may be read off from general properties about pushforwards on manifolds with corners [23], and in particular, we see that the interior singularities are smoothed by this mapping. The second composition is analyzed the same way. This proves the proposition. □

6. Compactification constructions

We now turn to the other major theme of this paper, which is the various ways one might compactify the conformally compact spaces M_j or their product $X = M_1 \times M_2$. The general problem of finding good compactifications of Riemannian manifolds, and in particular of (locally) symmetric spaces, has been an area of active research. We refer to [15] and [9] for more discussion of this in the symmetric space setting, and to [2] and [28] and [7] for some beautiful results in some general geometric settings.

There are two different compactication constructions we shall discuss here, the geodesic compactification (sometimes also called the conic compactification), as well as the Martin compactification. Each has a key role in understanding different aspects of the global geometry and function theory of a space. We define these now in turn.

When Z is a complete, simply connected manifold of nonpositive curvature (i.e. a Cartan-Hadamard manifold), then the geodesic compactification $\overline{Z}$ is obtained by adjoining to Z an ideal boundary $\partial\overline{Z}$, points of which are equivalence classes of geodesic rays. Two geodesic rays $\gamma(t)$, $\tilde{\gamma}(t)$, $t \geq 0$, are said to be equivalent if $d(\gamma(t), \tilde{\gamma}(t))$ remains bounded as $t \to \infty$. Thus in $\mathbb{R}^n$ any two parallel lines are identified, as are any two geodesics in the ball model of hyperbolic space $\mathbb{H}^n$ which converge to the same point on S^{n-1}. If $q \in \partial\overline{Z}$, then a neighborhood system at q is given by sets of the form $(Z \setminus B_R(p')) \cap \exp_p([T, \infty) \times \mathcal{U})$, where $\mathcal{U}$ is an open set in the unit sphere S_pZ in T_pZ. A point on this set is on some geodesic $\gamma(t)$ emanating from p with $\gamma'(0) \in \mathcal{U}$ and $t \geq T$ as well as in the exterior of the ball $B_R(p')$. Thus whenever Z is Cartan-Hadamard, then $\overline{Z}$ is homeomorphic to a closed ball $\overline{B^n}$, and its boundary $\partial\overline{Z}$ is identified with the unit tangent sphere S_pZ for any $p \in Z$. There are fairly obvious modifications of this construction when Z is not necessarily simply connected, or only has nonpositive curvature outside a compact set, and then the structure of $\overline{Z}$ is more complicated, but only on account of the topology of $\operatorname{int} Z$.

It is of substantial interest to understand when the compactification $\overline{Z}$ carries more structure than its initial definition as a topological space. In particular, we would like to determine when $\overline{Z}$ is naturally defined as a smooth manifold with boundary, or with corners. For example, when $Z = \mathbb{R}^n$ or $\mathbb{H}^n$, then as we have already stated $\overline{Z} \approx \overline{B^n}$, but obviously in these two examples, $\overline{Z}$ is 'really' a smooth closed ball. An interesting and underappreciated feature of this construction is that although it is possible to identify each of these compactifications topologically with a ball, the smooth structures at the boundary, or more precisely, the specific rings of functions which we are calling smooth, are different in the two cases, as we now explain.

When considering whether $\overline{Z}$ admits a natural smooth structure, the first key point is the regularity of the transition maps, as we now describe. Remaining within the context of Cartan-Hadamard manifolds for simplicity, the transition maps are the homeomorphisms between the unit tangent spheres at any two points $p, p' \in Z$,

$$S_p Z \longrightarrow \partial\overline{Z} \longleftarrow S_{p'} Z$$

provided through the geodesic spray from these two points. When Z has curvatures bounded between two negative constants $-a^2$ and $-b^2$, then from [2] these transition maps are of Hölder class $\mathcal{C}^{0,\alpha}$, $\alpha = a/b$, and accordingly, in this generality, $\overline{Z}$ has only a Hölder structure. However, for either of the examples above these transition maps are smooth; this is also true for general conformally compact manifolds for suitable localized versions of these transition maps (see [18]). The other key point concerns the choice of a class of defining functions for $\partial\overline{Z}$ (or, if $\overline{Z}$ is to be regarded as a smooth manifold with corners, then for subsets of it which are identified with the various boundary hypersurfaces). Smoothness of the transition maps and choice of defining functions together determine the $\mathcal{C}^\infty$ structure on $\overline{Z}$.

There is considerable flexibility in choosing (equivalence classes of) defining functions. For $\mathbb{R}^n$ it is most natural to use the radial compactification with defining function for $\partial\overline{\mathbb{R}^n}$, $\rho_1 = 1/|z| = 1/\operatorname{dist}(z, 0)$, $z \in \mathbb{R}^n$, the inverse of the polar distance variable. On the other hand, the Poincaré ball model for $\mathbb{H}^n$ suggests that the more natural choice now is $\rho_2 = \exp(-\operatorname{dist}(z, 0))$, 0 any fixed point in $\mathbb{H}^n$. These two defining functions for $\partial\overline{B^n}$ are quite different, since $\rho_1 = -1/\log \rho_2$. We say that $(\overline{B^n}, \rho_1)$ is the 'log blow-up' of $(\overline{B^n}, \rho_2)$. For later reference, we say that

- $\star$ A defining function defined as the reciprocal of the Riemannian distance function ($\rho = 1/d$) or its exponential ($\rho = e^{-1/d}$) is said to be of polynomial or exponential type, respectively.

The difference between these polynomial and exponential type defining functions appears most clearly in the spaces of functions with polyhomogeneous behaviour at the boundary, since these functions are characterized by their asymptotic developments using monomials of the form $\rho^z(-\log \rho)^\ell$, $z \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, where ρ is a boundary defining function. If ρ is replaced by $r = -1/\log \rho$, then one of these monomials becomes $e^{-z/r} r^{-\ell}$, which decays exponentially when $\operatorname{Re} z > 0$. In other words, functions which are polyhomogeneous with respect to exponential-type defining functions, with all exponents having positive real part, are residual, with respect to the log blow-up structure. The guiding principle for which defining functions to choose as primary is determined by the asymptotic behaviour of eigenfunctions of the Laplacian; one prefers eigenfunctions not to be automatically residual! The choices made above for $\overline{\mathbb{R}^n}$ and $\overline{\mathbb{H}^n}$, respectively, are vindicated by the analysis of manifolds with 'scattering metrics', see [25], and with conformally compact metrics, see [20] and [18].

As indicated above, the geodesic compactification of the conformally compact manifold (M, g) is just $\overline{M}$ with its usual smooth structure. Now consider what happens for the product of two such manifolds:

Proposition 6.1. *If $X = M_1 \times M_2$ is the product of two conformally compact manifolds, then its geodesic compactification $\overline{X}$ has boundary $\partial\overline{X}$ which may be identified with the simplicial join of ∂M_1 and ∂M_2, obtained from $\partial\overline{M}_1 \times \partial\overline{M}_2 \times [0, \pi/2]_\theta$ by collapsing each submanifold $\partial\overline{M}_1 \times \{q_2\} \times \{\pi/2\}$ and $\{q_1\} \times \partial\overline{M}_2 \times \{0\}$ to a point. This space $\overline{X}$ has polynomial type boundary defining functions for its hypersurface boundary faces.*

The precise meaning of this last statement will be clarified at the end of the proof.

Proof. Let us assume for simplicity that there are no forward trapped geodesics, i.e. geodesics which remain within a compact set as $t \to +\infty$. We remark on the general case at the end of the proof.

Every geodesic on X is of the form $(\gamma_1(\alpha_1 t), \gamma_2(\alpha_2 t))$, where γ_j is a (unit speed) geodesic on M_j and $\alpha_1^2 + \alpha_2^2 = 1$, $\alpha_1, \alpha_2 \geq 0$. Two such geodesics $\gamma(t)$ and $\tilde{\gamma}(t)$ are equivalent, i.e. remain within a bounded distance of one another as $t \to \infty$, if and only both their components $\gamma_j(\alpha_j t)$, $\tilde{\gamma}_j(\tilde{\alpha}_j t)$, remain within a bounded distance of one another in M_j. This in turn occurs only when $\alpha_j = \tilde{\alpha}_j$ and γ_j is equivalent to $\tilde{\gamma}_j$ on M_j, $j = 1, 2$. (Actually, note that if $\alpha_1 = \tilde{\alpha}_1 = 0$, then we do not require that γ_1 and $\tilde{\gamma}_1$ are equivalent, and similarly when $\alpha_2 = \tilde{\alpha}_2 = 0$.) Now write $\alpha_1 = \tilde{\alpha}_1 = \cos\theta$, $\alpha_2 = \tilde{\alpha}_2 = \sin\theta$, $0 \leq \theta \leq \pi/2$. There are three subcases.

- If $0 < \theta < \pi/2$, these equivalence classes are parametrized by pairs of points $(q_1, q_2) \in \partial\overline{M}_1 \times \partial\overline{M}_2$.
- If $\theta = 0$, these equivalence classes are parametrized by points $q_1 \in \partial\overline{M}_1$.
- If $\theta = \pi/2$, these equivalence classes are parametrized by points $q_2 \in \partial\overline{M}_2$.

Notice that in the second and third cases here, any two 'horizontal' geodesics $\gamma(t) = (\gamma_1(t), p_2)$, $\tilde{\gamma}(t) = (\gamma_1(t), p_2')$ with the same (or equivalent) first factors are identified, and similarly for any two 'vertical' geodesics in the product. This phenomenon is what causes the collapsing.

To understand the smooth structure of this compactification, recall that if x_j is a boundary defining function and if $\gamma_j(t)$ is a geodesic converging to the boundary on $\overline{M}_j$, then $x_j(\gamma_j(t)) \sim C_j e^{-t}$. Hence for a geodesic $\gamma(t) = (\gamma_1(\cos\theta\, t), \gamma_2(\sin\theta\, t))$ on X as above, we have

$$\lim_{t \to +\infty} \frac{\log x_1(\gamma(t))}{\log x_2(\gamma(t))} = \frac{\cos\theta}{\sin\theta}.$$

Hence $\log x_1 / \log x_2$ records the 'angle' at which geodesics tend to infinity in the two factors. It is thus reasonable to introduce

$$\rho_j = -1/\log x_j, \quad j = 1, 2,$$

as boundary defining functions on $\overline{M}_j$ as we have described in the introduction, so as to obtain the logarithmic blowup $(\overline{M}_j)_{\log}$ of $\overline{M}_j$. We now define the manifold with corners

$$\widetilde{X} = [(\overline{M}_1)_{\log} \times (\overline{M}_2)_{\log}; \partial(\overline{M}_1)_{\log} \times \partial(\overline{M}_2)_{\log}] \tag{6.1}$$

obtained by the usual (spherical) blowing up of the corner $\partial(\overline{M}_1)_{\log} \times \partial(\overline{M}_2)_{\log}$ in the product $(\overline{M}_1)_{\log} \times (\overline{M}_2)_{\log}$. This manifold has three boundary hypersurfaces: $\partial\overline{M}_1 \times M_2$, $M_1 \times \partial\overline{M}_2$, and the new face $\partial\overline{M}_1 \times \partial\overline{M}_2 \times (0, \pi/2)$. The last statement of the proposition about the polynomial-type defining function refers to the fact that we have needed to pass to the logarithmic blowups here. Furthermore, the earlier discussion shows that this new blown up face constitutes the main portion of the boundary of the geodesic compactification $\overline{X}$ of X, and $\overline{X}$ itself is obtained from $\widetilde{X}$ by collapsing each of the fibres $\partial\overline{M}_1 \times \{q_2\} \times \{\pi/2\}$ and $\{q_1\} \times \partial\overline{M}_2 \times \{0\}$ to a point.

Now consider the general case, when either M_1 or M_2 has forward trapped geodesics. Although the geodesic compactification of each factor makes unambiguous sense [18], a moment's reflection should reveal that the definition we have used here, in terms of equivalence classes of geodesic rays (where two rays are equivalent if they remain within a bounded distance of one another) is problematic for product spaces. For example, if γ_1 is forward trapped in M_1 and γ_2 converges to $\partial\overline{M}_2$, then the geodesics $(\gamma_1(\cos\theta\, t), \gamma_2(\sin\theta\, t))$ are mutually distinct for different values of $\theta \in [0, \pi/2]$. This would seem to indicate that the true geodesic compactification of X would require adding sets of the form $I \times \partial\overline{M}_2$ for every forward trapped geodesic in M_1, and similarly with the two factors interchanged. One way out of this impasse is to only allow geodesics γ_j on M_j which lie outside some specified compact subset. In any case, rather than enter into these subtleties, we shall refer to the compactification $\overline{X}$ defined in the statement of this proposition as the geodesic compactification of X. $\square$

The other compactification we consider here is due to Martin [17], and uses the function theory of Δ_g to associate a set of ideal boundary points ∂Z to Z. Notably, it may be carried out in great generality for pairs (Z, H) where H is a semibounded self-adjoint elliptic operator on a space Z, though we shall always assume here that H is the Laplacian. Let $\lambda_0 = \inf \operatorname{spec}(H)$. Then for every $\lambda \in \mathbb{R} \setminus (\lambda_0, \infty)$, there is a compactification $\overline{Z}_M(\lambda)$. Actually, it follows from the construction that $\overline{Z}_M(\lambda)$ is identified with $\overline{Z}_M(\lambda')$ for any two numbers $\lambda, \lambda' < \lambda_0$, so one needs to consider only $\overline{Z}_M(\lambda_0)$ and any other $\overline{Z}_M(\lambda)$.

Fix $\lambda < \lambda_0$. By [30], the set of solutions u to the equation $(H - \lambda)u = 0$ which remain everywhere positive is nonempty; we denote this $\mathbb{R}^+$-invariant set by $\mathcal{P}_+(\lambda)$. The structure of this positive cone is encoded in the slice

$$\mathcal{P}_+^p(\lambda) = \{u \in \mathcal{P}_+(\lambda) : u(p) = 1\}$$

for any fixed $p \in Z$. It follows readily from the Harnack inequality and elliptic estimates that for any sequence $u_j \in \mathcal{P}_+^p(\lambda)$ there is a subsequence $u_{j'}$ converging to a point in this space, and so $\mathcal{P}_+^p(\lambda)$ is compact. It is also obviously convex. Let $\mathcal{E}$ denote the set of its extreme points. Then for any $u \in \mathcal{P}_+^p(\lambda)$, the Krein-Milman theorem gives a measure $dm_u(e)$ supported on $\mathcal{E}$ such that $u = \int_{\mathcal{E}} dm_u(e)$. This is the generalized Poisson representation theorem! The set $\mathcal{E}$, or $\mathcal{E}(Z, \lambda)$ is called the minimal Martin boundary of Z. If $u \in \mathcal{E}$, then whenever $v \in \mathcal{P}_+^p(\lambda)$ and $v \leq u$ then $v = cu$ for some constant $0 < c \leq 1$, and this justifies the moniker 'minimal'.

If Z were to naturally embed in $\mathcal{P}^p_+(\lambda)$, then the closure of its image would be an obvious way to compactify it. Unfortunately, this is not the case, but instead we consider the resolvent kernel $R(\lambda; z, w)$. This is a solution of $(H_z - \lambda)R(\lambda; z, w) = 0$ when $z \neq w$, but is singular at $z = w$ and in addition, $R(\lambda; p, w) \neq 1$. Thus we define

$$u_w(z) = R(\lambda, z, w)/R(\lambda, p, w),$$

so that $u_w(p) = 1$ and u_w is a regular solution of $(H - \lambda)u_w = 0$ on $X \setminus \{w\}$. Now let w_j be any sequence of points in Z which leaves any compact set. Then some subsequence $u_{w_j'}$ converges to an element of $\mathcal{P}^p_+(\lambda)$. The (full) Martin boundary is the set of equivalence classes of these sequences, or equivalently, is the set of all possible functions u obtained as limits in this fashion. We label the different boundary points by $q \in \partial_M Z(\lambda)$, and write $u_q(z)$ for the corresponding limiting solution. The Martin compactification $\overline{Z}_M$ (or $\overline{Z}_M(\lambda)$) is the union of Z and $\partial_M Z(\lambda)$. There is a metric on the set of functions $u_w(z)$, $w \in \overline{Z}_M$, given by

$$d_p(u_w, u_{w'}) = \int_{B_1(p)} |u_w(z) - u_{w'}(z)|\, dV_g.$$

Thus $\overline{Z}_M$ not only a topological space, but a metric space.

This definition must be modified when $\lambda = \lambda_0$ and there is an L^2 eigenfunction u_0 with eigenvalue λ_0, since then the resolvent kernel $R(\lambda_0, z, w)$ does not exist. In this case $u_0 > 0$, and in fact $\mathbb{R}^+ \cdot u_0 = \mathcal{P}^p_+(\lambda_0)$, i.e. the only positive solutions are positive multiples of u_0. Hence it is consistent to let $\overline{Z}_M(\lambda_0)$ be the one point compactification of Z then. Otherwise, if λ_0 is not in the point spectrum, then the definition is the same as before.

