Dean Baskin

Qual Problems, Fall 2006

Fall 2006 Qual, Part I: 1, 2, 3, 4; Part II: 1

I.1 Let G be an unbounded open set in $(0, \infty)$. Define

$$D = \{x \in (0, \infty) : nx \in G \text{ for infinitely many } n\}.$$

Prove that D is dense in $(0, \infty)$.

Consider the sets

$$D_k = \{x \in (0, \infty) : nx \in G \text{ for some } n > k\}.$$

We claim that D_k is open and dense for each k.

That D_k is open follows from the openness of G: If $x \in D_k$, then there is some $n > k$ such that $nx \in G$. G is open, so there is some $\epsilon > 0$ such that $(nx - \epsilon, nx + \epsilon) \subset G$. Thus if $|y - x| < \epsilon/n$, we must have $y \in D_k$ as well, and so D_k is open.

Moreover, that D_k is dense follows from the unboundedness of G: For $x \in (0, \infty)$ and $\epsilon > 0$, we claim that the set

$$\cup_{n>k}(nx - n\epsilon, nx + n\epsilon)$$

contains a half-line. This follows from the claim that if N is large enough, then

$$(n(x - \epsilon), n(x + \epsilon)) \cap ((n + 1)(x - \epsilon), (n + 1)(x + \epsilon)) \neq \emptyset$$

for all $n > N$, because then we would have $(N(x - \epsilon), \infty)$ is contained in that union. To see the claim, just observe that if $n > \frac{x-\epsilon}{2\epsilon}$, then

$$n(x + \epsilon) - (n + 1)(x - \epsilon) = 2\epsilon n - (x - \epsilon) > (x - \epsilon) - (x - \epsilon) > 0,$$

so that the two intervals must overlap (since clearly $(n + 1)(x + \epsilon) > n(x - \epsilon)$). Thus the set

$$\cup_{n>k}(nx - n\epsilon, nx + n\epsilon)$$

contains a half-line, and so must have non-trivial intersection with G. In particular, for each $x \in (0, \infty)$ and $\epsilon > 0$, there is some $y \in (x - \epsilon, x + \epsilon)$ and $n > k$ such that $ny \in G$, and so D_k is dense.

Observe now that

$$D = \{x : nx \in G \text{ for infinitely many } n\} = \cap_k D_k.$$

We showed above that each D_k is open and dense, and then the Baire category theorem tells us that D must be dense. ∎

I.2 Suppose that $f \in L^1([0,1])$ but $f \notin L^2([0,1])$. Find a complete orthonormal basis $\{\phi_n\}$ for $L^2([0,1])$ such that each $\phi_n \in C^0([0,1])$ and such that

$$\int_0^1 f(x)\phi_n(x)dx = 0 \text{ for all } n.$$

We begin with the observation that if S is a dense subspace of $L^2([0,1])$ (or indeed, any Hilbert space) then we may extract a complete orthonormal basis ϕ_n of L^2 such that $\phi_n \in S$. This reduces the problem to showing that

$$S = \{\phi \in C([0,1]) : \int f\phi = 0\}$$

is a dense subset of $L^2([0,1])$. This is the main idea of this problem.

The secondary idea of this problem is that S is the kernel of a linear functional that is not continuous and so it cannot be closed (so we have a chance). We'll show that it is actually dense.

For $\phi \in C([0,1])$, let $T\phi = \int f\phi$. $f \in L^1([0,1])$ so this makes sense, but $f \notin L^2([0,1])$, so this cannot be continuous with respect to the $L^2([0,1])$-norm. This is because $C([0,1])$ is dense in L^2, so it would have to be continuous on all of L^2, and so would have to be given by an element of L^2 by Riesz-Fischer. We may thus find $g_n \in C([0,1])$ such that $\|g_n\|_{L^2} = 1$ but $Tg_n \to \infty$.

Now, given $h \in C([0,1])$, consider

$$h_n = h - \frac{Th}{Tg_n} g_n.$$

Observe first that $Th_n = 0$, so $h_n \in \ker T$. Now note that

$$\|h - h_n\|_{L^2} = \|\frac{Th}{Tg_n} g_n\|_{L^2} = \frac{|Th|}{|Tg_n|} \to 0,$$

so $h_n \to h$ in L^2. This tells us that we may approximate any continuous function in L^2-norm by an element of $\ker T$. We may approximate any L^2 function by a continuous one, so we know that the set S must be dense in L^2, finisihing the proof.

