Ph.D. Qualifying Exam, Real Analysis

September 2005, part I

Do all the problems.

1 (Quickies)

a. Let $\mathcal{B}$ denote the set of all Borel probability measures on $[0, 1]$? What are the extreme points of this set?

b. Suppose that u is a continuous linear functional on $\mathcal{C}^\infty(\mathbb{T})$ (i.e. a distribution on the circle), which has the property that $\langle u, \phi\rangle \geq 0$ whenever $\phi \geq 0$, $\phi \in \mathcal{C}^\infty(S^1)$. Show that u is a measure.

Hint: Show that $\langle u, \phi\rangle$ can be estimated by a constant times $\sup|\phi|$.

2

a. Suppose that $P(\xi_1, \dots, \xi_n)$ is a polynomial on $\mathbb{R}^n$ such that for some constants $C_1, C_2 > 0$,

$$|P(\xi)| \geq C_1|\xi| \qquad \text{when} \quad |\xi| \geq C_2.$$

Let $P(\partial)$ be the differential operator defined by replacing each ξ_j by $\partial/\partial x_j$. Suppose that $P(\partial)u = f$ in $\mathbb{R}^n$, that $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and that $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$. Prove that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

b. Prove that for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{\phi(x)}{x + i\epsilon}\, dx$$

exists, and that moreover the value of this limit depends continuously on ϕ in some $\mathcal{C}^k$ norm.

3 Show that if $g \in L^1(\mathbb{T})$, $\mu \in M(\mathbb{T})$ (a finite measure on $\mathbb{T}$), and $\mu(x + \alpha\pi) - \mu(x) = g dt$, for some irrational α, then μ is absolutely continuous.

4 Let $f \in C^\infty(\mathbb{R})$ (the space of infinitely differentiable functions on the line). Assume that for every $x \in \mathbb{R}$, $f^{(n)}(x) = 0$ for at least one $n \geq 0$. Prove that f is a polynomial.

Hints: Use Baire's theorem to show that there exists a dense open set G such that the restriction of f to any of its interval components agrees (on that interval) with a polynomial (i.e., for some n, which may depend on the component, $f^{(n)}(x) = 0$ identically). Use the Baire category theorem again.

5 Convolution and smoothness:

a. Let $f, g \in L^2(\mathbb{T})$. Prove that $f * g \in C(\mathbb{T})$.

b. Assume $f \in C^k(\mathbb{T})$ and $g \in C^l(\mathbb{T})$. Prove that $f * g \in C^{k+l}(\mathbb{T})$.

c. Construct a function $\psi \in C(\mathbb{T})$ such that $\psi * \psi * \cdots * \psi$ (k times) is not differentiable for any k.

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Do all the problems.

1 (Quickies)

a. Describe a norm $\| \|_0$ on $\mathbb{R}^3$ such that the unit vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ have norm 1 while $\|(1,1,1)\|_0 < \frac{1}{100}$.
Hint: Think in terms of the unit ball.

b. Let $f_n(t) = \sum_{j\in\mathbb{Z}} \hat{f}_n(j)e^{ijt}$ where $|\hat{f}_n(j)| \le |j|^{-\log j}$ for $|j| > 75$, uniformly in n. Assume that for all j, $\lim_n \hat{f}_n(j)$ exists, and denote it c_j. Prove that $g = \sum c_j e^{ijt} \in C^\infty(\mathbb{T})$ and that f_n converges to g in the topology of $C^k(\mathbb{T})$ for every $k > 0$.

2 Prove that every measurable homomorphism φ of $\mathbb{T} = \mathbb{R} \bmod 2\pi\mathbb{Z}$ into the multiplicative group $\mathbb{T}^* = \{z\colon |z| = 1\} \subset \mathbb{C}$ is given by $\varphi(t) = e^{int}$ with $n \in \mathbb{N}$.
Hint: Prove, and then use, the fact that φ is continuous.

3 The Hardy–Littlewood maximal function of a function $f \in L^1(\mathbb{R})$ is defined by:

$$M_f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|dt.$$

a. Proof that, for $f \neq 0$, M_f is not integrable, but is of weak-L^1-type, that is

$$\mu\left(\{x;\, M_f(x) > \lambda\}\right) \le \frac{c}{\lambda}.$$

b. Identify the function

$$m_f(x) = \limsup_{h\to 0+} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|dt.$$

4 $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the circle group. Let $k \in L^1(\mathbb{T})$ and let K be the integral operator on $L^2(\mathbb{T})$ defined by $K\colon f \mapsto \frac{1}{2\pi}\int k(x-t)f(t)dt$.

a. Prove that K is compact and normal (i.e. commutes with its adjoint). When is it actually self-adjoint?

b. What is the spectrum of K and what are the corresponding eigenfunctions and eigenvalues?

c. If we replace $\mathbb{T}$ by $\mathbb{R}$, then is the analogous operator on $L^2(\mathbb{R})$ (with $k \in L^1(\mathbb{R})$) compact?

5 Suppose that for some p, $1 < p < \infty$, $f_n \in L^p([0,1])$ and $\|f_n\|_p \leq 1$, uniformly in n. Assuming that $f_n(x) \to 0$ a.e.; prove that $f_n \to 0$ weakly in L^p.