Ph.D. Qualifying Exam, Real Analysis
September 2006, part I

Do all the problems.

1. Let $G$ be an unbounded open set in $(0, \infty)$. Define

$$D = \{x \in (0, \infty) : nx \in G \text{ for infinitely many } n\}.$$ 

Prove that $D$ is dense in $(0, \infty)$.

2. Suppose that $f \in L^1([0, 1])$ but $f \notin L^2([0, 1])$. Find a complete orthonormal basis $\{\phi_n\}$ for $L^2([0, 1])$ such that each $\phi_n \in C^0([0, 1])$ and such that

$$\int_0^1 f(x)\phi_n(x) \, dx = 0 \quad \forall n.$$ 

3. Let $H_1$ and $H_2$ be two separable Hilbert spaces. Suppose that $A : H_1 \to H_2$ is a continuous injective linear map. Suppose that $\{v_j\}$ is a bounded sequence in $H_1$ such that $Av_j$ converges strongly in $H_2$ to some element $w$. Prove that there exists an element $v \in H_1$ such that $v_j$ converges weakly to $v$ and $Av = w$.

4. Some problems about linear functionals:
   a. Let $T : C^0([0, 1]) \to \mathbb{C}$ be defined by $T(f) = f(1/2)$. Is $T$ continuous with respect to the $L^2$ norm? Explain why or why not.
   b. Let $P$ denote the set of polynomials of arbitrary degree, and consider their restrictions to $[0, 1]$ so as to consider $P \subset C^0([0, 1])$. Define the linear functional on $P$, $T_k(f) = a_k$ where $a_k$ is the coefficient of $x^k$. Does $T_k$ extend as a continuous linear functional to all of $C^0([0, 1])$?
      Hint: Consider $f_n(x) = (1 - x)^n, n \geq k$.

5. Let $Q = [0, 1] \times [0, 1]$ and denote by $X$ be the set of all closed nonempty subsets of $Q$. Define

$$d(A, B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where for any $C \in X$, $C_\delta = \{x \in Q : \text{dist } (x, C) < \delta\}$. Prove that $(X, d)$ is a compact metric space.

Hints: First prove that the subset of elements $A \in X$ where $A$ is finite is dense in $X$. Next, if $\{A_n\}$ is a decreasing nested sequence of closed subsets of $Q$, $\cap A_n = A$, prove that $A_n \to A$. Finally, if $\{B_n\}$ is an arbitrary Cauchy sequence in $(X, d)$, consider $A_n = \cup_{k \geq n} A_k$. 
Ph.D. Qualifying Exam, Real Analysis
September 2006, part II

Do all the problems.

1. Let $H$ be a Hilbert space with an orthogonal decomposition into finite dimensional subspaces, $H = \bigoplus_{j=1}^{\infty} H_j$. Thus each $v \in H$ can be written uniquely as $v = \sum_{j=1}^{\infty} v_j$ with $v_j \in H_j$. Let $c = (c_1, c_2, \ldots)$ where each $c_j > 0$, and define the subset

$$A_c = \{ v : ||v_j|| \leq c_j \} \subset H.$$

a. Prove that $c \in \ell^2$ if and only if $A_c$ is compact in $H$.

b. Prove that every compact subset $K \subset H$ is contained in some $A_c$ for some $c \in \ell^2$.

2. Let $\mu$ be a finite measure on $\mathbb{R}$. Define its Fourier transform $\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} d\mu(x)$. Prove that

$$|\mu(\{x\})| \leq \limsup_{|\xi| \to \infty} |\hat{\mu}(\xi)|.$$

3. Suppose that $p, q, r \in [1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Prove that for every $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, their convolution is an element of $L^r(\mathbb{R})$, and that

$$\int_{-\infty}^{\infty} |f \ast g|^r \ dx \leq \left( \int_{-\infty}^{\infty} |f|^p \ dx \right)^{\frac{r}{p}} \left( \int_{-\infty}^{\infty} |g|^q \ dx \right)^{\frac{r}{q}}.$$

Hint: Use interpolation.

4. Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function such that $g(x+1) = g(x)$ for every $x$. Define

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} g(2^k x).$$

Show that there exists $A > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq A |x - y| \log |x - y|.$$

Hint: If $2^{-n-1} \leq |x - y| \leq 2^{-n}$, divide the series for $f$ into two parts.

5. Suppose that $f$ is a function from the natural numbers $\mathbb{N}$ to $\mathbb{R}^+$ such that for every value of $n, m \in \mathbb{N}$, $f(n+m) \leq f(n) + f(m)$. Prove that

$$\lim_{n \to \infty} \frac{f(n)}{n}$$

exists and equals

$$\inf_{n>0} \frac{f(n)}{n}.$$