If $Z = \mathbb{R}^n$, then $\lambda_0 = 0$ and it is well-known that $\partial_M \mathbb{R}^n(0)$ is a single point, so that $(\overline{\mathbb{R}^n})_M(0)$ is the one-point compactification S^n. On the other hand, for $\lambda < 0$, the extreme points of $\mathcal{P}^0_+(\lambda)$ are the exponentials $e^{x\cdot\xi}$, $|\xi|^2 = -\lambda$, and this sphere of radius $\sqrt{-\lambda}$ is the full Martin boundary, and so $(\overline{\mathbb{R}^n})_M(\lambda) = \bar{B}^n$. If $Z = \mathbb{H}^n$ and $n = k + 1$, then $\lambda_0 = k^2/4$ and $\overline{Z}_M(\lambda)$ is the closed ball $\overline{B^n}$ for all $\lambda \leq \lambda_0$. The minimal positive eigenfunctions are the ones of the form $x^{k/2+\sqrt{k^2/4-\lambda}}$ for all possible choices of upper half-space coordinates (i.e. choice of which point on the boundary of the ball to send to infinity). From [20] it follows that when M is conformally compact, $\overline{M}_M(\lambda)$ is still equal to $\overline{M}$. To survey other cases relevant to us in which the Martin compactification is known, when Z is Cartan-Hadamard with curvatures pinched between two negative constants $-a^2$ and $-b^2$, then Anderson and Schoen [2] and Ancona [1] proved that $\overline{Z}_M(\lambda) = \overline{B^n}$ as $\mathcal{C}^\alpha$ manifolds, $\alpha = a/b$). There has been recent significant progress in determining the Martin compactifications for general symmetric spaces of noncompact type; definitive results are proved in the recent monograph [9], and this has an extensive bibliography of the literature on these developments.

Of particular relevance to us here is the work of Giulini and Woess [8] where the Martin compactification of the product of two hyperbolic spaces is determined. Their result is that now $\overline{X}_M(\lambda)$ is identified with $[\overline{\mathbb{H}^{n_1}} \times \overline{\mathbb{H}^{n_2}}; \partial\overline{\mathbb{H}^{n_1}} \times \partial\overline{\mathbb{H}^{n_2}}]$, the blow-up of the product of the two balls along the corner; recall that this space appeared as an intermediate picture in the description of the geodesic compactification. Their proof uses rather involved global heat-kernel estimates, and one of the motivations of this paper was to demonstrate how this result, and the analogous one for products of conformally compact spaces, may be obtained in a more straightforward manner using resolvent estimates and stationary phase.

To conclude this section on compactifications, we state once again that our primary interest is in obtaining a compactification of the double-space X^2 (where $X = M_1 \times M_2$) which is natural with respect to the structure of the resolvent. More specifically, we wish that if λ is in the resolvent set, then $R_X(\lambda)$ should have at most polyhomogeneous singularities at the boundary hypersurfaces of this compactification (apart from its usual diagonal singularity). In the next sections we shall determine this compactification by examining the asymptotics of $R_X(\lambda; z, w)$ as the points $z = (z_1, z_2)$ and $w = (w_1, w_2)$ diverge in all possible directions. Following this, we shall see that the Martin compactification is essentially a 'slice' of this resolvent compactification, and it is easily determined from this asymptotic of the resolvent. This is done in the final section. We shall see there that when either H_1 or H_2 has eigenvalues below the continuous spectrum, then the Martin compactification of X is obtained by substantially collapsing part of the boundary of the geodesic compactification $\overline{X}$. This indicates that the Martin boundary (of X) is a much cruder object than the resolvent compactification (of X^2), and that one should regard the resolvent compactification as the primary object of interest.

7. Asymptotics

In the remainder of this paper we shall give a more detailed description of the structure of the resolvent $R(\mu)$ on X. As a first approach to this we adopt the more traditional viewpoint and derive the asymptotic behavior of $R(\mu)f$ when $f \in \dot{\mathcal{C}}^\infty(\overline{X})$. (Functions vanishing to all orders at ∂X are a suitable analogue of the space of Schwartz functions.) More generally, it makes absolutely no difference if we allow f to be the sum of an element of $\dot{\mathcal{C}}^\infty(\overline{X})$ and a distribution of compact support. In particular, all of the calculations below apply when $f = \delta_p$, $p \in X = M_1 \times M_2$; indeed, this is the basis for our identification of the Martin boundary of X. However, we shall simply assume that f is Schwartz, and also, in this section, that μ is in the resolvent set for H.

Recall our convention that when μ_j is in the resolvent set for H_j, then the imaginary part of $\sqrt{\mu_j - k_j^2/4}$ is negative, and by the results of § 3, $R_j(\mu_j)f$ is then of the form

$$x_j^{k_j/2 + i\sqrt{\mu_j - k_j^2/4}} g_j, \ g_j \in \mathcal{C}^\infty(\overline{M}_j).$$

Also, if ϕ_{ji} is an L^2 eigenfunction of of H_j with eigenvalue λ_{ji}, then

$$\phi_{ji} = x_j^{k_j/2 + i\sqrt{\lambda_{ji} - k_j^2/4}} \psi_{ji}, \ \psi_{ji} \in \mathcal{C}^\infty(\overline{M}_j);$$

the Schwartz kernel of Π_{ji} is $\phi_{ji}(z_j) \otimes \overline{\phi_{ji}}(z_j')$ if the eigenvalue is simple, and is a finite sum of such terms otherwise.

We start with a slightly stronger result concerning the structure of $R_{12}(\mu_1, \mu_2) = R_1(\mu_1) \otimes R_2(\mu_2)$ applied to $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, considered as a function of both μ_1 and μ_2.

Lemma 7.1. *Let*

$$S = (\mathbb{C} \setminus \operatorname{spec}(H_1))_{\mu_1} \times (\mathbb{C} \setminus \operatorname{spec}(H_2))_{\mu_2}.$$

For $f \in \dot{\mathcal{C}}^\infty(\overline{X})$*,* $R_{12}f$*, considered as a function on* $\overline{X} \times S$*, satisfies*

$$R_{12}f = x_1^{k_1/2+i\sqrt{\mu_1-k_1^2/4}} x_2^{k_2/2+i\sqrt{\mu_2-k_2^2/4}} g,\ g \in \mathcal{C}^\infty(\overline{X} \times S), \tag{7.1}$$

with natural extensions corresponding to the meromorphic extension of the R_j*.*

Remark 7.2. This follows directly from the corresponding statement in each factor if $f = f_1 \otimes f_2$, $f_j \in \dot{\mathcal{C}}^\infty(\overline{M}_j)$.

Proof. The kernel of R_{12} is polyhomogeneous on $(\overline{M}_1)^2_0 \times (\overline{M}_2)^2_0 \times S$. Let π_L, π_R be the projections to the left and right factors of $\overline{M}_1 \times \overline{M}_2$. That is, if $\pi_{L,j}$, resp. $\pi_{R,j}$, denote the projection of $(\overline{M}_j)^2_0$ to its left, resp. right, factor, and id_S the identity map on S, then $\pi_L = \pi_{L,1} \times \pi_{L,2} \times \mathrm{id}_S$, $\pi_R = \pi_{R,1} \times \pi_{R,2} \times \mathrm{id}_S$. Note that π_L, π_R, are b-fibrations. Then R_{12} applied to f is given by the push-forward of $R_{12}\pi_R^* f$ under the map π_L. (Here we regard f as a function on $\overline{X} \times S$, i.e. we even allow S-dependence.) By the usual push-forward formula [23], treating $S_{(\mu_1,\mu_2)}$ as a parameter,

$$\begin{aligned} R_{12}(\mu_1,\mu_2)f &= x_1^{k_1/2+i\sqrt{\mu_1-k_1^2/4}} x_2^{k_2/2+i\sqrt{\mu_2-k_2^2/4}} g_{(\mu_1,\mu_2)}, \\ g_{(\mu_1,\mu_2)} &\in \mathcal{C}^\infty(\overline{M}_1 \times \overline{M}_2), \end{aligned} \tag{7.2}$$

and the function $g(z_1,z_2,\mu_1,\mu_2) = g_{(\mu_1,\mu_2)}(z_1,z_2)$ appearing above satisfies (7.1), proving the lemma. □

Although we use this result throughout the section, we usually state arguments for simplicity as if R_1 *and* R_2 *are applied separately to* f*.* In one particular case, when analyzing $R(\mu)f$ for μ real, below $\mathrm{spec}(H)$, and when either H_1 or H_2 have L^2 eigenvalues, we need a stronger result, where f is not required to be Schwartz. We postpone discussion of that case until it is required, see the arguments preceeding Theorem 7.12.

The asymptotic behavior of $R(\mu)f(z)$ must be analyzed in three separate regions: near $\partial\overline{M}_1 \times M_2$, near $M_1 \times \partial\overline{M}_2$ and near the corner $\partial\overline{M}_1 \times \partial\overline{M}_2$. Using coordinates $z = (z_1, z_2)$, $z_j = (x_j, y_j)$, these correspond to $x_1 \to 0$, $x_2 \geq c > 0$, or $x_1 \geq c > 0$, $x_2 \to 0$ or $x_1, x_2 \to 0$, respectively.

7.1. Asymptotics at $M_1 \times \partial\overline{M}_2$ and at $\partial\overline{M}_1 \times M_2$.

We first describe the uniform behaviour of $R(\mu)f$ on $M_1 \times \overline{M}_2$, i.e. at infinity in M_2. This is given as an asymptotic series in powers of $-1/\log x_2$; we shall think of this later as an asymptotic expansion on the logarithmic blow-up $(\overline{M}_2)_{\log}$ of $\overline{M}_2$, as defined earlier in §6.

We state the result as two lemmas, first assuming that H_1 has no L^2 eigenvalues, and then removing this condition.

Lemma 7.3. *Suppose* $f \in \dot{\mathcal{C}}^\infty(\overline{X})$*,* $\mu \in \mathbb{C} \setminus [k^2/4, +\infty)$*, and* H_1 *has no* L^2 *eigenfunctions. Then* $R(\mu)f$ *has the following asymptotic expansion on* $M_1 \times (\overline{M}_2)_{\log}$*:*

$$R(\mu)f = x_2^{k_2/2+i\sqrt{\mu-k^2/4}}(-1/\log x_2)^{3/2} g,\ g \in \mathcal{C}^\infty(M_1 \times (\overline{M}_2)_{\log}). \tag{7.3}$$

Moreover, there exists a constant $c \neq 0$ *such that*

$$g|_{M_1 \times \partial\overline{M}_2} = c\,(\mu - k^2/4)^{3/4} \left(S_1(k_1^2/4) \otimes P_2^t(\mu - k_2^2/4)\right) f, \tag{7.4}$$

where

$$S_1(k_1^2/4) = \frac{d}{d\tau}\Big|_{\tau=0} R_1(k_1^2/4 + \tau^2),$$

$P_2(\mu - k_2^2/4)$ *is the Poisson transform on* M_2 *at the spectral parameter* $\mu - k_2^2/4$, *and* P_2^t *its transpose.*

Remark 7.4. The Schwartz kernel $S_1(k_1^2/4; z_1, z_1')$ of $S_1(k_1^2/4)$ is a familiar function on hyperbolic space: it is a non-square-integrable eigenfunction for H_1, with threshold eigenvalue $k_1^2/4$, invariant under the group of rotations which fix z_1', and is known as a spherical function. Its smoothness near $z_1 = z_1'$ can be seen directly: simply differentiate the equality

$$(H_1 - k_1^2/4 - \tau^2)R_1(k_1^2/4 + \tau^2) = I$$

with respect to τ and set $\tau = 0$ to get $(H_1 - k_1^2/4)S_1(k_1^2/4) = 0$, and apply elliptic regularity. Alternately, using the construction of Theorem 3.3, note that the parts of R_1 which are holomorphic near $k_1^2/4$ do not contribute to this operator; since the term R_1' which carries the diagonal singularity in the decomposition of R_1 in Theorem 3.1 is of this form, we see again that the Schwartz kernel of $S_1(k_1^2/4)$ is smooth on M_1^2 and polyhomogeneous on $(\overline{M}_1)_0^2$.

Using slightly different notation for the spectral parameter momentarily, the Schwartz kernel $P_2(\zeta; z_2, y_2')$ of the Poisson transform for $H_2 - \zeta(k_2 - \zeta)$ is polyhomogeneous on $[M_2 \times \partial M_2; \Delta\iota_{\partial M_2}]$, the blow-up of the diagonal of the boundary in $M_2 \times \partial M_2$. It is simply the leading term of the expansion of the resolvent at this face, i.e.

$$P_2(\zeta; z_2, y_2') = \lim_{x_2' \to 0} (x_2')^{-\zeta} R_2(\zeta; z_2, z_2').$$

Its transpose $P_2^t(\mu - k_2^2/4; y_2, z_2')$ is obtained by reversing the roles of the variables z_2 and z_2'.

Proof. We analyze $R(\mu)f$, using the representation (2.1) for $R(\mu)$, by shifting γ so that it passes through $\inf \operatorname{spec}(H_1) = k_1^2/4$, and so that the minimum of $\operatorname{Im} \sqrt{\mu - k_2^2/4 - \mu_1}$ along this path is attained at that point, see Figure 3, and then applying (complex) stationary phase.

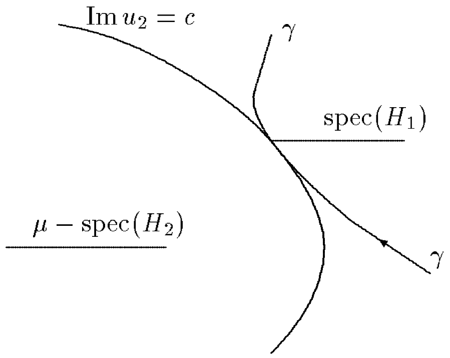

FIGURE 3. Contour of integration, γ, used to describe behavior in $M_1 \times \overline{M}_2$. Here $c = \operatorname{Im} \sqrt{\mu - k^2/4}$, and $u_2 = \sqrt{\mu - k_2^2/4 - \mu_1}$.

To set things up for stationary phase, first note that $R_2(\mu_2)$ maps $\dot{\mathcal{C}}^\infty(\overline{M}_2)$ to $x_2^{k_2/2 + i\sqrt{\mu_2 - k_2^2/4}}\mathcal{C}^\infty(\overline{M}_2)$. Therefore, the oscillatory factor $e^{(k_2/2 + i\sqrt{\mu_2 - k_2^2/4}) \log x_2}$,

$\mu_2 = \mu - \mu_1$ appears in the integrand. On the other hand, $R_1(\mu_1)$ is *not* smooth at $\mu_1 = k_1^2/4$, but instead has an expansion in powers of $\sqrt{\mu_1 - k_1^2/4}$. Thus it decomposes into the sum of odd and even powers, respectively, $R_1(\mu_1) = R_1^{\mathrm{odd}}(\mu_1) + R_1^{\mathrm{even}}(\mu_1)$, and we write $R_1^{\mathrm{odd}}(\mu_1) = \sqrt{\mu_1 - k_1^2/4}\, G_1(\mu_1 - k_1^2/4)$ where G_1 is smooth. Because the phase $\sqrt{\mu - \mu_1 - k_2^2/4}\,\log x_2$ is not stationary at $k_1^2/4$, the smooth part R_1^{even} contributes only terms decaying faster than any power of $-1/\log x_2$. Thus we are left only with

$$\int_\gamma e^{(k_2/2 + i\sqrt{\mu-\mu_1-k_2^2/4})\log x_2} \sqrt{\mu_1 - k_1^2/4}\, G_1(\mu_1 - k_1^2/4)\tilde{g}(\mu_1, z, z')\, d\mu_1,$$

where $\tilde{g}$ is smooth in all variables. Actually, it suffices to take this integral only over some compact segment of γ containing $k_1^2/4$ and where the decomposition of R_1 into even and odd parts is valid; using the uniform weighted L^2 estimates of §3, the integral over the remaining portion of γ contributes a term vanishing like $x_2^{k_2/2+m}$, where $m > 0$ can be made as large as wished by increasing the length of the compact segment, and is therefore negligible in the asymptotics. Changing the variable of integration to $\tau = \sqrt{\mu_1 - k_1^2/4}$, so that $\mu_1 = k_1^2/4 + \tau^2$, leads to

$$\int e^{(k_2/2 + i\sqrt{\mu - k^2/4 - \tau^2})\log x_2} 2\tau^2 G_1(\tau^2)\tilde{g}(k_1^2/4 + \tau^2, z, z')\, d\tau.$$

The phase function $\phi(\tau) = \sqrt{\mu - k^2/4 - \tau^2}\,\log x_2$ is now stationary at $\tau = 0$; furthermore, $\phi''(0) = -1/\sqrt{\mu - k^2/4}$, and so the further change of variables $\sigma = \tau\sqrt{-\log x_2}/(\mu - k^2/4)^{1/4}$ reduces this integral to

$$x_2^{k_2/2 + i\sqrt{\mu - k^2/4}}(\mu - k^2/4)^{3/4}(-1/\log x_2)^{3/2} \int e^{i\sigma^2/2}\, \sigma^2\, \hat{g}(\sigma, z, z')\, d\sigma$$

where $\hat{g}$ is obtained from $\tilde{g}$ by replacing τ by σ. Applying the stationary phase for phase functions with nonnegative imaginary part, see [14, Theorem 7.7.5], we obtain that

$$R(\mu)f = x_2^{k_2/2 + i\sqrt{\mu - k^2/4}}(-1/\log x_2)^{3/2} g, \tag{7.5}$$

where g is $\mathcal{C}^\infty$ on the logarithmic blow-up $M_1 \times (\overline{M}_2)_{\log}$ of $M_1 \times \overline{M}_2$, i.e. in the variables $(z_1, -1/\log x_2, y_2)$ for $x_1 \geq c > 0$. (Of course, g is continuous on $M_1 \times \overline{M}_2$; it is the lower order terms in the asymptotics that necessitate the change of the smooth structure.) In fact,

$$g|_{M_1 \times \partial\overline{M}_2} = c\,(\mu - k^2/4)^{3/4} x_2^{-k_2/2 - i\sqrt{\mu - k^2/4}} G_1(0) \otimes R_2(\mu - k_1^2/4) f\Big|_{M_1 \times \partial\overline{M}_2},$$

where c is a constant arising from the stationary phase lemma.

