I.1 Let $G$ be an unbounded open set in $(0, \infty)$. Define

$$D = \{x \in (0, \infty) : nx \in G \text{ for infinitely many } n\}.$$ 

Prove that $D$ is dense in $(0, \infty)$.

Consider the sets

$$D_k = \{x \in (0, \infty) : nx \in G \text{ for some } n > k\}.$$ 

We claim that $D_k$ is open and dense for each $k$.

That $D_k$ is open follows from the openness of $G$: If $x \in D_k$, then there is some $n > k$ such that $nx \in G$. $G$ is open, so there is some $\epsilon > 0$ such that $(nx - \epsilon, nx + \epsilon) \subseteq G$. Thus if $|y - x| < \epsilon/n$, we must have $y \in D_k$ as well, and so $D_k$ is open.

Moreover, that $D_k$ is dense follows from the unboundedness of $G$: For $x \in (0, \infty)$ and $\epsilon > 0$, we claim that the set

$$\bigcup_{n > k} (nx - n\epsilon, nx + n\epsilon)$$

contains a half-line. This follows from the claim that if $N$ is large enough, then

$$(n(x - \epsilon), n(x + \epsilon)) \cap ((n + 1)(x - \epsilon), (n + 1)(x + \epsilon)) \neq \emptyset$$

for all $n > N$, because then we would have $(N(x - \epsilon), \infty)$ is contained in that union. To see the claim, just observe that if $n > \frac{x - \epsilon}{2\epsilon}$, then

$$n(x + \epsilon) - (n + 1)(x - \epsilon) = 2\epsilon n - (x - \epsilon) > (x - \epsilon) - (x - \epsilon) > 0,$$

so that the two intervals must overlap (since clearly $(n + 1)(x + \epsilon) > n(x - \epsilon)$). Thus the set

$$\bigcup_{n > k} (nx - n\epsilon, nx + n\epsilon)$$

contains a half-line, and so must have non-trivial intersection with $G$. In particular, for each $x \in (0, \infty)$ and $\epsilon > 0$, there is some $y \in (x - \epsilon, x + \epsilon)$ and $n > k$ such that $ny \in G$, and so $D_k$ is dense.

Observe now that

$$D = \{x : nx \in G \text{ for infinitely many } n\} = \bigcap_k D_k.$$ 

We showed above that each $D_k$ is open and dense, and then the Baire category theorem tells us that $D$ must be dense. ■
I.2 Suppose that $f \in L^1([0,1])$ but $f \notin L^2([0,1])$. Find a complete orthonormal basis \{\phi_n\} for $L^2([0,1])$ such that each $\phi_n \in C^0([0,1])$ and such that

$$\int_0^1 f(x)\phi_n(x)\,dx = 0$$

for all $n$.

We begin with the observation that if $S$ is a dense subspace of $L^2([0,1])$ (or indeed, any Hilbert space) then we may extract a complete orthonormal basis $\phi_n$ of $L^2$ such that $\phi_n \in C^0([0,1])$ and such that

$$\int_0^1 f(x)\phi_n(x)\,dx = 0$$

for all $n$.

We finish with the observation that if $S$ is a dense subspace of $L^2([0,1])$ (or indeed, any Hilbert space) then we may extract a complete orthonormal basis $\phi_n$ of $L^2$ such that $\phi_n \in C^0([0,1])$ and such that

$$\int_0^1 f(x)\phi_n(x)\,dx = 0$$

for all $n$.

The secondary idea of this problem is that $S$ is the kernel of a linear functional that is not continuous and so it cannot be closed (so we have a chance). We’ll show that it is actually dense.

For $\phi \in C([0,1])$, let $T\phi = \int f\phi$. $f \in L^1([0,1])$ so this makes sense, but $f \notin L^2([0,1])$, so this cannot be continuous with respect to the $L^2([0,1])$-norm. This is because $C([0,1])$ is dense in $L^2$, so it would have to be continuous on all of $L^2$, and so would have to be given by an element of $L^2$ by Riesz-Fischer. We may thus find $g_n \in C([0,1])$ such that $\|g_n\|_{L^2} = 1$ but $Tg_n \to \infty$.

Now, given $h \in C([0,1])$, consider

$$h_n = h - \frac{T\phi}{Tg_n}g_n.$$

Observe first that $Th_n = 0$, so $h_n \in \ker T$. Now note that

$$\|h - h_n\|_{L^2} = \|T\phi - Tg_n\|_{L^2} = \frac{|Th|}{|Tg_n|} \to 0,$$

so $h_n \to h$ in $L^2$. This tells us that we may approximate any continuous function in $L^2$-norm by an element of $\ker T$. We may approximate any $L^2$ function by a continuous one, so we know that the set $S$ must be dense in $L^2$, finishing the proof.