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I.3 Let H_1 and H_2 be two separable Hilbert spaces. Suppose that $A : H_1 \to H_2$ is a continuous injective linear map. Suppose that $\{v_j\}$ is a bounded sequence in H_1 such that Av_j converges strongly in H_2 to some element w. Prove that there exists an element $v \in H_1$ such that v_j converges weakly to v and $Av = w$.

Hilbert spaces are reflexive, and so we know that the unit ball is weakly compact. This means that there must be a weakly convergent subsequence of the v_j, say $v_{j(k)} \to v$ weakly. We already know that bounded operators are also continuous in the weak topology, so we know that $Av_{j(k)} \to Av$ weakly. $Av_{j(k)} \to w$ strongly and so weakly, and weak limits are unique, so we must have $Av = w$.

It only remains to prove that $v_j \to v$ weakly. This is where we use the injectivity of the map A. Indeed, for any subsequence of v_j, we may find a weakly convergent sub-subsequence $v_{j(k(i))}$ such that $Av_{j(k(i))} \to w$, and so $v_{j(k(i))} \to u$ weakly, where u is such that $Au = w$. A is injective, so we must have that $u = v$. In other words, every subsequence of v_j has a weakly convergent sub-subsequence that converges to v, and so we must have that $v_j \to v$ weakly. ∎

I.4 Some problems about linear functionals:

1. Let $T : C^0([0,1]) \to \mathbb{C}$ be defined by $T(f) = f(1/2)$. Is T continuous with respect to the L^2 norm? Explain why or why not.
2. Let P denote the set of polynomials of arbitrary degree, and consider their restrictions to $[0,1]$ so as to consider $P \subset C^0([0,1])$. Define the linear functional on P, $T_k(f) = a_k$, where a_k is the coefficient of x^k. Does T_k extend as a continuous linear functional to all of $C^0([0,1])$? Hint: Consider $f_n(x) = (1-x)^n$, $n \geq k$.

1. T is not continuous with respect to the L^2 norm. Indeed, let $\phi \in C^0([0,1])$ be a compactly supported function such that $\phi(0) = \phi(1) = 0$ and $\phi(1/2) = 1$. Extend ϕ to be 0 outside of $[0,1]$, and let

$$\phi_j(x) = \phi(j(x - \frac{1}{2}) + \frac{1}{2}),$$

so that $\phi_j(\frac{1}{2}) = 1$ for all j, but $\operatorname{supp} \phi_j \subset [\frac{1}{2} - \frac{1}{j}, \frac{1}{2} + \frac{1}{j}]$. We then have that $T(\phi_j) = 1$ for all j, but

$$\|\phi_j\|_{L^2} = \frac{1}{\sqrt{j}},$$

so T cannot be bounded with respect to the L^2 norm.

2. As suggested by the hint, we consider $f_n(x) = (1-x)^n$. We know from the binomial theorem that for $n \geq k$,

$$T_k(f_n) = \binom{n}{k},$$

which tends to infinity as $n \to \infty$. On the other hand, we have that

$$\|)\|_{C^0([0,1]}f_n = 1$$

because $|1-x| \leq 1$ for $x \in [0,1]$. Thus T_k is not bounded with respect to the C^0 norm on P, and so cannot extend to a functional on $C^0([0,1])$.

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II.1 Let H be a Hilbert space with an orthogonal decomposition into finite dimensional subspaces, $H = \bigoplus_{j=1}^{\infty} H_j$. We denote this decomposition $\mathcal{H} = \{H_j\}_{j=1}^{\infty}$. Thus each $v \in H$ can be written uniquely as $v = \sum_{j=1}^{\infty} v_j$ with $v_j \in H_j$. Let $c = (c_1, c_2, \ldots)$ where each $c_j > 0$, and define the subset

$$A_{c,\mathcal{H}} = \{v : \|v_j\| \le c_j\} \subset H.$$

1. Prove that $c \in \ell^2$ if and only if $A_{c,\mathcal{H}}$ is compact in H.
2. Prove that every compact subset $K \subset H$ is contained in some $A_{c,\mathcal{H}}$ for some $\mathcal{H}$ and $c \in \ell^2$.