We identify the operators here slightly more explicitly to write this formula in final form. First, according to [20], for each $z_1 \in M_1$,

$$x_2^{-k_2/2 - i\sqrt{\mu - k^2/4}} R_2(\mu - k_2^2/4) f\Big|_{\partial\overline{M}_2}$$

$$= \int_{\partial\overline{M}_2} P_2^t(\mu - k_2^2/4; y_2, z_2') f(z_1, z_2')\, dz_2' \equiv P_2^t(\mu - k_2^2/4)(f)(z_1, y_2).$$

In addition, it follows directly from the discussion above that $G_1(0) = S_1(k_1^2/4)$. This finishes the proof. □

When H_1 has L^2 eigenvalues, then shifting the contour γ through the corresponding poles of the resolvent gives a contribution $-\Pi_{1i} \otimes R_2(\mu - \lambda_{1i})$ from the residues of the integrand. (For simplicity we assume that the ramification point $k_1^2/4$ is not a pole of the meromorphic continuation of R_1 to the Riemann surface, though this can be handled too by the arguments preceeding Theorem 7.12.) Hence in this case the proof of the preceeding lemma yields:

Lemma 7.5. *Suppose* $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\mu \in \mathbb{C}\backslash[k^2/4, +\infty)$, *and* $k_1^2/4$ *is not a pole of the meromorphic continuation of* R_1 *to* Σ_1. *Then* $R(\mu)f$ *has the following asymptotic expansion on* $M_1 \times (\overline{M}_2)_{\log}$*:*

$$R(\mu)f = x_2^{k_2/2+i\sqrt{\mu-k^2/4}}(-1/\log x_2)^{3/2} g + \sum_{i=1}^{N_1} x_2^{k_2/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}} g_i, \tag{7.6}$$

where $g, g_i \in \mathcal{C}^\infty(M_1 \times (\overline{M}_2)_{\log})$, g *is given by* (7.4), *and*

$$g_i|_{M_1\times\partial\overline{M}_2} = -(\Pi_{1i} \otimes P_2^t(\mu - \lambda_{1i}))f. \tag{7.7}$$

In particular, $g_i = \phi_{1i} \otimes \tilde{g}_i$, *with* $\tilde{g}_i \in \mathcal{C}^\infty(\overline{M}_2)$ *and* ϕ_{1i} *an eigenfunction of* H_1 *with eigenvalue* λ_{1i}.

Remark 7.6. This lemma shows that near $M_1 \times \partial\overline{M}_2$, only the L^2 eigenvalues of H_1 play a role in the asymptotics, but perhaps surprisingly, *not* those of H_2. Notice that all terms in (7.6) which come from the L^2 eigenvalues of H_1 dominate the term coming from the continuous spectrum since

$$\operatorname{Im}\sqrt{\mu - k^2/4} < \operatorname{Im}\sqrt{\mu - \lambda_{1i} - k_2^2/4} < 0.$$

In addition, the term corresponding to the lowest eigenvalue λ_{10} of H_1 dominates all the other terms. This will be important later in the determination of the Martin boundary in the presence of bound states.

The asymptotics of $R(\mu)f$ at the other face $\partial\overline{M}_1 \times M_2$ is completely analogous. The same calculations lead to the expression

$$\begin{gathered} R(\mu)f = x_1^{k_1/2+i\sqrt{\mu-k^2/4}}(-1/\log x_1)^{3/2} g + \sum_{i=1}^{N_1} x_1^{k_1/2+i\sqrt{\mu-\lambda_{2i}-k_1^2/4}} g_i, \\ \text{where } \ g, g_i \in \mathcal{C}^\infty((\overline{M}_1)_{\log} \times M_2). \end{gathered} \tag{7.8}$$

In addition,

$$g|_{\partial\overline{M}_1\times M_2} = c\,(\mu - k^2/4)^{3/4}\left(P_1^t(\mu - k_1^2/4) \otimes S_2(k_2^2/4)\right) f, \tag{7.9}$$

where P_1^t is the transpose of the Poisson transform on M_1 and S_2 is the 'spherical function' at eigenvalue $k_2^2/4$ for M_2.

We note once again that even if both H_1 and H_2 have L^2 eigenvalues, only those of H_2 contribute to the asymptotics when $x_1 \to 0$, $x_2 \geq c > 0$ and only those of H_1 contribute to the asymptotics when $x_2 \to 0$, $x_1 \geq c > 0$. This indicates that the asymptotics at the corner $\partial\overline{M}_1 \times \partial\overline{M}_2$, where both $x_1, x_2 \to 0$ must be more complicated, at least in the presence of bound states, because it intermediates this transition.

7.2. **Asymptotics at $\partial\overline{M}_1 \times \partial\overline{M}_2$ in the absence of L^2 eigenvalues.** We now proceed to the analysis of $R(\mu)f$ at this corner. We have already seen the necessity of logarithmic blow ups at the side faces of the product. Correspondingly, the asymptotics at the corner necessitate that we pass to the log blow-ups of both factors $\overline{M}_j$, which we denote $(\overline{M}_j)_{\log}$. In terms of these, recall from (6.1) the space $\widetilde{X}$ defined by blowing up $(\overline{M}_1)_{\log} \times (\overline{M}_2)_{\log}$ at its corner $\partial(\overline{M}_1)_{\log} \times \partial(\overline{M}_2)_{\log}$. We now show that this space is the correct one on which to consider asymptotics of $R(\mu)f$.

We use coordinates of the form (ρ_j, y_j) on $(\overline{M}_j)_{\log}$ near the boundary, where $\rho_j = -1/\log x_j$ and y_j are coordinates on $\partial(\overline{M}_j)_{\log}$, $j = 1, 2$. Denoting their pullbacks to $\widetilde{X}$ by the same symbols, we see that ρ_j vanishes simply both at the side face which covers $\partial(\overline{M}_j)_{\log}$ and also at the front face obtained by blowing up the corner $\rho_1 = \rho_2 = 0$. In the region of $\widetilde{X}$ where $\rho_1/\rho_2 < C$, for some $C > 0$, we can use

$$\rho_2,\ s = \rho_1/\rho_2,\ y_1, \text{ and } y_2$$

as a nonsingular coordinate system. Equivalently, in terms of the original boundary defining functions x_j, in any region where $\log x_2/\log x_1 < C$ for some $C > 0$, we use the projective coordinate

$$s = \frac{-1/\log x_1}{-1/\log x_2} = \frac{\log x_2}{\log x_1}$$

along the new 'front' face of $\widetilde{X}$ covering the corner, and $-1/\log x_2$ as a defining function for this face. Thus $s \to 0$ upon approach to the lift of $\partial\overline{M}_1 \times \overline{M}_2$, while $s \to \infty$ on approach to the lift of $\overline{M}_1 \times \partial\overline{M}_2$. The function

$$\rho = (\rho_1^{-2} + \rho_2^{-2})^{-1/2}$$

is a *total* boundary defining function on $\widetilde{X}$, i.e. it vanishes simply along each of the of the boundary hypersurfaces. The factor ρ^{-1} appears frequently in the exponents in our results below. Note that the function $\rho_1 = -1/\log x_1$ also vanishes simply on the boundary hypersurfaces lying in $s < C$, and so we use this simpler function for the initial local calculation and then change to ρ for the global statements. The trivial identity

$$x_2 = e^{\log x_2} = e^{s \log x_1} = x_1^s$$

will be used repeatedly.

We first assume that neither H_1 nor H_2 has L^2 eigenvalues, and calculate the asymptotics of $R(\mu)f$ on $\tilde{X}$.

Proposition 7.7. *Suppose $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\mu \in \mathbb{C} \setminus [k^2/4, +\infty)$, and H_1, H_2 have no L^2 eigenfunctions. Then, with $\rho_i = -1/\log x_i$, $\rho^{-1} = \sqrt{\rho_1^{-2} + \rho_2^{-2}}$, $R(\mu)f$ has the following asymptotic expansion on $\widetilde{X}$:*

$$R(\mu)f = x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho) h. \tag{7.10}$$

Here h polyhomogeneous on $\widetilde{X}$, with leading asymptotic order $1/2$ on the front face and $3/2$ on the side faces of $\widetilde{X}$. Moreover, the restriction of the leading term of h

to the boundary (which may be called the principal symbol of h) is

$$\begin{aligned} &a(\mu,s)\left(P_1^t(\mu_1^0(s))\otimes P_2^t(\mu-\mu_1^0(s))\right)f \text{ on the front face,}\\ &a'(\mu)\left(S_1(k_1^2/4)\otimes P_2^t(\mu-k_2^2/4)\right)f \text{ on the lift of } \overline{M}_1\times\partial\overline{M}_2,\\ &a''(\mu)\left(P_1^t(\mu-k_1^2/4)\otimes S_2(k_2^2/4)\right)f \text{ on the lift of } \partial\overline{M}_1\times\overline{M}_2,\end{aligned} \tag{7.11}$$

with $\mu_1^0(s)$ given by

$$\mu_1^0(s)=\frac{\mu-k_2^2/4+s^2k_1^2/4}{1+s^2}, \tag{7.12}$$

a, a', a'' nonvanishing.

Remark 7.8. The explicit form of a is given in (7.16), up to a non-zero constant.

Proof. Away from the front face, this proposition merely restates Lemma 7.3, so we only need to consider the asymptotics at the front face. The integrand $(R_1(\mu_1)\otimes R_2(\mu-\mu_1))\,f$ in the contour integral representation of $R(\mu)f$ has the form

$$x_1^{k_1/2+i\sqrt{\mu_1-k_1^2/4}}x_2^{k_2/2+i\sqrt{\mu-\mu_1-k_2^2/4}}g,$$

or equivalently,

$$\exp\left\{\left[k_1/2+sk_2/2+i\left(\sqrt{\mu_1-k_1^2/4}+s\sqrt{\mu-\mu_1-k_2^2/4}\right)\right]\log x_1\right\}g, \tag{7.13}$$

where g is $\mathcal{C}^\infty$ in μ,μ_1 and on $\overline{M}_1\times\overline{M}_2$. The expressions inside these square roots assume values in $\mathbb{C}\setminus[0,\infty)$, and we recall the square roots have *negative* imaginary parts in this region.

We again do a stationary phase analysis of the integral of (7.13), and for this it is necessary to choose the contour of integration so that, if we set

$$F(\mu_1)=F_s(\mu_1)=\sqrt{\mu_1-k_1^2/4}+s\sqrt{\mu-\mu_1-k_2^2/4},$$

then the supremum of $\operatorname{Im}F(\mu_1)$ along γ is as negative as possible. Both μ and s are parameters here, and we must choose the contour differently corresponding to the different points of the front face of $\widetilde{X}$, i.e. the different values of s. Since F is analytic outside $[k_1^2/4,+\infty)\cup(\mu-[k_2^2/4,+\infty))$, its critical points and those of its imaginary part are the same. In fact, F has a unique critical point, located at

$$\mu_1^0=\mu_1^0(s)=\frac{\mu-k_2^2/4+s^2k_1^2/4}{1+s^2} \tag{7.14}$$

(cf. (7.12)), and this is a saddle point of the harmonic function $\operatorname{Im}F(\mu_1)$. (We return to this and shall explain it in greater detail below.) Moreover, this critical point always lies on the straight line segment connecting the two ramification points $k_1^2/4$ and $\mu-k_2^2/4$, and tends to the former as $s\to+\infty$ and to the latter as $s\to 0$. Next,

$$F(\mu_1^0)=\sqrt{(\mu-k^2/4)(1+s^2)},\qquad F'(\mu_1^0)=0,$$

and

$$F''(\mu_1^0)=-\frac{1}{4}s^{-2}(1+s^2)^{5/2}(\mu-k^2/4)^{-3/2}.$$

Thus we may choose an appropriate (parametrized) contour $\gamma(t)$ so that $\operatorname{Im}F(\gamma(t))$ attains a non-degenerate maximum when $\gamma(t)=\mu_1^0$. The stationary phase lemma may now be applied as before, although the singular change of variables is no longer required (in $s > 0$); we deduce that the asymptotics of $R(\mu)f$ have the form

$$
\begin{aligned}
&\exp\left(\left(k_1/2 + sk_2/2 + i\sqrt{(\mu - k^2/4)(1+s^2)}\right)\log x_1\right)(-1/\log x_1)^{1/2} g \\
&= x_1^{k_1/2} x_2^{k_2/2}\left(\exp\sqrt{(\log x_1)^2 + (\log x_2)^2}\right)^{-i\sqrt{\mu - k^2/4}}(-1/\log x_1)^{1/2} g.
\end{aligned}
\tag{7.15}
$$

At the points of the front face of $\widetilde{X}$ where $\log x_2/\log x_1 = s$ this coefficient function g is a nonvanishing multiple of

$$
\begin{aligned}
s(1+s^2)^{-5/4}(\mu - k^2/4)^{3/4} g(\mu_1^0(s)) &= s\,(1+s^2)^{-5/4}((\mu - k^2/4)^{-3/2})^{-1/2} \\
\times x_1^{-k_1/2 - i\sqrt{\mu_1^0(s) - k_1^2/4}}\, & x_2^{-k_2/2 - i\sqrt{\mu - \mu_1^0(s) - k_2^2/4}}\left(R_1(\mu_1^0(s)) \otimes R_2(\mu - \mu_1^0(s))\right) f;
\end{aligned}
$$

since the powers of x_j here are exactly the leading terms in the 'outgoing' expansions of the resolvents R_j, we may take a limit and obtain finally

$$
c\,s\,(1+s^2)^{-5/4}(\mu - k^2/4)^{3/4}\left(P_1^t(\mu_1^0(s)) \otimes P_2^t(\mu - \mu_1^0(s))\right) f.
\tag{7.16}
$$

Note that although this computation is done separately for each value of s, this final expression depends smoothly on s and in fact g is smooth on $\widetilde{X}$ up to the front face.

To be precise, we still need to examine what happens near and at the corner $s = 0$, $-1/\log x_2 = 0$. However, this is hardly different from the discussion at $\partial\overline{M}_1 \cap M_2$. Indeed, we simply introduce $\tau = \sqrt{\mu - \mu_1 - k_2^2/4}$ as the new smooth variable of integration, and then stationary phase can be performed uniformly in s down to $s = 0$, with the limiting contour being the same as for the discussion at $\partial\overline{M}_1 \cap M_2$, showing that $R(\mu)f$ is actually polyhomogeneous. In particular, note that the expansions (7.4), (7.9) and (7.16) at the front and side faces of $\widetilde{X}$ match up at the corners, in agreement with the polyhomogeneity. The one point to note is that, for example, there is a factor of $(-1/\log x_1)^{1/2}$ in (7.15), whereas (7.8) has a factor of $(-1/\log x_1)^{3/2}$. To reconcile this, observe that (7.8) has an extra factor of s, and

$$
s(-1/\log x_1)^{1/2} = \frac{-\log x_2}{-\log x_1}(-1/\log x_1)^{1/2} = (-\log x_2)(-1/\log x_1)^{3/2},
$$

so in fact the powers match up. This completes the proof. $\square$

We remark that if the contour γ does not go through the critical point $\mu_1^0(s)$, then it must necessarily contain points where the integrand is larger; however, at those points the phase function itself, *not just its imaginary part*, will fail to be stationary, and so stationary phase gives a decay rate $O((-1/\log x)^\infty)$; this merely indicates that the contour has not been chosen optimally and a better result is possible.

