Ph.D. Qualifying Exam, Real Analysis

Fall 2011, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.
   a. Suppose that \((X, \mathcal{A}, \mu)\) is a finite measure space and \(f : X \to \mathbb{R}\) is an \(\mathcal{A}\)-measurable function. Prove that \(\|f\|_{L^\infty} = \lim_{p \to \infty} \|f\|_{L^p}\).
   b. Let \(\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\), and identify it with the unit circle \(\{z \in \mathbb{C} : |z| = 1\}\) in the complex plane. For \(\delta \in (0, 1)\), let \(\Omega_\delta = \{z \in \mathbb{C} : 1 - \delta < |z| < 1\}\). Suppose that \(\phi \in C(\mathbb{T})\) and there exist \(\delta \in (0, 1)\) and \(f \in C(\Omega_\delta)\), \(f\) holomorphic on \(\Omega_\delta\) such that \(f|_{\mathbb{T}} = \phi\). Let \((\mathcal{F}\phi)_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-i\theta}\phi(\theta) d\theta\) be the \(n\)th Fourier coefficient of \(\phi\). Show that for all \(k > 0\) there exists \(C > 0\) such that \(|(\mathcal{F}\phi)_n| \leq C|n|^{-k}\) for \(n \leq -1\).

2 Suppose that \(f : \mathbb{N} \to \mathbb{C}\) is a bounded function (where \(\mathbb{N}\) is the set of the positive integers), and define \(M_f \in \mathcal{L}(\ell^p)\), \(1 \leq p < \infty\), by \(M_f(a_n)_{n=1}^\infty = (f(n)a_n)_{n=1}^\infty\).
   a. State and prove a necessary and sufficient condition in terms of \(f\) for \(M_f\) being a compact operator.
   b. Can there be any points in the spectrum of \(M_f\) that are not eigenvalues? Give an example or prove the contrary.

3 In this problem \(X, Y\) are Hilbert spaces, \(\mathcal{L}(X, Y)\) the set of bounded linear operators from \(X\) to \(Y\). Prove or disprove the following statements: (a) There exists \(T \in \mathcal{L}(X, Y)\) such that \(T\) is a bijection, but the set theoretic inverse \(T^{-1}\) is not in \(\mathcal{L}(Y, X)\). (b) There exists \(T \in \mathcal{L}(X, Y)\) such that \(T\) is injective, but there is no left inverse \(S \in \mathcal{L}(Y, X)\) for \(T\) (i.e. there is no \(S \in \mathcal{L}(Y, X)\) such that \(ST\) is the identity on \(X\)). (c) There exists \(T \in \mathcal{L}(X, Y)\) such that \(T\) is surjective, but there is no right inverse \(S \in \mathcal{L}(Y, X)\) for \(T\) (i.e. there is no \(S \in \mathcal{L}(Y, X)\) such that \(TS\) is the identity on \(Y\)).

4 Suppose that \(X\) is a separable reflexive Banach space.
   a. Show that \(B = \{x \in X : \|x\| \leq 1\}\) is compact in the weak topology on \(X\).
   b. Give (and prove) a necessary and sufficient condition for \(\{x \in X : \|x\| = 1\}\) to be compact in the weak topology on \(X\).
   c. Prove that \(B = \{x \in X : \|x\| \leq 1\}\) is sequentially compact in the weak topology of \(X\), i.e. every sequence in \(B\) has a weakly convergent subsequence.

5 Recall that \(\text{SL}_2(\mathbb{R}) \subset \text{GL}_2(\mathbb{R})\), consisting of matrices of determinant 1, acts on \(\mathbb{R}^2\) by \(\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2 \ni (A, x) \mapsto Ax \in \mathbb{R}^2\). Find all Baire measures \(\mu\) on \(\mathbb{R}^2\) (i.e. Borel measures finite on compact sets) which are invariant under the \(\text{SL}_2(\mathbb{R})\)-action, i.e. such that \(\mu(S) = \mu(AS)\) (with \(AS = \{Ax : x \in S\}\)) for all \(A \in \text{SL}_2(\mathbb{R})\) and \(S\) Borel.
Suppose $A \subset \mathbb{R}$ is Borel, $T$ is a dense subset of $\mathbb{R}$ and $\tau_t(A) \setminus A$ has Lebesgue measure zero for each $t \in T$, where $\tau_t : \mathbb{R} \to \mathbb{R}$ is the translation $x \mapsto x + t$. Prove that either $A$ or $\mathbb{R} \setminus A$ has Lebesgue measure zero.