I.3 Let $H_1$ and $H_2$ be two separable Hilbert spaces. Suppose that $A : H_1 \to H_2$ is a continuous injective linear map. Suppose that $\{v_j\}$ is a bounded sequence in $H_1$ such that $Av_j$ converges strongly in $H_2$ to some element $w$. Prove that there exists an element $v \in H_1$ such that $v_j$ converges weakly to $v$ and $Av = w$.

Hilbert spaces are reflexive, and so we know that the unit ball is weakly compact. This means that there must be a weakly convergent subsequence of the $v_j$, say $v_{j(k)} \to v$ weakly. We already know that bounded operators are also continuous in the weak topology, so we know that $Av_{j(k)} \to Av$ weakly. $Av_{j(k)} \to w$ strongly and so weakly, and weak limits are unique, so we must have $Av = w$. 

2
It only remains to prove that \( v_j \to v \) weakly. This is where we use the injectivity of the map \( A \). Indeed, for any subsequence of \( v_j \), we may find a weakly convergent sub-subsequence \( v_{j(k(i))} \) such that \( Av_{j(k(i))} \to w \), and so \( v_{j(k(i))} \to u \) weakly, where \( u \) is such that \( Au = w \). \( A \) is injective, so we must have that \( u = v \). In other words, every subsequence of \( v_j \) has a weakly convergent sub-subsequence that converges to \( v \), and so we must have that \( v_j \to v \) weakly. ■

### I.4 Some problems about linear functionals:

1. Let \( T : C^0([0,1]) \to \mathbb{C} \) be defined by \( T(f) = f(1/2) \). Is \( T \) continuous with respect to the \( L^2 \) norm? Explain why or why not.

2. Let \( P \) denote the set of polynomials of arbitrary degree, and consider their restrictions to \([0,1]\) so as to consider \( P \subset C^0([0,1]) \). Define the linear functional on \( P \), \( T_k(f) = a_k \), where \( a_k \) is the coefficient of \( x^k \). Does \( T_k \) extend as a continuous linear functional to all of \( C^0([0,1]) \)? Hint: Consider \( f_n(x) = (1 - x)^n, n \geq k \).

1. \( T \) is not continuous with respect to the \( L^2 \) norm. Indeed, let \( \phi \in C^0([0,1]) \) be a compactly supported function such that \( \phi(0) = \phi(1) = 0 \) and \( \phi(1/2) = 1 \). Extend \( \phi \) to be 0 outside of \([0,1]\), and let

\[
\phi_j(x) = \phi(j(x - \frac{1}{2}) + \frac{1}{2}),
\]

so that \( \phi_j(\frac{1}{2}) = 1 \) for all \( j \), but \( supp \phi_j \subset [\frac{1}{2} - \frac{1}{j}, \frac{1}{2} + \frac{1}{j}] \). We then have that \( T(\phi_j) = 1 \) for all \( j \), but

\[
\|\phi_j\|_{L^2} = \frac{1}{\sqrt{j}},
\]

so \( T \) cannot be bounded with respect to the \( L^2 \) norm.

2. As suggested by the hint, we consider \( f_n(x) = (1 - x)^n \). We know from the binomial theorem that for \( n \geq k \),

\[
T_k(f_n) = \binom{n}{k},
\]

which tends to infinity as \( n \to \infty \). On the other hand, we have that

\[
\|f_n\|_{C^0([0,1])} = 1
\]

because \( |1 - x| \leq 1 \) for \( x \in [0,1] \). Thus \( T_k \) is not bounded with respect to the \( C^0 \) norm on \( P \), and so cannot extend to a functional on \( C^0([0,1]) \). ■
II.1 Let $H$ be a Hilbert space with an orthogonal decomposition into finite dimensional subspaces, $H = \bigoplus_{j=1}^{\infty} H_j$. We denote this decomposition $\mathcal{H} = \{H_j\}_{j=1}^{\infty}$. Thus each $v \in H$ can be written uniquely as $v = \sum_{j=1}^{\infty} v_j$ with $v_j \in H_j$. Let $c = (c_1, c_2, \ldots)$ where each $c_j > 0$, and define the subset

$$A_{c,\mathcal{H}} = \{v : \|v_j\| \leq c_j\} \subset H.$$ 

1. Prove that $c \in \ell^2$ if and only if $A_{c,\mathcal{H}}$ is compact in $H$.

2. Prove that every compact subset $K \subset H$ is contained in some $A_{c,\mathcal{H}}$ for some $\mathcal{H}$ and $c \in \ell^2$.