This proposition already indicates that s plays a dual role, both as a coordinate on the space $\widetilde{X}$ and also as a spectral parameter. The function $\mu_1^0(s)$ identifies the front face of $\widetilde{X}$ with the line joining the two threshold values $k_1^2/4$ and $\mu - k_2^2/4$ in the spectral plane. This role becomes even more pronounced in determining asymptotics at the front face in the presence of bound states, for then the optimal locus for the contour γ determines which of the residue terms corresponding to the different eigenvalues must be included in the asymptotics. In addition, for certain real values of μ, there are a finite number of exceptional values of s at which $\mu_1^0(s)$

equals either λ_{1i} or $\mu - \lambda_{2i}$; then the optimal contour must pass through this pole, and this creates an additional singularity. We explain all of this now.

7.3. Asymptotics at $\partial\overline{M}_1 \times \partial\overline{M}_2$ in the presence of L^2 eigenvalues for $\mu \in \mathbb{C} \setminus \mathbb{R}$.

Thus suppose that H_1 or H_2 have L^2 eigenvalues. If $\mu \in \mathbb{R}$, the asymptotics of $R(\mu)f$ will have additional singularities compared to $\mu \notin \mathbb{R}$, so we first analyze $\mu \notin \mathbb{R}$. As we have already seen, these eigenvalues correspond to poles of the integrand in the integral representation of the resolvent and the residues at these poles may contribute to the asymptotics. We first make the following definition that divides up the μ-plane according to the phase function F.

Definition 7.9. Let

$$F(\mu_1) = F_s(\mu_1) = \sqrt{\mu_1 - k_1^2/4} + s\sqrt{\mu - \mu_1 - k_2^2/4}.$$

For each s, let $\mathcal{N}_s$ and $\mathcal{P}_s$ denote the regions in $\Lambda(\mu)$ where $\operatorname{Im} F(\mu_1) < \operatorname{Im} F(\mu_1^0(s))$ and $\operatorname{Im} F(\mu_1) > \operatorname{Im} F(\mu_1^0(s))$, respectively, where $\mu_1^0(s)$ is the unique critical point of F given by (7.12). The region $\mathcal{P}_s$ decomposes further into a union of two pieces, $\mathcal{P}_s^\ell \cup \mathcal{P}_s^r$, which have closures containing $\mu - k_2^2/4$ and $k_1^2/4$, respectively. (See Figure 4 for a picture.)

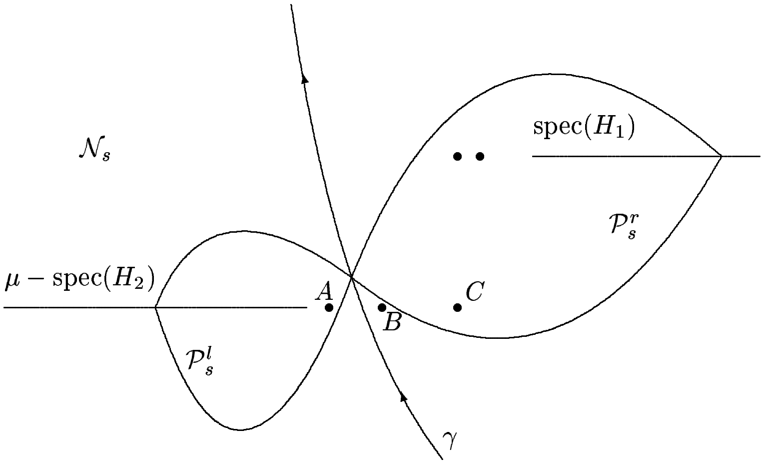

FIGURE 4. The regions $\mathcal{P}_s^\ell$, $\mathcal{P}_s^r$ and $\mathcal{N}_s$ for small $s > 0$. The points A, B and C denote $\mu - \lambda_{23}$, $\mu - \lambda_{22}$ and $\mu - \lambda_{21}$ respectively. The contour γ had to be shifted through point C, but not yet past A. The point B can made to lie on either side of γ; the residue there gives a smaller asymptotic term than the critical point $\mu_1(s)$ of F.

We prove the following result below.

Theorem 7.10. *Suppose* $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\mu \in \mathbb{C} \setminus \mathbb{R}$. *Let* ρ_i, ρ *and* s *be as above. Then* $R(\mu)f$ *has the following asymptotic expansion on* $\widetilde{X}$*:*

$$\begin{gathered}R(\mu)f = x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho)h \\ + \sum_{i=1}^{N_1} x_2^{k_2/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (\phi_{1i} \otimes h_{1i})\chi_{1i} + \sum_{i=1}^{N_2} x_1^{k_1/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (h_{2i} \otimes \phi_{2i})\chi_{2i},\end{gathered} \tag{7.17}$$

where h *has the properties stated in Proposition 7.7, and the other terms have the following properties.*

(i) $h_{1i} \in \mathcal{C}^\infty(\overline{M}_2)$, $h_{2i} \in \mathcal{C}^\infty(\overline{M}_1)$ *satisfy*

$$\begin{aligned}(\phi_{1i} \otimes h_{1i})|_{\overline{M}_1 \times \partial \overline{M}_2} &= -(\Pi_{1i} \otimes P_2^t(\mu - \lambda_{1i}))f, \ \textit{resp.} \\ (h_{2i} \otimes \phi_{2i})|_{\partial \overline{M}_1 \times \overline{M}_2}. &= -(P_1^t(\mu - \lambda_{2i}) \otimes \Pi_{2i})f.\end{aligned} \tag{7.18}$$

(ii) $\chi_{ji}(s) \in \mathcal{C}^\infty(\mathbb{R}^+)$ *vanishes for* s *large and equals* 1 *for* s *small, so that*
   (a) *If* $\lambda_{1i} \in \mathcal{P}_s^r$, *then* $\chi_{1i}(s) = 0$, *and if* $\lambda_{2i} \in \mathcal{P}_s^\ell$, *then* $\chi_{2i}(s) = 0$.
   (b) *If* $\lambda_{1i} \in \mathcal{P}_s^\ell$, *respectively if* $\lambda_{2i} \in \mathcal{P}_s^r$, *then* $\chi_{1i}(s) = 1$, *resp.* $\chi_{2i}(s) = 1$.
   (c) *If* $\lambda_{ji} \in \mathcal{N}_s$, *then the first term in (7.17) dominates the one corresponding to* ϕ_{ji}, *hence the choice of* χ_{ji} *is irrelevant.*

Proof. We first note that the additional terms coming from these residues (when shifting the contour) have the form

$$x_2^{k_2/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} \phi_{1i} \otimes \tilde{g}_{1i}, \ \tilde{g}_{1i} \in \mathcal{C}^\infty(\overline{M}_2), \tag{7.19}$$

with analogous terms when the roles of $\overline{M}_1$ and $\overline{M}_2$ have been switched. Here

$$\phi_{1i} \in x_1^{k_1/2 + i\sqrt{\lambda_{1i} - k_1^2/4}} \mathcal{C}^\infty(\overline{M}_1).$$

The first main observation is that the terms (7.19) dominate the previous one (7.15) coming from the continuous spectrum only when

$$\operatorname{Im}\left(\sqrt{\lambda_{1i} - k_1^2/4} + s\sqrt{\mu - \lambda_{1i} - k_2^2/4}\right) > \operatorname{Im}\left(\sqrt{\mu - k^2/4}\sqrt{1 + s^2}\right),$$

or in other words when

$$\operatorname{Im} F(\lambda_{1i}) > \operatorname{Im} F(\mu_1^0(s)). \tag{7.20}$$

In all other cases, these terms coming from the bound states are lower order in the expansion (7.15).

To analyze this, we must examine the geometry of the function F a bit more closely. Set

$$\Lambda(\mu) = \mathbb{C} \setminus \left([k_1^2/4, \infty) \cup (\mu - [k_2^2/4, \infty)\right),$$

and let $L \subset \Lambda(\mu)$ be the open line segment connecting the two threshold values $k_1^2/4$ and $\mu - k_2^2/4$; this segment is the image of the map $(0, \infty) \ni s \to \mu_1^0(s)$. The function F depends implicitly on both μ and s; we shall hold μ fixed throughout and study how F changes as s varies. We have already noted that $\operatorname{Im} F$ has a saddle point at $\mu_1^0(s)$, which is clear since nondegenerate critical points of harmonic functions are always saddle points.

As indicated above, the union of $\mathcal{N}_s$ (introduced above in Definition 7.9) with the slits has only one component, while $\mathcal{P}_s$ has two components, each of which has compact closure intersecting precisely one of the two slit axes and which intersect at

$\mu_1^0(s)$, and which are labeled $\mathcal{P}_s^\ell$ and $\mathcal{P}_s^r$, respectively, the ℓ and r denoting whether these regions touch the slits extending toward the left or right. Finally, recall that the 'optimal' contour of integration γ is any contour which remains entirely within the region $\mathcal{N}_s$, except when it passes through the saddle point.

Now consider the locations of the poles λ_{1i} and λ_{2i} relative to the contour γ and these regions $\mathcal{N}_s$ and $\mathcal{P}_s$. For any s, these two collections of poles are separated by the initial choice of contour γ in (2.1). However, in deforming γ to a γ' which lies entirely within $\mathcal{N}_s$ it may be necessary to shift past some of these poles. This is not necessary if all of the λ_{1i} lie within $\mathcal{N}_s \cup \mathcal{P}_s^r$ and all of the $\mu - \lambda_{2i}$ lie within $\mathcal{N}_s \cup \mathcal{P}_s^\ell$. When s is sufficiently small, then in fact all of the λ_{1i} lie within $\mathcal{N}_s \cup \mathcal{P}_s^r$; all of the $\mu - \lambda_{2i}$ lie in this region too, and none lie in $\mathcal{P}_s^\ell$. On the other hand, when $s \to \infty$, both the λ_{1i} and the λ_{2i} lie within $\mathcal{N}_s \cup \mathcal{P}_s^\ell$. As s increases from 0 to ∞, the region $\mathcal{P}_s^\ell$ gradually engulfs each of the points λ_{1i}, and so the optimal contour (which must always lie in $\mathcal{N}_s$) must cross these before this happens. A similar phenomenon occurs for the λ_{2i}. □

7.4. Asymptotics at $\partial\overline{M}_1 \times \partial\overline{M}_2$ in the presence of L^2 eigenvalues for $\mu \in \mathbb{R}$.

Remark 7.11. The reader should be cautioned that this section is the most technical. While the results here are certainly necessary for a complete understanding of the resolvent asymptotics, they may perhaps be omitted on first reading.

When $\mu \in \mathbb{R}$ and there are bound states in the interval $(\mu - k_2^2/4, k_1^2/4)$, i.e. when

$$\mu < \min\left\{k_1^2/4 - \max\{\lambda_{1i}\}, k_2^2/4 - \max\{\lambda_{2i}\}\right\},$$

then the preceeding analysis must fail for certain values of s, namely those values where either $\mu_1^0(s) = \lambda_{1i}$ or $\mu - \mu_1^0(s) = \lambda_{2i'}$ for some i, i'. For then an optimal contour would need to pass directly through a pole. It is clear that there must be some sort of jump in the behaviour at these points, because for nearby values of s this phenomenon does not occur and the previous analysis may be used to get the asymptotics; the residue at the pole would be included in these asymptotics if s varies slightly to one side, but are not included if it varies slightly to the other. We now analyze the precise form of the singularity of $R(\mu)f$ at the front face of $\widetilde{X}$ at those values of s where this occurs. This requires a series of blow-ups, culminating in the space (7.23). In order to motivate it, it is better to proceed with our calculations directly; we state the resulting theorem at the end of this subsection.

To be definite, suppose that s_0 satisfies $\mu_1^0(s_0) = \lambda_{1i}$ for some i. Let γ be the original contour defining $R(\mu)$, as in Figure 1, and let γ_s be an optimal contour for any value s near s_0. We may arrange that $\gamma_s(0) = \mu_1^0(s)$, so $\gamma_{s_0}(0) = \lambda_{1i}$. When $s > s_0$ (closer to the lift of $\overline{M}_1 \times \partial\overline{M}_2$) the contour has crossed past λ_{1i}, and the residue term is present, whereas when $s < s_0$ (closer to the lift of $\partial\overline{M}_1 \times \overline{M}_2$) this residue term is not included.

We proceed as follows. First consider what happens when $s \to s_0$ from the left, i.e. with $s \leq s_0$. Equation (7.15) describes the asymptotics when $s < s_0$, and we are interested in the behavior in the limit. Write

$$R_2(\mu - \mu_1) = R_2(\mu - \lambda_{1i}) + (\mu_1 - \lambda_{1i})\tilde{R}_2(\mu - \mu_1),$$

i.e. expand R_2 to first order in Taylor series in μ_1 around $\mu_1^0(s_0) = \lambda_{1i}$. Thus,

$$\begin{aligned}\tilde{R}_2(\mu - \mu_1) &= \frac{R_2(\mu - \mu_1) - R_2(\mu - \lambda_{1i})}{\mu_1 - \lambda_{1i}} \\ &= -\int_0^1 R_2'(\sigma(\mu - \mu_1) + (1 - \sigma)(\mu - \lambda_{1i}))\, d\sigma,\end{aligned} \tag{7.21}$$

and in particular, $\lim_{\mu_1 \to \lambda_{1i}} \tilde{R}_2(\mu - \mu_1) = -R_2'(\mu - \lambda_{1i})$.

Our first remark is that the contour in

$$\int_{\gamma_s} (R_1(\mu_1) \otimes R_2(\mu - \lambda_{1i}))\, f\, d\mu_1$$

can be shifted farther from the spectrum of H_1 since now the R_2 term is independent of μ_1, so its contribution is negligible. Hence we only need to analyze $\int_{\gamma_s} \Big(R_1(\mu_1)(\mu_1 - \lambda_{1i}) \otimes \tilde{R}_2(\mu - \mu_1)\Big)\, d\mu_1$. But

$$\tilde{R}_1(\mu_1) \otimes \tilde{R}_2(\mu - \mu_1) = R_1(\mu_1)(\mu_1 - \lambda_{1i}) \otimes \tilde{R}_2(\mu - \mu_1)$$

is holomorphic for μ_1 near λ_{1i}, so we may move the contour to go through λ_{1i} when $s = s_0$. Moreover, from the usual parametrix identity (3.3), using the holomorphy of $\tilde{R}_1(\mu_1) = (\mu_1 - \lambda_{1i})R_1(\mu_1)$ as a bounded operator on L^2 near $\mu_1 = \lambda_{1i}$, $(\mu_1 - \lambda_{1i})R_1(\mu_1)$ has the standard asymptotics, i.e. $\Big(R_1(\mu_1)(\mu_1 - \lambda_{1i}) \otimes \tilde{R}_2(\mu - \mu_1)\Big)\, f$ has the form stated in Lemma 7.1 for $R_{12} f$. Note that $\tilde{R}_1(\lambda_{1i}) = \Pi_{1i}$.