1. First suppose that $c \in \ell^2$. We claim that the identity operator on $A_{c,\mathcal{H}}$ can be approximated by finite rank operators I_k. Indeed, let

   $$I_k x = \sum_{j \le k} \Pi_{H_j} x,$$

   where Π_{H_j} is the orthogonal projector onto H_j. Note then that

   $$(I - I_k)x = \sum_{j > k} \Pi_{H_j} x.$$

   Using that $x \in A_{c,\mathcal{H}}$ gives us that

   $$\|(I - I_k)x\|^2 = \sum_{j>k} \|\Pi_{H_j} x\|^2 \le \sum_{j>k} c_j^2 \to 0$$

   as $k \to \infty$ because $c \in \ell^2$. Thus $I|_{A_{c,\mathcal{H}}}$ is the norm limit of the finite rank operators I_k, and so $A_{c,\mathcal{H}}$ must be compact.

   Now, if $A_{c,\mathcal{H}}$ is compact, we know it must be bounded. Thus there is some constant C such that $\|x\|^2 \le C$ for all $x \in A_{c,\mathcal{H}}$. For each j, let $v_j \in H_j$ be such that $\|v_j\| = c_j$. Consider the sequence

   $$u_k = \sum_{j \le k} v_j.$$

   We know that $u_k \in A_{c,\mathcal{H}}$, and that

   $$C \ge \|u_k\|^2 = \sum_{j \le k} \|v_j\|^2 = \sum_{j \le k} c_j^2,$$

   so we must have that

   $$\sum c_j^2 \le C,$$

   so $c \in \ell^2$.

2. We will prove the statement that given any orthogonal decomposition of H into finite dimensional subspaces and any compact set $K \subset H$, there is another orthogonal

decomposition $\mathcal{H}$ of H into finite dimensional subspaces, with each subspace being a finite sum of the original summands, and a sequence $c \in \ell^2$ such that $K \subset A_{c,\mathcal{H}}$.

In fact, we will even be able to assume that c is the sequence $c_j = \frac{1}{j}$. Indeed, suppose that we can find N_j such that if we let

$$E_j = \bigoplus_{i=N_{j-1}+1}^{N_j} H_i,$$

and we let Π_j be the orthogonal projection onto $\bigoplus_{i=i}^{N_j} H_i$, then

$$\|(I - \Pi_j)x\| < 1/2(j+1)$$

for all $x \in K$. Note that we immediately have an orthogonal decomposition

$$H = \bigoplus_{j=1}^{\infty} E_j.$$

If $x \in K$, then we write $x = \sum x_j$, where $x_j \in E_j$. Then

$$\|x_j\| = \|\Pi_j x - \Pi_{j-1} x\| \leq \|x - \Pi_j x\| + \|x - \Pi_{j-1} x\| \leq \frac{1}{j},$$

so that

$$K \subseteq A_{\{1/j\},\{E_j\}}.$$

We must thus only show that we may find such a sequence of N_j. Fix a j. K is compact, so it is totally bounded and we may find $y_1, \ldots, y_m \in K$ such that for all $x \in K$, there is some i with

$$\|x - y_i\| < \frac{1}{8(j+1)}.$$

Moreover, we know that

$$H = \bigoplus_{i=1}^{\infty} H_i,$$

so by taking N_j large, we may find

$$x_1, \ldots, x_m \in \bigoplus_{i=1}^{N_j} H_i$$

such that $\|x_i - y_i\| < \frac{1}{8(j+1)}$ for all $i \leq m$. In particular, we have that

$$x_i = \bigoplus_{k=1}^{N_j} H_k$$

for $i = 1, \ldots, m$ and for each $x \in K$, there is some i with

$$\|x_i - x\| \leq \frac{1}{4(j+1)}.$$

To finish the proof of the claim, we need only observe that

$$\|x - \Pi_j x\| \leq \|x - x_i\| + \|\Pi_j(x - x_i)\| \leq \frac{1}{2(j+1)}.$$

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