1. First suppose that $c \in \ell^2$. We claim that the identity operator on $A_{c,\mathcal{H}}$ can be approximated by finite rank operators $I_k$. Indeed, let

$$I_k x = \sum_{j \leq k} \Pi_{H_j} x,$$

where $\Pi_{H_j}$ is the orthogonal projector onto $H_j$. Note then that

$$(I - I_k) x = \sum_{j > k} \Pi_{H_j} x.$$ 

Using that $x \in A_{c,\mathcal{H}}$ gives us that

$$\| (I - I_k) x \|^2 = \sum_{j > k} \| \Pi_{H_j} x \|^2 \leq \sum_{j > k} c_j^2 \to 0$$

as $k \to \infty$ because $c \in \ell^2$. Thus $I|_{A_{c,\mathcal{H}}}$ is the norm limit of the finite rank operators $I_k$, and so $A_{c,\mathcal{H}}$ must be compact.

Now, if $A_{c,\mathcal{H}}$ is compact, we know it must be bounded. Thus there is some constant $C$ such that $\|x\|^2 \leq C$ for all $x \in A_{c,\mathcal{H}}$. For each $j$, let $v_j \in H_j$ be such that $\|v_j\| = c_j$. Consider the sequence

$$u_k = \sum_{j \leq k} v_j.$$ 

We know that $u_k \in A_{c,\mathcal{H}}$, and that

$$C \geq \|u_k\|^2 = \sum_{j \leq k} \|v_j\|^2 = \sum_{j \leq k} c_j^2,$$ 

so we must have that

$$\sum_{j \leq k} c_j^2 \leq C,$$

so $c \in \ell^2$.

2. We will prove the statement that given any orthogonal decomposition of $H$ into finite dimensional subspaces and any compact set $K \subset H$, there is another orthogonal
decomposition $\mathcal{H}$ of $H$ into finite dimensional subspaces, with each subspace being a finite sum of the original summands, and a sequence $c \in \ell^2$ such that $K \subset A_{c,\mathcal{H}}$. In fact, we will even be able to assume that $c$ is the sequence $c_j = \frac{1}{j}$. Indeed, suppose that we can find $N_j$ such that

$$E_j = \bigoplus_{i=N_{j-1}+1}^{N_j} H_i,$$

and we let $\Pi_j$ be the orthogonal projection onto $\bigoplus_{i=N_1}^{N_j} H_i$, then

$$\| (I - \Pi_j) x \| < \frac{1}{2(j + 1)}$$

for all $x \in K$. Note that we immediately have an orthogonal decomposition

$$H = \bigoplus_{j=1}^{\infty} E_j.$$

If $x \in K$, then we write $x = \sum x_j$, where $x_j \in E_j$. Then

$$\| x_j \| = \| \Pi_j x - \Pi_{j-1} x \| \leq \| x - \Pi_j x \| + \| x - \Pi_{j-1} x \| \leq \frac{1}{j},$$

so that

$$K \subseteq A_{\{1/j\}, \{E_j\}}.$$

We must thus only show that we may find such a sequence of $N_j$. Fix a $j$. $K$ is compact, so it is totally bounded and we may find $y_1, \ldots, y_m \in K$ such that for all $x \in K$, there is some $i$ with

$$\| x - y_i \| < \frac{1}{8(j + 1)}.$$

Moreover, we know that

$$H = \bigoplus_{i=1}^{\infty} H_i,$$

so by taking $N_j$ large, we may find

$$x_1, \ldots, x_m \in \bigoplus_{i=1}^{N_j} H_i$$

such that $\| x_i - y_i \| < \frac{1}{8(j + 1)}$ for all $i \leq m$. In particular, we have that

$$x_i = \bigoplus_{k=1}^{N_j} H_k$$
for \( i = 1, \ldots, m \) and for each \( x \in K \), there is some \( i \) with

\[
\| x_i - x \| \leq \frac{1}{4(j+1)}.
\]

To finish the proof of the claim, we need only observe that

\[
\| x - \Pi_j x \| \leq \| x - x_i \| + \| \Pi_j (x - x_i) \| \leq \frac{1}{2(j+1)}.
\]