Thus, the holomorphic function $\Big(\tilde{R}_1(\mu_1) \otimes R_2(\mu_2)\Big)\, f$ has asymptotics

$$\Big(\tilde{R}_1(\mu_1) \otimes R_2(\mu_2)\Big)\, f = x_1^{k_1/2} x_2^{k_2/2} x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu_2 - k_2^2/4}} a(\mu_1, \mu_2),$$

where $a(\mu_1, \mu_2)$ is $\mathcal{C}^\infty$ on $U_{\mu_1} \times U_{\mu - \mu_2} \times \overline{M}_1 \times \overline{M}_2$, and holomorphic in μ_1 and μ_2, where U is a neighborhood of λ_{1i}. Hence

$$\begin{aligned}&\Big(\tilde{R}_1(\mu_1) \otimes \tilde{R}_2(\mu - \mu_1)\Big)\, f = \\ &\frac{\Big(\tilde{R}_1(\mu_1) \otimes R_2(\mu - \mu_1)\Big)\, f - \Big(\tilde{R}_1(\mu_1) \otimes R_2(\mu - \lambda_{1i})\Big)\, f}{\mu_1 - \lambda_{1i}} \\ &= x_1^{k_1/2} x_2^{k_1/2} \cdot (\mu_1 - \lambda_{1i})^{-1} \cdot \Big(x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \mu_1 - k_2^2/4}} a(\mu_1, \mu - \mu_1) \\ &\qquad - x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} a(\mu_1, \mu - \lambda_{1i}) \Big)\end{aligned}$$

Now add and subtract $x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \mu_1 - k_2^2/4}} a(\mu_1, \mu - \lambda_{1i})$ in the numerator of this fraction on the right, and note that the term involving the holomorphic function

$$\frac{a(\mu_1, \mu - \mu_1) - a(\mu_1, \mu - \lambda_{1i})}{\mu_1 - \lambda_{1i}}$$

has the same asymptotics as if λ_{1i} were not a pole, i.e. the asymptotics given in Proposition 7.7; thus we only need to consider

$$\int_{\gamma_s} \frac{x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \mu_1 - k_2^2/4}} - x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \lambda_{1i} - k_2^2/4}}}{\mu_1 - \lambda_{1i}}\, a(\mu_1, \mu - \lambda_{1i})\, d\mu_1.$$

We can also write $a(\mu_1, \mu - \lambda_{1i}) = a(\lambda_{1i}, \mu - \lambda_{1i}) + (\mu_1 - \lambda_{1i})a_1(\mu_1, \mu - \lambda_{1i})$, where a_1 is holomorphic in μ_1 and smooth on $\overline{M}_1 \times \overline{M}_2$; the term involving a_1 again yields an expression with the same asymptotics as in Proposition 7.7, i.e. as if λ_{1i} were not a pole. Hence we reduce to

$$a(\lambda_{1i}, \mu - \lambda_{1i}) \int_{\gamma_s} \frac{x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \mu_1 - k_2^2/4}} - x_1^{i\sqrt{\mu_1 - k_1^2/4}} x_2^{i\sqrt{\mu - \lambda_{1i} - k_2^2/4}}}{\mu_1 - \lambda_{1i}} \, d\mu_1.$$

This depends on f only via

$$\begin{aligned} a(\lambda_{1i}, \mu - \lambda_{1i}) &= -x_1^{-k_1/2 - i\sqrt{\lambda_{1i} - k_1^2/4}} x_2^{-k_2/2 - i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (\Pi_{1i} \otimes R_2(\mu - \lambda_{1i}))f \\ &= -x_1^{-k_1/2 - i\sqrt{\lambda_{1i} - k_1^2/4}} (\Pi_{1i} \otimes P_2^t(\mu - \lambda_{1i}))f. \end{aligned}$$

The factor $x_1^{-i\sqrt{\lambda_{1i} - k_1^2/4}}$ here corresponds to the precise decay of the eigenfunction ϕ_{1i}. The remaining integral is an explicit function which depends only on s, x_1 and x_2 (and of course μ). Note that in the region $x_2 > 0$, it is simply a smooth function of s, and x_2 (or equivalently, s and x_1, keeping in mind that $s = \log x_2 / \log x_1$ is near $s_0 \in (0, +\infty)$. Thus we only need to understand the asymptotics of this function as $x_1, x_2 \to 0$.

Factor out $x_1^{iF(\mu_1)} = x_1^{iF(\mu_1^0(s))} x_1^{i(F(\mu_1) - F(\mu_1^0(s)))}$ to get

(7.22)
$$x_1^{iF(\mu_1^0(s))} \int_{\gamma_s} e^{-i(F(\mu_1) - F(\mu_1^0(s)))/\rho_1} \frac{1 - e^{-is(\sqrt{\mu - \lambda_{1i} - k_2^2/4} - \sqrt{\mu - \mu_1 - k_2^2/4})/\rho_1}}{\mu_1 - \lambda_{1i}} \, d\mu_1,$$

where we wrote the integrand in terms of $\rho_1 = -1/\log x_1 \geq 0$ to keep the signs easier to follow. Note that $\operatorname{Im}(F(\mu_1) - F(\mu_1^0(s))) \leq 0$ along γ_s.

So far *any* contour γ_s that stays in $\mathcal{N}_s$ has been suitable for our calculations. Now we impose an additional condition, namely that there exists a fixed interval $[-\epsilon, \epsilon]$ such that for $s_0 - \delta \leq s \leq s_0$ γ_s stays in the region where $\operatorname{Im} \sqrt{\mu - \mu_1 - k_2^2/4} \geq \operatorname{Im} \sqrt{\mu - \lambda_{1i} - k_2^2/4}$. Since this region is one side of a parabola, see Figure 3, this condition can be easily satisfied. This condition ensures that the real part of the exponent in the numerator in (7.22) is non-positive, hence bounded even as $\rho_1 \to 0$. In addition, the fraction which is the final factor in the integrand in (7.22) is bounded by $C\rho_1^{-1}$ for some $C > 0$, since dividing the denominator by ρ_1 yields a bounded function.

The key point is that

$$F(\mu_1) - F(\mu_1^0(s)) = (\mu_1 - \mu_1^0(s))^2 F_2(\mu_1 - \mu_1^0(s), \mu_1^0(s)),$$

with F_2 non-zero at $(0, \mu_1^0(s))$, since $\mu_1^0(s)$ is a non-degenerate critical point of F, while

$$\sqrt{\mu - \lambda_{1i} - k_2^2/4} - \sqrt{\mu - \mu_1 - k_2^2/4} = (\mu_1 - \lambda_{1i}) F_1(\mu_1 - \lambda_{1i})$$

with F_1 nonvanishing and holomorphic near 0. This implies that the first exponent is a smooth function of $(\mu_1 - \mu_1^0(s))^2/\rho_1$, with a negative real part of the same order of magnitude as $\rho_1 \to 0$, hence Schwartz as $(\mu_1 - \mu_1^0(s))^2/\rho_1 \to \infty$, and the second one is a smooth function of $(\mu_1 - \lambda_{1i})/\rho_1$ with a negative real part. However, we can only make the conclusion about being Schwartz here if the real part is of the same order of magnitude as $(\mu_1 - \lambda_{1i})/\rho_1$. Since γ_s is on one side of a parabola, this is impossible to accomplish if γ_s is to be smooth (for then it is tangent to the parabola). Hence, we need to break up γ_s into two integrals which will yield boundary terms. These boundary terms cancel when $s \neq s_0$.

The geometry is as follows. Start with the space $Z_0 = [0,1)_{\rho_1} \times (s_0 - \delta, s_0]_s \times \mathbb{R}_t$. We first blow up $t = 0$, $\rho_1 = 0$ parabolically (recall that t is the parameter along γ_s; $\gamma_s(0) = \mu_1^0(s)$), so that in the interior of the front face s, $t/\rho_1^{1/2}$ and ρ_1 become valid coordinates (see Figure 5). Then $(F(\mu_1) - F(\mu_1^0(s)))/\rho_1$ is smooth near the interior of the front face, and $e^{i(F(\mu_1)-F(\mu_1^0(s)))/\rho_1}$ is smooth on the blown up space with infinite order vanishing off the front face. Away from $s = s_0$, $e^{-is(\sqrt{\mu-\lambda_{1i}-k_2^2/4}-\sqrt{\mu-\mu_1-k_2^2/4})/\rho_1}$ is rapidly decreasing, hence the standard push-forward theorem for polyhomogeneous functions [23] yields the same result as stationary phase. In fact, we should think of integrating densities, hence we need to change $d\mu_1$, i.e. dt, to $\rho^{1/2}d(t/\rho^{1/2})$, giving the stationary phase asymptotics $\rho^{1/2}$ times a smooth function of $(s, \rho^{1/2})$ (multiplied by the exponential $x_1^{iF(\mu_1^0(s))}$ that we took outside the integral in (7.22)), though the terms of the form ρ times a smooth function of ρ and s in fact cancel (these are the boundary terms of the integration). This is an alternative way of thinking about complex stationary phase when the real part of the exponent is non-positive and at least comparable to or larger than the imaginary part, as suggested to us by Richard Melrose. It simplifies our previous arguments here, although unfortunately it does not directly apply when considering asymptotics at the continuous spectrum, as we do in the next section.

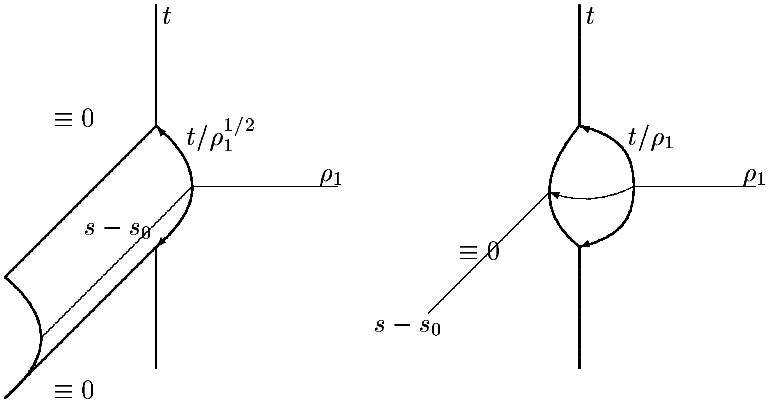

FIGURE 5. The space Z_0 blown up at $t = 0$, $\rho_1 = 0$ parabolically is shown on the left, with $\equiv 0$ denoting the boundary hypersurfaces where $e^{i(F(\mu_1)-F(\mu_1^0(s)))/\rho_1}$ is rapidly decreasing, i.e. where $t/\rho_1^{1/2} \to \pm\infty$. The space $[Z_0; F_1']$ is shown on the right, with $\equiv 0$ denoting where $e^{-is(\sqrt{\mu-\lambda_{1i}-k_2^2/4}-\sqrt{\mu-\mu_1-k_2^2/4})/\rho_1}$ is rapidly decreasing, hence second factor of the integrand of (7.22) is $(\mu - \lambda_{1i})^{-1}$ up to a rapidly decreasing term. The 'equator' of the quarter sphere corresponds to $\rho_1 = 0$, $t/\rho_1 = 0$, i.e. $(s - s_0)/\rho_1$ is the variable along it.

At $s = s_0$ we also need to resolve the geometry of the second factor in the integrand of (7.22). Had we not performed the parabolic blow-up above, it would suffice to blow up

$$F_1' = \{(\rho_1, s, t) :\ t = 0,\ \rho_1 = 0,\ s = s_0\} \subset Z_0$$

in the usual spherical (homogeneous) sense. Indeed, the exponent in the second factor as well as the denominator are smooth functions of $(\mu_1 - \lambda_{1i})/\rho_1$, hence of t/ρ_1 and $(s - s_0)/\rho_1$, and the exponential is rapidly decreasing as $|t/\rho_1| + |(s - s_0)/\rho_1| \to \infty$ since

$$\operatorname{Im}(\sqrt{\mu - \lambda_{1i} - k_2^2/4} - \sqrt{\mu - \mu_1 - k_2^2/4}) \leq -C'|\mu_1 - \lambda_{1i}| \leq -C(|t| + |s - s_0|).$$

Thus, the second factor is polyhomogeneous on this blown up space, of order -1 on the front face, order 0 off the front face.

Unfortunately these two blow ups, i.e. the parabolic one of $t = 0$, $\rho_1 = 0$, and the spherical one of $t = 0$, $\rho_1 = 0$, $s = s_0$, conflict with each other, and we need to find a common resolution. One common resolution is the following: we blow up the boundary $\rho_1 = 0$ of Z_0 by introducing $\hat{\rho} = \rho_1^{1/2}$ as our new boundary defining function. We write $(Z_0)_{1/2}$ for the blown up space. The spherical blow-ups of

$$F_1 = \{(\hat{\rho}, s, t) :\ \hat{\rho} = 0,\ t = 0\} \text{ and } F_2 = \{(\hat{\rho}, s, t) :\ \hat{\rho} = 0,\ t = 0,\ s = s_0\}$$

commute with each other since F_2 is a p-submanifold of F_1, so $[(Z_0)_{1/2}; F_1, F_2] = [(Z_0)_{1/2}; F_2, F_1]$. Let ff_j denote the front face of the blow-up of F_j, $j = 1, 2$. Near the interior of the front face of the second blow-up, i.e. ff_2, $t/\hat{\rho}$, $(s - s_0)/\hat{\rho}$ and $\hat{\rho}$ are valid coordinates. Now blow up the submanifold

$$F_3 = \{\hat{\rho}, (s - s_0)/\hat{\rho}, t/\hat{\rho}) :\ t/\hat{\rho} = 0,\ (s - s_0)/\hat{\rho} = 0,\ \hat{\rho} = 0\}$$

(which is a single point) to obtain

$$Z = [(Z_0)_{1/2}; F_1, F_2, F_3];$$

see Figure 6. This introduces, in particular, the front face of the spherical blow up of

$$F_1' = \{(\rho_1, t, s) :\ t = 0,\ \rho_1 = 0,\ s = s_0\} \subset Z_0$$

in Z_0, except that the boundary defining functions differ. In other words, if we blow up the front face of $[Z_0; F_1']$ to admit $\rho_1^{1/2}$ as a smooth function, then a neighborhood of the interior of the front face is naturally diffeomorphic to a neighborhood of the interior of the front face ff_3 of the blow-up of F_3. This can be seen explicitly since local coordinates which are valid in this region are given by $(t/\hat{\rho})/\hat{\rho} = t/\rho$, $((s - s_0)/\hat{\rho})/\hat{\rho} = (s - s_0)/\rho$ and $\hat{\rho} = \rho^{1/2}$. Since upon blowing up F_2, the lifts of F_1 and F_3 are disjoint, their blow-ups commute. Thus, we can rewrite Z as $[(Z_0)_{1/2}; F_2, F_3, F_1]$. Since the first factor of the integrand of (7.22) is polyhomogeneous on $[(Z_0)^{1/2}; F_1]$, while the second factor is polyhomogeneous on $[(Z_0)_{1/2}; F_2, F_3]$, we deduce that the integrand is (one-step) polyhomogeneous on Z.

While the geometry of the integrand only requires these blow-ups, we need further blow-ups to create a b-fibration for the push-forward. Unfortunately, with the blow-ups discussed above, all we can hope for is to create a b-fibration with base given by a double blow-up of $(Y_0)_{1/2} = [0, 1)_{\hat{\rho}} \times (s_0 - \delta, s_0]$ (which is already a blow-up, in the sense of change of the $\mathcal{C}^\infty$ structure, of $Y_0 = [0, 1)_{\rho_1} \times (s_0 - \delta, s_0]$). Namely, one first blows up

$$G_2 = \{(\hat{\rho}, s) :\ s = s_0,\ \hat{\rho} = 0\},$$

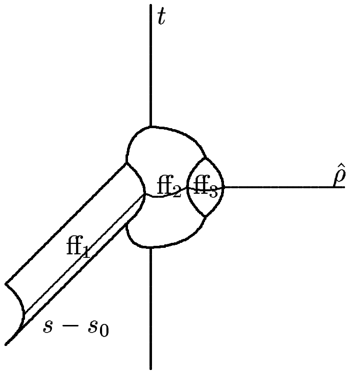

FIGURE 6. The space $Z = [(Z_0)_{1/2}; F_1, F_2, F_3]$, with ff_j denoting the front face of the blow-up of F_j. The space Y can be seen on the picture as the intersection of the figure with $t = 0$, with contour lines given by the 'equatorial lines' on each of the ff_j as well as the $\hat\rho$ axis.

in $(Y_0)_{1/2}$, and then

$$G_3 = \{(\hat\rho, s) : \ (s - s_0)/\hat\rho = 0, \ \hat\rho = 0\}$$

in $[(Y_0)_{1/2}; G_2]$; we denote this space by

$$Y = [(Y_0)_{1/2}; G_2, G_3]. \tag{7.23}$$

It is then straight-forward to carry out appropriate blow-ups in our resolved space Z to obtain a new space Z' such that $Z' \to Y$ is a b-fibration. Hence we can apply the push-forward theorem of [23], and deduce that the integral of (7.22) yields a polyhomogeneous function on Y. More specifically, the result is a one-step polyhomogeneous function on Y (keep in mind that $\hat\rho$ is the boundary defining function!) with order 1 on the lift of $\hat\rho = 0$, and order 0 on each of the two new front faces. Indeed, while in this set-up one would also expect logarithmic terms, they vanish since writing the integrand of (7.22) as the sum of an even and an odd function of t, they would arise from the odd part (as an indefinite integral), which however cancels in the definite integral over $\mathbb{R}$.

The asymptotics from $s \geq s_0$, $s \to s_0$, can be seen similarly. The optimal contours in $s > s_0$ have been shifted through λ_{1i}, so there is automatically a contribution from the pole. We are interested in letting s decrease to s_0. Since in this region the real part of the exponent of $x_1^{i\sqrt{\mu_1 - k_1^2/4}}$ in the expansion of $R_1(\mu_1)$ is less than that of $x_1^{i\sqrt{\lambda_{1i} - k_1^2/4}}$, we now need to expand an operator associated to H_1 rather than one associated to H_2 around λ_{1i}. This is a little more delicate, as we discuss below. The parametrix identity gives

$$R_1(\mu_1) = P_1(\mu_1) - E(\mu_1)P_1(\mu_1) + E(\mu_1)R_1(\mu_1)F(\mu_1),$$

with all terms meromorphic. The first two terms are actually holomorphic, so we can shift the integral to the optimal location, and even let it go through λ_{1i}. Thus, we only need to consider the last term. Here the kernel of $E(\mu_1)$ is polyhomogeneous on the standard double space $\overline{M}_1 \times \overline{M}_1$, with rapid vanishing on the right face. We expand the holomorphic function $E_1(\mu_1)$ as

$$E(\mu_1) = E_1(\lambda_{1i}) + (\mu_1 - \lambda_{1i})\tilde{E}_1(\mu_1)$$

similarly to the expansion for $R_2(\mu - \mu_1)$ above. The contour of the integral $\int_{\gamma_s} E_1(\lambda_{1i})R_1(\mu_1)F(\mu_1) \otimes R_2(\mu - \mu_1)f\,d\mu_1$ can be shifted farther from $\operatorname{spec}(H_1)$, hence it is negligible compared to the other terms. So it remains to deal with

$$\int_{\gamma_s} \tilde{E}_1(\lambda_{1i})(\mu_1 - \lambda_{1i})R_1(\mu_1)F(\mu_1) \otimes R_2(\mu - \mu_1)f\,d\mu_1.$$

But the integrand above again has the standard asymptotics by virtue of the holomorphy of $(\mu_1 - \lambda_{1i})R_1(\mu_1)$, hence completely analogous calculation apply as for $s \leq s_0$ above.