2. Two short problems.

a. Let $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$ be the circle, and let $C^1(\mathbb{T})$ denote the Banach space of complex valued continuously differentiable functions on $\mathbb{T}$ with the norm $\|f\|_{C^1} = \sup |f| + \sup |f'|$. Suppose that $u : C^1(\mathbb{T}) \to \mathbb{C}$ is continuous and has the property that $\phi \in C^1(\mathbb{T})$, $\phi \geq 0$ on $\mathbb{T}$, imply $u(\phi) \geq 0$. Show that there is a finite Borel measure $\mu$ on $\mathbb{T}$ such that $u(\phi) = \int \phi d\mu$ for all $\phi \in C^1(\mathbb{T})$.

b. Suppose that $\mathcal{H}$ is a Hilbert space, $\mathcal{L}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, and for $t \in \mathbb{R}$ define $U(t) = f_t(A)$ via the functional calculus where $f_t(s) = e^{its}$. Show that $U(t) - I$ is compact for all $t \in \mathbb{R}$ if and only if $A$ is compact.

3. Let $\mathcal{L}(L^2(\mathbb{R}))$ denote the set of bounded linear operators on $L^2(\mathbb{R})$. Consider the following operators $\Lambda_s$, $s \in \mathbb{R}$, on $L^2(\mathbb{R})$: $(\Lambda_s f)(\xi) = (1 + |\xi|^2)^{is/2} f(\xi)$. Prove or disprove each of the following statements for the map $\Lambda : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}))$ given by $\Lambda(s) = \Lambda_s$: (a) $\Lambda$ is continuous when $\mathcal{L}(L^2(\mathbb{R}))$ is equipped with the norm topology. (b) $\Lambda$ is continuous when $\mathcal{L}(L^2(\mathbb{R}))$ is equipped with the strong operator topology. (c) $\Lambda$ is continuous when $\mathcal{L}(L^2(\mathbb{R}))$ is equipped with the weak operator topology.

4. Let $\mathcal{D}'(\mathbb{T})$ denote the set of distributions on $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$, i.e. the dual of $C^\infty(\mathbb{T})$, equipped with the ‘weak-* topology’, i.e. the weakest topology in which the maps $E_\phi : \mathcal{D}'(\mathbb{T}) \to \mathbb{C}$, $E_\phi(u) = u(\phi)$ are continuous for all $\phi \in C^\infty(\mathbb{T})$.

a. Show that the multiplication map $M : C(\mathbb{T}) \times C(\mathbb{T}) \to C(\mathbb{T})$ given by $(M(f, g))(x) = f(x)g(x)$ has no continuous extension to a map $\hat{M} : \mathcal{D}'(\mathbb{T}) \times \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$.

b. Show that, on the other hand, the convolution map $*: C(\mathbb{T}) \times C(\mathbb{T}) \to C(\mathbb{T})$ given by $*(f, g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-y)g(y) \, dy$ (usually written as $(f * g)(x)$) has a unique continuous extension to a map $*: \mathcal{D}'(\mathbb{T}) \times \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$.

5. For $s \geq 0$, let $H^s(\mathbb{T})$ be the space of $L^2$ functions $f$ on the circle $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int e^{-inx} f(x) \, dx$ satisfy $\sum (1 + n^2)^s |\hat{f}_n|^2 < \infty$, and define the norm on $H^s(\mathbb{T})$ by $\|f\|^2_s = (2\pi)^{-1} \sum (1 + n^2)^s |\hat{f}_n|^2$.

Consider $A = -\frac{d^2}{dx^2}$, $A \in \mathcal{L}(H^2(\mathbb{T}), L^2(\mathbb{T}))$, suppose $V \in \mathcal{L}(L^2(\mathbb{T}))$, and let $L = A + V \in \mathcal{L}(H^2(\mathbb{T}), L^2(\mathbb{T}))$. With $R(\lambda) = (\lambda I - L)^{-1} : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ when $\lambda I - L : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible, show that $C \ni \lambda \mapsto R(\lambda)$ is a meromorphic operator-valued function, and that there exists a half-plane $\Omega = \{ \lambda \in \mathbb{C} : \text{Re} \lambda < -C \}$ such that $R(\lambda)$ is holomorphic on $\Omega$. 

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