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Spring 2006 Qual, Part I: 1, 2, 3, 5; Part II: 1, 2

I.1

1. Let $f_n \in L^p([0,1])$, where $1 < p < \infty$. Suppose that $\|f_n\|_p \leq 1$ and moreover that $f_n(x) \to 0$ for a.e. $x \in [0,1]$. Prove that $f_n \to 0$ weakly in L^p.
2. Let $v_1, \ldots, v_N$ be a finite sequence of unit vectors in a Hilbert space $\mathcal{H}$. Suppose that there exists a number $a \in (0,1)$ such that
$$\langle v_i, v_j \rangle \leq -a, \ \forall i \neq j.$$
Find an upper bound for N in terms of a.
3. Let $h \in L^2(\mathbb{S}^1)$ and assume that $h(t) \neq 0$ for a.e. $t \in \mathbb{S}^1$. Prove that the subspace
$$V = \{P(t)h(t) : P \text{ a trigonometric polynomial}\} \subset L^2(\mathbb{S}^1)$$
is dense.

1. To show that $f_n \to 0$ weakly in L^p, it is enough to show that $\int_0^1 f_n g \to 0$ for each $g \in L^q([0,1])$, as L^q is the dual of L^p.

   Let $g \in L^q$ and fix $\epsilon > 0$. Recall from measure theory that there is some $\delta > 0$ such that if $A \subset [0,1]$ has $m(A) < \delta$, then
$$\left(\int_A |g|^q \right)^{1/q} < \frac{\epsilon}{2}.$$

   $f_n \to 0$ almost everywhere, so by Lusin's theorem there is a set $A \subset [0,1]$ with $m(A) < \delta$ such that $f_n \to 0$ uniformly on $[0,1] \backslash A$. In other words, there is some N such that for all $n \geq N$ we have
$$|f_n(x)| \leq \frac{\epsilon}{2\|g\|_q} \text{ on } [0,1]\backslash A.$$

   We then have that
$$\begin{aligned} |\langle f_n, g\rangle| &\leq \int_A |f_n g| + \int_{[0,1]\backslash A} |f_n g| \\ &\leq \|f_n\|_p \left(\int_A |g|^q \right)^{1/q} + \left(\int_{[0,1]\backslash A} |f_n|^p \right)^{1/p} \|g\|_q \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2\|g\|_q} \cdot \|g\|_q = \epsilon. \end{aligned}$$

2. Consider the element $v = \sum_{i=1}^{N} v_i$. We then have that

$$0 \leq \langle v, v \rangle = \sum_{i=1}^{N} v_i + 2\sum_{i \neq j} \langle v_i, v_j \rangle$$
$$\leq N - 2\binom{N}{2} a = N - N(N-1)a,$$

and so $N(N-1)a \leq N$. In other words, $(N-1)a \leq 1$, or

$$N \leq 1 + \frac{1}{a} = \frac{a+1}{a}.$$

3. To show that V is dense in $L^2(\mathbb{S}^1)$, we will use the fact that the trigonometric polynomials are dense here. We proceed by showing that if $f \in L^2(\mathbb{S}^1)$ is orthogonal to V, then $f = 0$. Suppose that $f \perp V$, so that

$$\int_{\mathbb{S}^1} P(x)h(x)\overline{f(x)}dx = 0$$

for all trigonometric polynomials P. In particular, we have that $h\bar{f}$ is orthogonal to the space of trigonometric polynomials. These are dense, so we must have that $h\bar{f} = 0$. $h(x) \neq 0$ almost everywhere, so we can conclude that $f = 0$ almost everywhere, i.e. $f = 0$ in $L^2(\mathbb{S}^1)$.

■

I.2 Let $\{f_k\}$ be a sequence of real-valued functions defined on $[-1,1]$ such that

$$|f_k(x) - f_k(y)| \leq \sqrt{|x-y|} + \frac{1}{k}$$

for all $k \geq 0$ and $x, y \in [-1,1]$. suppose also that each $f_k(0) = 0$. Prove that some subsequence of the the f_k converges uniformly to a continuous function f on $[-1,1]$.

First note that the f_k are uniformly bounded by 2. We will construct a convergent subsequence of the f_k via a diagonalization process analogous to a hands-on proof of Arzela-Ascoli on an interval.

The sequence $\{f_k(1)\}$ is bounded, so we may choose a convergent subsequence. In fact, we may choose a subsequence $f_{k_0(j)}$ such that the values of $f_{k_0(j)}(x)$ converge at the points $x = -1, 0, 1$. Now, given a subsequence $f_{k_{n-1}(j)}$, we may choose a subsequence of this subsequence, call it $f_{k_n(j)}$ such that its values at

$$\left\{ \frac{j}{2^n} : j \in \{0, \pm 1, \dots, \pm 2^n\} \right\}$$

converge. Without loss of generality, we may assume that the r-th element of the r-th subsequence, $f_{k_r(r)}$ satisfies

$$\left| f_{k_r(r)}\left(\frac{j}{2^r}\right) - \text{ its limit} \right| \leq \frac{1}{2^r}$$

for all $j \in \{0, \pm 1, \dots, \pm 2^n\}$. By a standard diagonalization argument, we obtain a subsequence (which, in an abuse of notation, we denote f_k) whose values converge at each dyadic rational. (To be more precise, the first element of our sequence is $f_{k_1(1)}$, and the r-th element is $f_{k_r(r)}$.)