We have introduced these new front faces in Y_0 in order to understand the asymptotics in terms of polyhomogeneous expansions. We should certainly discuss whether this complicated geometry is simply an artifact of our method, or whether it is necessary in the sense that polyhomogeneous asymptotics do not hold in a simpler space. We certainly need at least one of the front faces. Indeed, the residue term is order 0 in $s > s_0$, and is missing in $s < s_0$, where the asymptotic expansion is order 1 (in terms of $\rho_1^{1/2}$). Thus, it is possible to obtain polyhomogeneous asymptotics through $s = s_0$ only if there is some blow-up, and hence a new boundary face, along which the term of order 0 tends to zero on approach to the lift of $s < s_0$. In addition, the exponential in the second factor is smooth only after F_1' is blown up in Z_0, and so it appears that the spherical blow-up of $s = s_0$, $\rho_1 = 0$ in Y_0, or at least the presence of the front face of the last blow-up in $[(Y_0)_{1/2}; G_2; G_3]$, is necessary. It then remains to see whether the integral (7.22) is polyhomogeneous on $[Y_0; \{(\rho_1, s) :\ \rho_1 = 0,\ s = s_0\}]$. If it is, it must be order 0 on the front face and order 1/2 on the lift of $s < s_0$. Correspondingly it has order at most 1/2 on the blow-up of the corner, which is the interior of the front face $[(Y_0)_{1/2}; G_2]$, although with a different boundary defining function. Nonetheless, we would expect to see decay in the asymptotics at the front face of $[(Y_0)_{1/2}; G_2]$ away from G_3, and there is no such decay. Indeed, the exponential in the second factor of the integrand in (7.22) is rapidly decreasing in the inverse image of this region under the projection, so it can be disregarded. Thus the dominant term as $\rho_1 \to 0$ is an integral of the form $\int_{\gamma_s} e^{a\mu_1^2/\rho_1}(\mu_1 - \lambda_{1i})^{-1}\,d\mu_1$, $a = -iF''(\mu_1^0(s))/2 > 0$. Shifting the contour to be vertical, the integral becomes $i\int_{\mathbb{R}} e^{-at^2/\rho_1}(it + (\mu_1^0(s) - \lambda_{1i}))^{-1}\,dt$, with imaginary part

$$\int_{\mathbb{R}} e^{-at^2/\rho_1} \frac{\mu_1^0(s) - \lambda_{1i}}{t^2 + (\mu_1^0(s) - \lambda_{1i}))^2}\,dt.$$

Thus, with $\mu_1^0(s) - \lambda_{1i} = \rho_1^{1/2}S$, $S < 0$, and changing variables $T = t/\rho_1^{1/2}$,

$$\int_{\mathbb{R}} e^{-aT^2} \frac{S}{T^2 + S^2}\,dT < 0,$$

independent of $\rho_1 > 0$. Hence there is no decay as $\rho_1 \to 0$. The limit $S \to 0$ can also be analyzed; indeed, it is the distribution $(T + i0)^{-1}$ paired with e^{-aT^2}. The limit from $s > s_0$ is similar, but now we get $(T - i0)^{-1}$ paired with e^{-aT^2}. The difference, $2\pi i$, is exactly the residue corresponding to the pole.

Note moreover that the principal symbol of $R(\mu)f$ in $s < s_0$ is a non-vanishing f-independent multiple of $P_1^t(\mu_1^0(s)) \otimes P_2^t(\mu - \mu_1^0(s))f$, which in turn is the restriction of

$$x_1^{-k_1/2-i\sqrt{\mu_1^0(s)-k_1^2/4}} \left(R_1(\mu_1^0(s)) \otimes P_2^t(\mu - \mu_1^0(s))\right) f$$

to $x_1 = 0$. That is, the top term of the asymptotics of $R(\mu)f$, up to non-vanishing smooth factors and after factoring out $x_1^{k_1/2} x_2^{k_2/2} e^{-i\sqrt{\mu - k^2/4}/\rho}$, is

$$\rho_1^{1/2} x_1^{-k_1/2-i\sqrt{\mu_1^0(s)-k_1^2/4}} \left(R_1(\mu_1^0(s)) \otimes P_2^t(\mu - \mu_1^0(s))\right) f,$$

which behaves as $\frac{\rho_1^{1/2}}{\mu_1^0(s)-\lambda_{1i}} x_1^{-k_1/2-i\sqrt{\mu_1^0(s)-k_1^2/4}} \left(\Pi_{1i} \otimes P_2^t(\mu - \mu_1^0(s))\right) f$ as $s \to s_0$ (up to lower order terms). We have thus explicitly matched the coefficients from the two sides of the corner.

Altogether, we have proved the following theorem.

Theorem 7.12. *Suppose* $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\mu \in \mathbb{R} \setminus \operatorname{spec}(H)$. *Let* $\rho_i = -1/\log x_i$, $\rho^{-1} = \sqrt{\rho_1^{-2} + \rho_2^{-2}}$, $s = \rho_1/\rho_2$. *Then* $R(\mu)f$ *has the following asymptotic expansion on* $\widetilde{X}$*:*

(7.24)
$$R(\mu)f = x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho) h$$
$$+ \sum_{i=1}^{N_1} x_2^{k_2/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}} (\phi_{1i} \otimes h_{1i})\chi_{1i} + \sum_{i=1}^{N_2} x_1^{k_1/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}} (h_{2i} \otimes \phi_{2i})\chi_{2i},$$

where the terms have the following properties.

(i) $h_{1i} \in \mathcal{C}^\infty(\overline{M}_2)$, $h_{2i} \in \mathcal{C}^\infty(\overline{M}_1)$ *satisfy*

$$(7.25)\qquad \begin{aligned} (\phi_{1i} \otimes h_{1i})|_{\overline{M}_1 \times \partial\overline{M}_2} &= -(\Pi_{1i} \otimes P_2^t(\mu - \lambda_{1i}))f, \ \textit{resp.} \\ (h_{2i} \otimes \phi_{2i})|_{\partial\overline{M}_1 \times \overline{M}_2}. &= -(P_1^t(\mu - \lambda_{2i}) \otimes \Pi_{2i})f. \end{aligned}$$

(ii) χ_{1i} *is a* $\mathcal{C}^\infty$ *function in the variable* $S = (s - s_0)/\rho$, *where* s_0 *is given by*

$$(7.26)\qquad \lambda_{1i} = \mu_1^0(s_0) = \frac{\mu - k_2^2/4 + s_0^2 k_1^2/4}{1 + s_0^2},$$

with χ_{1i}' *compactly supported,* $\lim_{S\to+\infty} \chi_{1i}(S) = 1$, *and* $\lim_{S\to-\infty} \chi_{1i}(S) = 0$. *Similarly,* χ_{2i} *is a* $\mathcal{C}^\infty$ *function of* $S = (s - s_0)/\rho$, s_0 *given by*

$$(7.27)\qquad \lambda_{2i} = \mu - \mu_1^0(s_0) = \frac{s_0^2(\mu - k_1^2/4) + k_2^2/4}{1 + s_0^2},$$

χ_{2i}' *is compactly supported,* $\lim_{S\to+\infty} \chi_{2i}(S) = 0$, *and* $\lim_{S\to-\infty} \chi_{1i}(S) = 1$.

(iii) g *is polyhomogeneous on* $\widetilde{X}$ *doubly blown up at the finite number of submanifolds* $s = s_0$, $\rho_1 = 0$, *of the front face of* $\widetilde{X}$, *as in* (7.23), *with* s_0 *given by either of the equations* (7.26)-(7.27). *The order at the front faces of the blow-ups is* 0, *while the order on the old front face is still* $1/2$. *The principal symbol on the front faces (i.e. the restriction of the leading term of* h *to these) depends on* f *only via* $(\Pi_{1i} \otimes P_2^t(\mu - \lambda_{1i}))f$, *resp.* $(P_1^t(\mu - \lambda_{2i}) \otimes \Pi_{2i})f$*; it is indeed a non-vanishing multiple of these.*

8. Asymptotics inside the continuous spectrum

The arguments needed to analyze the asymptotics of $R(\mu \pm i0)f$ when $f \in \dot{\mathcal{C}}^\infty(X)$ and $\mu > k^2/4$ are not substantially different, and in many ways simpler because the contributions from the L^2 eigenfunctions below the continuous spectrum are always negligible away from the side-faces. Since threshold eigenvalues introduce additional complications similar to the discussion of the previous section, with the additional issue that the push-forward results of [23] are not directly applicable since the stationary phase arguments have a substantially different flavor when the phase is pure imaginary, we will assume that $k_j^2/4$ is not a pole of R_j. Thus, we have the following theorem.

Theorem 8.1. *Suppose* $f \in \dot{\mathcal{C}}^\infty(\overline{X})$, $\mu > k^2/4 = (k_1^2 + k_2^2)/4$, *and* $k_j^2/4$ *is not a pole of* R_j, $j = 1, 2$. *Then, with* $\rho_i = -1/\log x_i$, $\rho^{-1} = \sqrt{\rho_1^{-2} + \rho_2^{-2}}$, $R(\mu - i0)f$ *has the following asymptotic expansion on* $\widetilde{X}$*:*

$$\begin{aligned} R(\mu - i0)f &= x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho) h \\ &+ \sum_{i=1}^{N_1} x_2^{k_2/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (\phi_{1i} \otimes h_{1i}) + \sum_{i=1}^{N_2} x_1^{k_1/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (h_{2i} \otimes \phi_{2i}), \end{aligned} \tag{8.1}$$

where h *is one-step polyhomogeneous on* $\widetilde{X}$, *and* h_{1i} *and* h_{2i} *are one-step polyhomogeneous on* $\bar{M}_2$ *and* $\bar{M}_1$ *respectively. Moreover,* h *has order* $1/2$ *on the front face and* $3/2$ *on the side faces, while the* h_{1i} *is in* $\mathcal{C}^\infty(\overline{M}_2)$ *(i.e. has order* 0 *at* $\partial\overline{M}_2$*) and* $h_{2i} \in \mathcal{C}^\infty(\overline{M}_1)$*. The principal symbol (i.e. restriction of the leading term to the boundary) of* h *is given by* (7.11).

Remark 8.2. Note that due to the exponential decay of $x_1^{-k_1/2}\phi_{1i}$ for non-threshold eigenvalues λ_{1i}, namely

$$x_1^{-k_1/2}\phi_{1i} \in x_1^{\sqrt{k_1^2/4 - \lambda_{1i}}} \mathcal{C}^\infty(\bar{M}_1), \ k_1^2/4 - \lambda_{1i} > 0,$$

and the similar decay of $x_2^{-k_2/2}\phi_{2i}$, the second term is dominated by the first one everywhere but at the lift of $\bar{M}_1 \times \partial\bar{M}_2$, and similarly the third term is dominated by the first one everywhere but at the lift of $\partial\bar{M}_1 \times \bar{M}_2$. In particular, on a neigborhood U of the lift of $\bar{M}_1 \times \partial\bar{M}_2$, one has the asymptotics

$$R(\mu)f|_U = x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho) g + \sum_{i=1}^{N_1} x_2^{k_2/2 + i\sqrt{\mu - \lambda_{1i} - k_2^2/4}} (\phi_{1i} \otimes g_{1i}),$$

while on a neighborhood U' of the front face which is disjoint from $\partial\bar{M}_1 \times \bar{M}_2$ and $\bar{M}_1 \times \partial\bar{M}_2$ both the second and third terms are irrelevant:

$$R(\mu)f|_{U'} = x_1^{k_1/2} x_2^{k_2/2} \exp(-i\sqrt{\mu - k^2/4}/\rho) g.$$

Remark 8.3. The asymptotics is unchanged away from the corners of $\widetilde{X}$ even if $k_j^2/4$ is a pole of R_j, $j = 1, 2$, in the sense that these do not contribute to the asymptotics in the interior of the front face of $\widetilde{X}$, and give the standard contribution in each appropriate side face. This follows from stationary phase arguments using the integral representation (7.21). In fact, we can always reduce the calculation to the evaluation of an integral like (7.22), though here it is convenient to rewrite the fraction appearing in that formula as an integral analogous to (7.21).

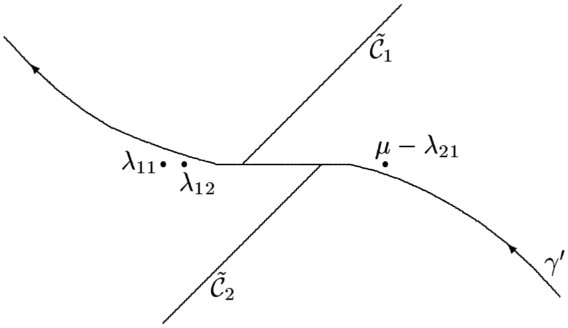

FIGURE 7. Contour of integration γ' to describe $R(\mu)$ inside the main branch of the continuous spectrum. The cuts $\mathcal{C}_1$ and $\mathcal{C}_2$, corresponding to the spectra of H_1 and H_2, respectively, have been rotated to the cuts $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$, as in Figure 2, to make the picture clearer, although the analytic continuation is not needed for our proof.

Proof. This is simply a limiting case of the argument of the previous section, but which has additional simplifying features. Thus, we simply consider a smooth contour γ' that runs along the real axis on the interval $[k_1^2/4, \mu - k_2^2/4]$, but avoids the poles of R_1 as well as μ minus the poles of R_2, see Figure 7. Such a contour is simply the limit of the contours γ' that we have considered before. Note, in particular, that with

$$F(\mu_1) = \sqrt{\mu_1 - k_1^2/4} + s\sqrt{\mu - \mu_1 - k_2^2/4}, \qquad s = \frac{\rho_1}{\rho_2}$$

as in the previous section, $\operatorname{Im} F(\mu_1) \le 0$ everywhere, and it is strictly negative when $\mu_1 \notin [k_1^2/4, \mu - k_2^2/4]$. Since the critical point of F lies in $[k_1^2/4, \mu - k_2^2/4]$, the choice of the contour outside this interval is irrelevant (except that it should be far from the real axis near infinity, as before, to ensure convergence). In particular, $\lambda_{ji} \in \mathcal{N}_s$, for all $s \in [0, +\infty)$, except if $s = 0$, when $\operatorname{Im} F(\mu - \lambda_{2i}) = 0$ (but $\operatorname{Im} F(\lambda_{1i}) < 0$); of course a similar statement holds at $s^{-1} = 0$ with λ_{2i} replaced by λ_{1i}. Thus, the residues of the poles of R_1 and R_2 also provide lower order contributions than the stationary phase term in most regions in $\widetilde{X}$, except that the poles of R_1 give the same order as the stationary phase term at $M_1 \times \partial \overline{M}_2$, and similarly for R_2. The standard stationary phase lemma now gives the desired results, much as in the previous section. To deal with the points $\mu_1 = k_1^2/4$ and $\mu_1 = \mu - k_2^2/4$, i.e. the endpoints of $[k_1^2/4, \mu - k_2^2/4]$, one introduces $\tau_1 = \sqrt{\mu_1 - k_1^2/4}$, resp. $\tau_2 = \sqrt{\mu - \mu_1 - k_2^2/4}$, just as in the previous section, and notes that in $0 \le s \le C$, the phase with respect to τ_1 is never stationary, while with respect to τ_2 it is only stationary at $s = 0$, hence the left end point gives a rapidly decreasing contribution in this region, while the right end point gives asymptotically non-trivial terms only at $s = 0$, exactly as expected. □

9. The resolvent compactification

We now turn to a problem originally posited as one of our main goals, namely to define a resolution $\overline{X}^2_{\mathrm{res}}$ of X^2, which we call the resolvent double space (or perhaps more suitably, the full product-0 double space); this is a manifold with corners on which $R(\mu)$ is polyhomogeneous when $\mu \notin \operatorname{spec}(H)$, and its existence sets the stage for all further development of the analytic properties of the Laplacian, including the scattering theory, on X.

To set this into context, recall from §5 that we have already defined the product-0 double space $\overline{X}^2_{\mathrm{p0}}$, and have used it to define the small calculus of product-0 pseudodifferential operators. We know from the analysis in §7 that $R(\mu)$ has nontrivial asymptotic behavior on the side faces of this double-space, i.e. at hypersurface boundaries which do not meet diag. However, it is not hard to see that the Schwartz kernel of $R(\mu)$ is not polyhomogeneous at the codimension two boundaries of $\overline{X}^2_{\mathrm{p0}}$, and so we must perform further blowups to resolve the singularities completely. These blowups all occur away from the front faces of $\overline{X}^2_{\mathrm{p0}}$, and so the elements of the small calculus $\Psi^*_{\mathrm{p0}}(X)$ could equally well have been characterized as having Schwartz kernels on this larger double space with conormal singularities along diag (extendible across the front faces), and which vanish to infinite order at all side faces. We have defined the small calculus on the smaller double space for simplicity only.

Recall also that in principle the resolvent should be obtained constructively on $\overline{X}^2_{\mathrm{res}}$, but since we have an alternate means to study $R(\mu)$ (i.e. via its contour integral representation), our goal here is reduced to showing that this operator is actually polyhomogeneous on $\overline{X}^2_{\mathrm{res}}$.

We start with the decomposition $R(\mu) = R'(\mu) + R''(\mu)$ from Proposition 5.3. Since we have already described the Schwartz kernel of $R'(\mu)$ precisely as a distribution on $\overline{X}^2_{\mathrm{p0}}$, let us turn to the other component. We know that $R''(\mu)$ can be written as a contour integral with an integrand as in Lemma 7.1 which is the product of powers of boundary defining functions and a function which is smooth on $\overline{X}^2_{\mathrm{p0}}$. The arguments of § 7 only use this much of the structure given by Lemma 7.1, and hence can be applied with $\tilde{R}_{12}$ from Proposition 5.3 replacing $R_{12}f$ in Lemma 7.1. Thus we must show how to interpret these results geometrically.

From §7 it is clear that in defining $\overline{X}^2_{\mathrm{res}}$ we must incorporate two main features: the exponential type defining functions ρ_{lf_j} and ρ_{rf_j} for the side faces of $(\overline{M}_j)^2_0$ must be replaced by the polynomial type defining functions

$$\mathcal{R}_{\mathrm{lf}_j} = -1/\log \rho_{\mathrm{lf}_j}, \qquad \mathcal{R}_{\mathrm{rf}_j} = -1/\log \rho_{\mathrm{rf}_j}, \qquad j = 1, 2;$$

moreover all pairwise quotients of these new defining functions, i.e. $\mathcal{R}_{\mathrm{lf}_1}/\mathcal{R}_{\mathrm{lf}_2}$ etc., must be smooth on this new double space, at least in the region where they are bounded.

Thus first define the partial logarithmic blow up $(\overline{M}_j)^2_{0,\log}$ of the 0-double space $(\overline{M}_j)^2_0$ by requiring that $\mathcal{R}_{\mathrm{lf}_j}$ and $\mathcal{R}_{\mathrm{rf}_j}$ be smooth on it, $j = 1, 2$. Note that we are *not* logarithmically blowing up the front face ff_j of $(\overline{M}_j)^2_0$. The product $(\overline{M}_1)^2_{0,\log} \times (\overline{M}_2)^2_{0,\log}$ has six boundary hypersurfaces: the two front faces

$$\mathrm{ff}_1 \times (\overline{M}_2)^2_{0,\log}, \qquad \text{and} \qquad (\overline{M}_1)^2_{0,\log} \times \mathrm{ff}_2,$$

and the four side faces

$$\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}, \quad \mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}, \quad (\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2, \quad (\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2 .$$

There are six codimension 2 corners coming from pairwise intersections of these side faces, and to make the ratios of defining functions for these side faces smooth at these corners we must blow up all of these submanifolds. Unfortunately, these corners do not all meet transversely, so we proceed in two stages. First blow up the two corners

$$(\mathrm{lf}_1 \cap \mathrm{rf}_1) \times (\overline{M}_2)^2_{0,\log}, \ (\overline{M}_1)^2_{0,\log} \times (\mathrm{lf}_2 \cap \mathrm{rf}_2). \tag{9.1}$$

This may be done in either order since these submanifolds are transverse. The effect of this step is that the remaining corners now lift to be disjoint. So now blow up the remaining collection of submanifolds, which cover

$$\begin{aligned} &(\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2), \ (\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2), \\ &(\mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2), \ (\mathrm{rf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{rf}_2). \end{aligned} \tag{9.2}$$

Again, because they are disjoint, this may be done in any order.

Definition 9.1. The resolvent double space $\overline{X}^2_{\mathrm{res}}$ is the manifold with corners obtained from the product-0 double space $\overline{X}^2_{\mathrm{p0}}$ by performing these two sequences of blow-ups, as just described.

The following lemma shows that this construction succeeds with the two crucial features we wished the resolvent double space to have.

Lemma 9.2. *The function*

$$S = (\log \rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})/(\log \rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1}), \tag{9.3}$$

and its inverse S^{-1} lift to be smooth functions on the portions of $\overline{X}^2_{\mathrm{res}}$ on which they are bounded. Similarly, the function

$$\mathcal{R} = \left((\mathcal{R}_{\mathrm{lf}_1}^{-1} + \mathcal{R}_{\mathrm{rf}_1}^{-1})^2 + (\mathcal{R}_{\mathrm{lf}_2}^{-1} + \mathcal{R}_{\mathrm{rf}_2}^{-1})^2\right)^{-1/2}$$

also lifts to be smooth on $\overline{X}^2_{\mathrm{res}}$. In fact, $\mathcal{R}$ is a product of defining functions for all the hypersurface boundaries of $\overline{X}^2_{\mathrm{res}}$ which cover the side faces of $\overline{X}^2_{\mathrm{p0}}$ or which arise from blowing up intersections of these side faces.

Proof. Let us consider the behaviour of S first. We may certainly assume that $\rho_{\mathrm{lf}_j}, \rho_{\mathrm{rf}_j} \leq 1/2$. Now write

$$S = \frac{\log(\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})}{\log(\rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1})} = \frac{\log \rho_{\mathrm{lf}_2}}{\log \rho_{\mathrm{lf}_1} + \log \rho_{\mathrm{rf}_1}} + \frac{\log \rho_{\mathrm{rf}_2}}{\log \rho_{\mathrm{lf}_1} + \log \rho_{\mathrm{rf}_1}}, \tag{9.4}$$

which is equal to

$$\frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{lf}_2}} \frac{\mathcal{R}_{\mathrm{rf}_1}}{\mathcal{R}_{\mathrm{lf}_1} + \mathcal{R}_{\mathrm{rf}_1}} + \frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{rf}_2}} \frac{\mathcal{R}_{\mathrm{rf}_1}}{\mathcal{R}_{\mathrm{lf}_1} + \mathcal{R}_{\mathrm{rf}_1}}. \tag{9.5}$$

Both terms in this last expression are positive, and hence their sum is bounded if and only if both terms are. Hence, it suffices to show that all of the quotients

$$\mathcal{R}_{\mathrm{lf}_i}/\mathcal{R}_{\mathrm{lf}_j}, \quad \mathcal{R}_{\mathrm{lf}_i}/\mathcal{R}_{\mathrm{rf}_j}, \quad \mathcal{R}_{\mathrm{rf}_i}/\mathcal{R}_{\mathrm{rf}_j} \qquad i, j = 1, 2,$$

and their inverses are smooth (when bounded) on $\overline{X}^2_{\mathrm{res}}$.

In fact, working in a region where (9.4) is bounded, we must show that both terms in (9.5) are smooth. But

$$\frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{lf}_2}}\frac{\mathcal{R}_{\mathrm{rf}_1}}{\mathcal{R}_{\mathrm{lf}_1}+\mathcal{R}_{\mathrm{rf}_1}}=\frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{lf}_2}}\frac{1}{(\mathcal{R}_{\mathrm{lf}_1}/\mathcal{R}_{\mathrm{rf}_1})+1},$$

with a similar expression for the other term. The blowup (9.1) makes $\mathcal{R}_{\mathrm{rf}_j}/\mathcal{R}_{\mathrm{lf}_j}$ and its inverse smooth where bounded, while the prefactors $\mathcal{R}_{\mathrm{lf}_1}/\mathcal{R}_{\mathrm{lf}_2}$ and $\mathcal{R}_{\mathrm{lf}_1}/\mathcal{R}_{\mathrm{rf}_2}$ are also smooth because of the second set of blowups (9.2).

As for the function $\mathcal{R}$, first note that

$$\mathcal{R}_j=\left(\mathcal{R}_{\mathrm{lf}_j}^{-2}+\mathcal{R}_{\mathrm{rf}_j}^{-2}\right)^{-1/2}=\frac{\mathcal{R}_{\mathrm{lf}_j}\mathcal{R}_{\mathrm{rf}_j}}{\sqrt{\mathcal{R}_{\mathrm{lf}_j}^2+\mathcal{R}_{\mathrm{rf}_j}^2}}.$$

This function lifts to be smooth on $[(\overline{M}_j)^2_0;\mathrm{lf}_j\cap\mathrm{rf}_j]$, i.e. the blowup of the intersection of side faces in $(\overline{M}_j)^2_0$, and in fact it is equal to a product of defining functions for all faces except the front face ff_j. Now it is easy to see that

$$\mathcal{R}=\frac{\mathcal{R}_1\mathcal{R}_2}{\sqrt{\mathcal{R}_1^2+\mathcal{R}_2^2}}$$

lifts smoothly to $\overline{X}^2_{\mathrm{res}}$ and is a product of defining functions over all non-front faces. This finishes the proof. □

Remark 9.3. It is also illuminating to consider the special case when X is a product of hyperbolic spaces. In fact, first recall the expression (3.6) for the distance function between pairs of points $\delta_j(z_j,z_j')$ on $M_j=\mathbb{H}^{k_j+1}$. Thus,

$$\rho_{\mathrm{lf}_j}\rho_{\mathrm{rf}_j}=e^{-\delta_j(z_j,z_j')}\ \text{in, say, } \delta_j(z_j,z_j')\geq 1;$$

the restriction on $\delta_j(z_j,z_j')$ only plays a role because δ_j is not smooth on the diagonal while $\rho_{\mathrm{lf}_j}\rho_{\mathrm{rf}_j}$ is. We thus have

$$S=\frac{\delta_2(z_2,z_2')}{\delta_1(z_1,z_1')}$$

and also

$$\mathcal{R}=\frac{1}{\sqrt{\delta_1(z_1,z_1')^2+\delta_2(z_2,z_2')^2}}=\frac{1}{\delta(z,z')}.$$

This shows that we could have predicted the correct blowup $\overline{X}^2_{\mathrm{res}}$ by considering only the behaviour of distance functions on X.

Theorem 9.4. *Suppose that neither* H_1 *nor* H_2 *have* L^2 *eigenvalues and let* $\mu\in\mathbb{C}\setminus\mathrm{spec}(H)$. *Then the Schwartz kernel of* $R(\mu)$ *takes the following form:*

$$\begin{aligned}&R(\mu)=R'(\mu)+R''(\mu),\ R'(\mu)\in\Psi^{-2}_{\mathrm{p0}}(\overline{X}),\\ &R''(\mu)=(\rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1})^{k_1/2}(\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{k_2/2}\exp(-i\sqrt{\mu-k^2/4}/\mathcal{R})F(\mu),\end{aligned}\tag{9.6}$$

Here $F(\mu)$ *is polyhomogeneous on* $\overline{X}^2_{\mathrm{res}}$, *with leading asymptotic order* 0 *on the lift of the two front faces* $\mathrm{ff}_1\times(\overline{M}_2)^2_{0,\log}$, $(\overline{M}_1)^2_{0,\log}\times\mathrm{ff}_2$, *and leading asymptotic order* 3/2 *on the lift of the four side faces* $\mathrm{lf}_1\times(\overline{M}_2)^2_0$, $\mathrm{rf}_1\times(\overline{M}_2)^2_0$, $(\overline{M}_1)^2_0\times\mathrm{lf}_2$, $(\overline{M}_1)^2_0\times\mathrm{rf}_2$, *as well as the front faces of the two blow-ups* (9.1), *and order* 1/2 *on* *the front faces of the four blow-ups* (9.2)*. Moreover, the restriction of the leading term of* $F(\mu)$ *to the boundary, i.e. its principal symbol, is*

$$\begin{aligned} &a(\mu,s)\,\big(P_1^t(\mu_1^0(S))\otimes P_2^t(\mu-\mu_1^0(S))\big)\ \textit{on the front face of the blow-up of}\\ &\qquad\qquad(\mathrm{lf}_1\times(\overline{M}_2)^2_{0,\log})\cap((\overline{M}_1)^2_{0,\log}\times\mathrm{lf}_2),\\ &a'(\mu)\,\big(S_1(k_1^2/4)\otimes P_2^t(\mu-k_2^2/4)\big)\ \textit{on the lift of}\ (\overline{M}_1)^2_0\times\mathrm{lf}_2,\\ &a''(\mu)\,\big(P_1^t(\mu-k_1^2/4)\otimes S_2(k_2^2/4)\big)\ \textit{on the lift of}\ \mathrm{lf}_1\times(\overline{M}_2)^2_0,\end{aligned}\tag{9.7}$$

with $\mu_1^0(S)$ *given by*

$$\mu_1^0(S)=\frac{\mu-k_2^2/4+S^2k_1^2/4}{1+S^2},\tag{9.8}$$

a,a',a'' *never zero,* S *as in* (9.3)*.*

If H_1 *or* H_2 *has* L^2 *eigenvalues, but* μ *is not real, then*

$$\begin{aligned} &R(\mu)=R'(\mu)+R''(\mu),\ \ R'(\mu)\in\Psi^{-2}_{\mathrm{p0}}(\overline{X}),\\ &R''(\mu)=(\rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1})^{k_1/2}(\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{k_2/2}\exp(-i\sqrt{\mu-k^2/4}/\mathcal{R})F(\mu)\\ &\qquad-\sum_{i=1}^{N_1}(\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{k_2/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}}(\Pi_{1i}\otimes R_2(\mu-\lambda_{1i}))\chi_{1i}\\ &\qquad-\sum_{i=1}^{N_2}(\rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1})^{k_1/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}}(R_1(\mu-\lambda_{2i})\otimes\Pi_{2i})\chi_{2i},\end{aligned}\tag{9.9}$$

$F(\mu)$ *is polyhomogeneous on* $\overline{X}^2_{\mathrm{res}}$ *as above, and* χ_{ji} *have the same properties as in Theorem 7.10, i.e.* $\chi_{ji}(S)\in\mathcal{C}^\infty(\mathbb{R}^+)$*, vanishes when* S *is large and is identically* 1 *when* S *is small, so that*

(i) *If* $\lambda_{1i}\in\mathcal{P}^r_S$*, then* $\chi_{1i}(S)=0$*, and if* $\lambda_{2i}\in\mathcal{P}^\ell_S$*, then* $\chi_{2i}(S)=0$*.*
(ii) *If* $\lambda_{1i}\in\mathcal{P}^\ell_S$*, respectively if* $\lambda_{2i}\in\mathcal{P}^r_S$*, then* $\chi_{1i}(S)=1$*, resp.* $\chi_{2i}(S)=1$*.*
(iii) *If* $\lambda_{ji}\in\mathcal{N}_S$*, then the first term in (7.24) dominates the one corresponding to* ϕ_{ji}*, hence the choice of* χ_{ji} *is irrelevant.*

Remark 9.5. While this full statement is rather complicated, only certain parts are relevant for many applications, such as determination of the mapping properties of $R(\mu)$ on weighted L^2 spaces. An immediate corollary, which is much simpler to state and which often suffices, is that the pushforward of $R''(\mu)$ to the simple product space $\overline{X}^2$ is of the form $(x_1x_1')^{k_1/2}(x_2x_2')^{k_2/2}e^{-i\sqrt{\mu-k^2/4}/\mathcal{R}}\tilde{F}$, where $\tilde{F}(z,z')$ is bounded on $\overline{X}^2$. There is an obvious analogue of this for Theorem 9.8 below.

Proof. As alluded to above, we use the decomposition $R(\mu)=R'(\mu)+R''(\mu)$ of Proposition 5.3. Since $R'(\mu)$ satisfies the statement of the theorem, it remains to analyze $R''(\mu)$.

Now, $\tilde{R}_{12}(\mu_1,\mu_2)$ is the product of a smooth function on $\overline{X}^2_{\mathrm{p0}}$ with

$$\prod_{j=1}^2(\rho_{\mathrm{lf}_j}\rho_{\mathrm{rf}_j})^{k_j/2+i\sqrt{\mu_j-k_j^2/4}};$$

in particular, it is smooth up to the ff_j. It is thus straightforward to see that $R''(\mu)$ is smooth at the two 'front faces' of the product, $\mathrm{ff}_1\times(\overline{M}_2)^2_0$ and $(\overline{M}_1)^2_0\times\mathrm{ff}_2$. This is because the exponents in the expansions of $\tilde{R}_{12}(\mu_1, \mu - \mu_1)$ at these faces do not depend on μ_1. It remains to analyze the behaviour at the remaining 'side faces'.

In fact, as mentioned above the computations of the last section give the asymptotics of $R''(\mu)$ at $(\mathrm{lf}_1 \times (\overline{M}_2)^2_0) \cup ((\overline{M}_1)^2_0 \times \mathrm{lf}_2)$, and in particular at the corner $(\mathrm{lf}_1 \times (\overline{M}_2)^2_0) \cap ((\overline{M}_1)^2_0 \times \mathrm{lf}_2)$. To proceed further, the salient observation is that while those stationary phase computations were motivated by the geometric picture of letting the variable $(z_1, z_2) \in M_1 \times M_2$ converge to the boundary (in one of three ways), they can also be interpreted purely analytically as a derivation of asymptotics when integrating a function of the form $x_1^{\alpha_1} x_2^{\alpha_2} F$, where α_1 and α_2 are the appropriate exponents depending on μ and μ_1 and F depends smoothly on these variables. What we are asking now is the asymptotics of an integral of almost exactly the same form, but where each x_j is replaced by $\rho_{\mathrm{lf}_j} \rho_{\mathrm{rf}_j}$.

Now perform the same calculations as in the last section. It follows that the asymptotics of $R''(\mu)$ is polyhomogeneous on $(\overline{M}_1)^2_{0,\log} \times (\overline{M}_2)^2_{0,\log}$ in terms of the new defining functions $\mathcal{R}_{\mathrm{lf}_j}$, $\mathcal{R}_{\mathrm{rf}_j}$, provided we also adjoin

$$S = (\log \rho_{\mathrm{lf}_2} \rho_{\mathrm{rf}_2}) / (\log \rho_{\mathrm{lf}_1} \rho_{\mathrm{rf}_1})$$

to the smooth structure, when it is bounded, and similarly for its inverse. By the preceeding lemma, this function is smooth on $\overline{X}^2_{\mathrm{res}}$ when it is bounded, and similarly for its inverse, finishing the proof. □

To conclude this discussion, suppose that either H_1 or H_2 have bound states and μ is real. Then $R(\mu)$ will have additional singularities at certain submanifolds of $\overline{X}^2_{\mathrm{res}}$ where $\log(\rho_{\mathrm{lf}_2} \rho_{\mathrm{rf}_2}) / \log(\rho_{\mathrm{lf}_1} \rho_{\mathrm{rf}_1})$ assumes the same critical values s_0 as already appeared for $s = \log x_2 / \log x_1$ in part (iii) of Theorem 7.12. That is, these singularities happen at the submanifolds given by $S = s_0$ *inside* $\partial \overline{X}^2_{\mathrm{res}}$.

Lemma 9.6. *For $s_0 \neq 0$, the set given by $S = s_0$ is a codimension one p-submanifold of $\overline{X}^2_{\mathrm{res}}$. In particular, its its intersection with $\partial \overline{X}^2_{\mathrm{res}}$ is a finite union of codimension one p-submanifolds of $\partial \overline{X}^2_{\mathrm{res}}$.*

Proof. We first rewrite S in terms of coordinates on $\overline{X}^2_{\mathrm{res}}$ as discussed above. Thus,

$$S = \left(\frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{lf}_2}} + \frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{rf}_2}} \right) \frac{1}{(\mathcal{R}_{\mathrm{lf}_1} / \mathcal{R}_{\mathrm{rf}_1}) + 1} = (\sigma_{l_1 l_2} + \sigma_{l_1 r_2}) \frac{1}{\sigma_{l_1 r_1} + 1},$$

in the region where $\sigma_{l_1 l_2} = \frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{lf}_2}}$, $\sigma_{l_1 r_2} = \frac{\mathcal{R}_{\mathrm{lf}_1}}{\mathcal{R}_{\mathrm{rf}_2}}$ and $\sigma_{l_1 r_1} = \mathcal{R}_{\mathrm{lf}_1} / \mathcal{R}_{\mathrm{rf}_1}$ are valid coordinates, i.e. where they are bounded. Since the second factor is thus bounded from both below and from above, we conclude that at least one of $\sigma_{l_1 l_2}$ and $\sigma_{l_1 r_2}$ is non-zero at any point on $S = s_0$, and in addition $\partial_{\sigma_{l_1 l_2}} S \neq 0$, $\partial_{\sigma_{l_1 r_2}} S \neq 0$ there, so $S = s_0$ is a codimension one p-submanifold of $\overline{X}^2_{\mathrm{res}}$. □

We can thus blow up $S = s_0$ as on the single space, and make the following definition.

Definition 9.7. We let $(\overline{X}^2_{\mathrm{res}})'$ be the double blow up of $S = s_0$ inside $\partial \overline{X}^2_{\mathrm{res}}$, preceeded by the square root blow up of the boundary defining function.

Now the same proof as the one leading to Theorem 7.12 applies and yields the following result.

Theorem 9.8. *If H_1 and H_2 have L^2 eigenvalues and $\mu \in \mathbb{R} \setminus \mathrm{spec}(H)$ then, with the notation of Theorem 9.4,*

$$\begin{aligned} R(\mu) &= R'(\mu) + R''(\mu),\ \ R'(\mu) \in \Psi^{-2}_{\mathrm{p0}}(\overline{X}), \\ R''(\mu) &= (\rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1})^{k_1/2}(\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{k_2/2}\exp(-i\sqrt{\mu - k^2/4}/\mathcal{R})F(\mu) \\ &\qquad - \sum_{i=1}^{N_1} (\rho_{\mathrm{lf}_2}\rho_{\mathrm{rf}_2})^{k_2/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}}(\Pi_{1i} \otimes R_2(\mu - \lambda_{1i}))\chi_{1i} \\ &\qquad - \sum_{i=1}^{N_2} (\rho_{\mathrm{lf}_1}\rho_{\mathrm{rf}_1})^{k_1/2+i\sqrt{\mu-\lambda_{1i}-k_2^2/4}}(R_1(\mu - \lambda_{2i}) \otimes \Pi_{2i})\chi_{2i}, \end{aligned} \tag{9.10}$$

$F(\mu)$ is polyhomogeneous on $(\overline{X}^2_{\mathrm{res}})'$ and χ_{1i}, χ_{2i} are functions of $(S - s_0)/\mathcal{R}$ as in Theorem 7.12, s_0 given by $\mu^0_1(s_0) = \lambda_{1i}$ or $\mu - \mu^0_1(s_0) = \lambda_{2i}$. The principal symbol of F on the new front faces arising from the blow up corresponding to $S = s_0$ inside the lift of $(\mathrm{lf}_1 \times (\overline{M}_2)^2_{0,\log}) \cap ((\overline{M}_1)^2_{0,\log} \times \mathrm{lf}_2)$ to $\overline{X}^2_{\mathrm{res}}$ is given by a non-vanishing multiple of $\Pi_{1i} \otimes P^t_2(\mu - \lambda_{1i})$, resp. $P^t_1(\mu - \lambda_{2i}) \otimes \Pi_{2i}$ analogously to Theorem 7.12.

While these blow ups may appear somewhat complicated, the only relevant part of $\overline{X}^2_{\mathrm{res}}$ as far as the Martin boundary is concerned, is the inverse image U' of regions $X \times U$, $\overline{U}$ compact subset of X, $U = U_1 \times U_2 \subset \overline{M}_1 \times \overline{M}_2$ open, under the blow-down map $\beta : \overline{X}^2_{\mathrm{res}} \to X^2$. Thus, in the second factor we always stay away from the boundary, hence any blow-ups involving rf_1, rf_2 can be neglected. Thus, the intersection of U' with $\overline{X}^2_{\mathrm{res}}$ is a subset of

$$[(\overline{M}_1)_{\log} \times U_1 \times (\overline{M}_2)_{\log} \times U_2; \partial(\overline{M}_1)_{\log} \times U_1 \times \partial(\overline{M}_2)_{\log} \times U_2]$$

which is essentially the same as $\widetilde{X} \times U_1 \times U_2$ (i.e. they agree up to the rearranging of factors). Hence, in this set the asymptotics of the kernel of $R(\mu)$ is given by (9.7). In other words, it is essentially given by Theorem 7.12, with an extra variable on U added, and references to f dropped (this really corresponds to applying the resolvent to delta distributions at points z and letting z vary).

10. The Martin compactification: the proof of Theorem 1.2

With the information we have collected and proved about the resolvent $R(\mu)$, it is now a simple matter to determine the full Martin compactification of $X = M_1 \times M_2$ and prove Theorem 1.2. We refer back to §6 for details of how this construction is carried out in general, but we briefly recall the main ideas. Fix any $\mu \in \mathbb{R}$, $\mu < \inf \mathrm{spec}(H)$. In addition, fix $p \in X$ and let $w^{(\ell)} = (w^{(\ell)}_1, w^{(\ell)}_2) \in X$ be any sequence which leaves every compact set. Define

$$U^{(\ell)}(z) = \frac{R(\mu; z, w^{(\ell)})}{R(\mu; p, w^{(\ell)})};$$

this function is a solution of $(H - \mu)U^\ell = 0$ for $z \neq w^{(\ell)}$ and is normalized so that $U^{(\ell)}(p) = 1$. Any such sequence has a subsequence $U^{\ell'}$ which converges in the $\mathcal{C}^\infty$ topology on compact sets to a function U which satisfies $(H - \mu)U = 0$ on all of X and $U(p) = 1$. The Martin boundary $\overline{X}_M$ is defined to be the set of all possible limit eigenfunctions that arise in this manner.

Rephrasing our goals slightly, we consider the function

$$U(\mu,z,w)=\frac{R(\mu;z,w)}{R(\mu;p,w)},$$

initially only on the interior $X_z\times X_w$, and wish to show that it extends continuously to $X_z\times\widetilde{X}_w$. For then the limits of the various functions $U^{(\ell)}(z)$ above simply correspond to letting $w^{(\ell)}$ converge to any boundary point of $\widetilde{X}_w$.

First consider the case when neither H_1 nor H_2 has L^2 eigenvalues. In this case, by Theorem 9.4, the restriction of $R(\mu)$ to $X\times\widetilde{X}$ is given by

$$(x_1')^{k_1/2}(x_2')^{k_2/2}\exp(-i\sqrt{\mu-k^2/4}/\mathcal{R}_2)\,\mathcal{R}_2^{1/2}\mathcal{R}_{\mathrm{lf}_2}\mathcal{R}_{\mathrm{rf}_2}\,g, \tag{10.1}$$

where $g\in\mathcal{C}^\infty(X\times\widetilde{X})$ and has restriction to each face determined by (9.7), and we have switched to using the newer notation for the defining functions for the faces of $\widetilde{X}$, i.e. have replaced ρ's by $\mathcal{R}$'s. (Also, x_j' are defining functions for each of the two factors M_j in X_w.) This now determines a map

$$\begin{gathered}\partial\widetilde{X}\longrightarrow\partial\overline{X}_M,\\ \partial\widetilde{X}\ni w\longmapsto U(\mu,\cdot,w)\in\mathcal{C}^\infty(X_z),\end{gathered}\tag{10.2}$$

which is continuous and surjective. More explicitly, if $w^{(\ell)}$ is any sequence of points in X converging to $w\in\partial\widetilde{X}$, the continuity of U implies that $U^{(\ell)}(z)=U(\mu,z,w)$, hence $U(\mu,.,w)$ is a point in the Martin boundary. Conversely, suppose that $w^{(\ell)}$ is any sequence of points in X which leaves every compact subset of X, and such that $U^{(\ell)}$ converges uniformly on compact subsets of X. By passing to a subsequence $w^{(\ell')}$ we may assume that $w^{(\ell')}$ converges to some $w\in\partial\widetilde{X}$ since $\widetilde{X}$ is compact. Hence $\lim U^{(\ell')}(z)=U(\mu,w,z)$, so the map is indeed surjective.

By (9.7), U is given by

$$\frac{S_1(k_1^2/4;z_1,w_1)P_2^t(\mu-k_2^2/4;z_2,w_2)}{S_1(k_1^2/4;p_1,w_1)P_2^t(\mu-k_2^2/4;p_2,w_2)},\ (w_1,w_2)\in\overline{M}_1\times\partial\overline{M}_2,$$

$$\frac{P_1^t(\mu-k_1^2/4;z_1,w_1)S_2(k_2^2/4;z_2,w_2)}{P_1^t(\mu-k_1^2/4;p_1,w_1)S_2(k_2^2/4;p_2,w_2)},\ (w_1,w_2)\in\partial\overline{M}_1\times\overline{M}_2.$$

and

$$\frac{P_1^t(\mu_1^0(s);z_1,w_1)P_2^t(\mu-\mu_1^0(s);z_2,w_2)}{P_1^t(\mu_1^0(s);p_1,w_1)P_2^t(\mu-\mu_1^0(s);p_2,w_2)},\ (s,w_1,w_2)\in[0,+\infty)_s\times\partial\overline{M}_1\times\partial\overline{M}_2,$$

Thus, the injectivity of the map (10.2) depends on whether these are all different elements of $\mathcal{C}^\infty(X_z)$ as w varies. It is easy to see that the restriction of this map to the front face is indeed injective since $P_j^t(\nu;z_j,w_j)$ has a pole when $z_j\to w_j$ and is otherwise bounded, and so the points w_1 and w_2 are uniquely identified, as is the eigenparameter $\mu_1^0(s)$ and hence s itself. On the other hand, it is not necessarily true that the generalized spherical function $S_j(k_j^2/4;z_j,w_j^0)$ determines the point w_j^0. In case M_j is hyperbolic space, this is holds because of the rotational symmetry around w_j^0, and so must also hold for conformally compact metrics (globally) near to the hyperbolic one. In general, these portions of the Martin boundary are identified with $\mathcal{I}_1\times\overline{M}_2$ and $\overline{M}_1\times\mathcal{I}_2$, where

$$\mathcal{I}_j=\mathrm{image}\big(w_j^0\longrightarrow S_j(k_j^2/4;z_j,w_j^0)\big).$$

Next suppose that H_1 has L^2 eigenvalues. The asymptotics of the resolvent kernel in $X \times \widetilde{X}$ are now given by Theorem 9.8. Let λ_{11} be the bottom eigenvalue of H_1 which is hence simple, and let s_0 be the corresponding value of $s = \rho_1/\rho_2$ given by (7.26). In the region $s < s_0$, in particular near $X \times \partial\overline{M}_1 \times \overline{M}_2$, $R(\mu)$ has the same asymptotics as before, while on the front face of the double blow up of $s = s_0$ as well as in $s > s_0$ the leading part of the asymptotics is given by non-vanishing z-independent ($z \in X$) multiples of $x_1^{-k_1/2-i\sqrt{\lambda_{1i}-k_1^2/4}}\Pi_{11} \otimes P_2^t(\mu - \lambda_{1i})$. We again set

$$U(\mu, z, w) = \frac{R(\mu; z, w)}{R(\mu; p, w)},$$

and once again, U is again continuous on $X \times \widetilde{X}$. Now, however, $w \mapsto U(\mu, \cdot, w)$ factors through the map $\partial\widetilde{X} \to \partial\widetilde{X}_c$, where $\partial\widetilde{X}_c$ is the collapsed boundary of $\widetilde{X}$ given by

$$\partial\widetilde{X}_c = (\partial\overline{M}_1 \times \overline{M}_2) \cup ([0, s_0)_s \times \partial\overline{M}_1 \times \partial\overline{M}_2) \cup \partial\overline{M}_2,$$

and the collapse map $\partial\widetilde{X} \to \partial\widetilde{X}_c$ is the identity on the first two sets on the right hand side and is the projection $[s_0, +\infty) \times \partial\overline{M}_1 \times \partial\overline{M}_2 \to \partial\overline{M}_2$, resp. $\overline{M}_1 \times \partial\overline{M}_2 \to \partial\overline{M}_2$ in the other parts of $\partial\widetilde{X}$. Thus, we obtain a continuous surjection from $\partial\widetilde{X}_c$ to $\partial\overline{X}_M$.

Again, U can be written down explicitly. It is the same as in the case without L^2 eigenfunctions when $s < s_0$, and it is

$$\frac{P_2^t(\mu - k_2^2/4; z_2, w_2)}{P_2^t(\mu - k_2^2/4; p_2, w_2)},\ w_2 \in \partial\overline{M}_2.$$

Thus, the map $\partial\widetilde{X}_c \to \partial\overline{X}_M$ is injective over the collapsed part of the boundary, as can be seen by letting $z_2 \to w_2$, and otherwise we are exactly in the same situation as in the case where there are no L^2 eigenfunctions.

Similar arguments work if H_2 has eigenvalues, or even if both H_1 and H_2 do. In the latter case $\widetilde{X}$ collapses near both side faces. This finishes the proof of Theorem 1.2.

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