We now claim that in fact f_k must converge uniformly to a continuous function. To do this we will first show that f_k is uniformly Cauchy to obtain a limiting function f. We will then show that this f must be continuous.

Fix $\epsilon > 0$. Let N be such that $1/N < \frac{\epsilon}{4}$ and $2^{(1-N)/2} < \frac{\epsilon}{4}$. We claim that if $n, m \geq N$, then for all $x \in [-1, 1]$, $|f_n(x) - f_m(x)| < \epsilon$. Indeed, there is an integer j such that $|x - \frac{j}{2^N}| \leq \frac{1}{2^{N+1}}$. We then have that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq \left|f_n(x) - f_n(\frac{j}{2^N})\right| + \left|f_n(\frac{j}{2^N}) - f_m(\frac{j}{2^N})\right| + \left|f_m(\frac{j}{2^N}) - f_m(x)\right| \\
&\leq 2\sqrt{\frac{1}{2^{N+1}}} + \frac{2}{N} + \left|f_n(\frac{j}{2^N}) - f_m(\frac{j}{2^N})\right| \\
&\leq 2^{\frac{1-N}{2}} + \frac{2}{N} + 2 \cdot 2^{1-N} \leq \epsilon.
\end{aligned}$$

To rephrase the above in words, we may ensure that f_k are uniformly Cauchy on the dyadic rationals, and then the assumption about the f_k from the problem and the denseness of the dyadic rationals implies that the f_k are uniformly Cauchy. This gives us a limiting function f.

We now must show that f is continuous. Indeed, note that we must have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\
&\leq \epsilon + \sqrt{|x-y|} + \frac{1}{k} + \epsilon \\
&\leq \sqrt{|x-y|} + \frac{1}{k} + 2\epsilon.
\end{aligned}$$

By taking limits, we see that $|f(x) - f(y)| \leq \sqrt{|x-y|}$ and so must be continuous. ■

I.3

1. Construct a sequence $\{f_n\}$ of positive continuous functions on $\mathbb{R}$ such that $f_n(x)$ is bounded is $n \to \infty$ when $x \in \mathbb{Q}$, but $f_n(x)$ is unbounded for $x \in \mathbb{R} \backslash \mathbb{Q}$.
2. Prove that there is no sequence $\{g_n\}$ of positive continuous functions such that $g_n(x)$ is bounded when $x \in \mathbb{R} \backslash \mathbb{Q}$, but $g_n(x)$ is unbounded when $x \in \mathbb{Q}$.

1. The idea here is to construct a sequence of functions f_n with very large spikes so that for any rational number x, $f_n(x)$ is eventually 0. A picture here is very helpful, I think.

   We start by letting

$$h(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}] \\ 2 - 2x & x \in [\frac{1}{2}, 1] \end{cases},$$

extended to be periodic with period 1. Now, for $x \in \mathbb{R}$, we let

$$f_n(x) = (n+1)h(n! \cdot x).$$

We then have that $f_n(x) = 0$ for all $x \in \mathbb{Q}$ such that $n! \cdot x \in \mathbb{Z}$. In particular, if $x \in \mathbb{Q}$, we have that $f_n(x) = 0$ for all large enough x, and so $f_n(x)$ is bounded for $x \in \mathbb{Q}$.

It remains to show only that $f_n(x)$ is unbounded when $x \notin \mathbb{Q}$. Indeed, consider the sets

$$\begin{aligned} B_M &= \{x : f_n(x) \text{ eventually bounded by M }\} \\ &= \bigcup_k \bigcap_{n \geq k} \left(\bigcup_{j \in \mathbb{Z}} \left[\frac{j}{n!} - \frac{M}{2(n+1)!}, \frac{j}{n!} + \frac{M}{2(n+1)!} \right] \right) \\ &= \bigcup_k S_{M,k}. \end{aligned}$$

Observe that the sets $S_{M,k}$ are finite; indeed, it is easy to check that

$$S_{M,k} = \{\frac{j}{k!} : j \in \mathbb{Z}\}.$$

This tells us that

$$B_M = \bigcup_k S_{M,k} = \mathbb{Q},$$

and so

$$\{x : f_n(x) \text{ is bounded}\} = \bigcup_M B_M = \mathbb{Q}.$$

Thus $f_n(x)$ is unbounded for all $x \in \mathbb{R} \backslash \mathbb{Q}$.

2. We just showed that the set of points where $g_n(x)$ is bounded is

$$\cup_{M=1}^{\infty} \cup_k \cap_{n \geq k} \{x : g_n(x) \leq M\}.$$

$g_n(x)$ is continuous, so this is a countable union of closed sets, and so is a F_σ set. $\mathbb{R} \backslash \mathbb{Q}$ is not an F_σ set, so we cannot have such a sequence.

Here is a quick proof that $\mathbb{R} \backslash \mathbb{Q}$ is not an F_σ. If it were, then $\mathbb{Q}$ would be a G_δ set. Both $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$ would then both be dense G_δ sets, so the Baire category theorem tells us that their intersection should be dense. But this is impossible because they have empty intersection.

■

I.5 Let C be a closed convex set in a Hilbert space $\mathcal{H}$. Prove that C contains a unique element of minimal norm.

Let

$$\eta = \inf_{z \in C} \|z\|.$$

If $\eta = 0$, then there is a sequence $z_j \in C$, $\|z_j\| \to 0$, so $z_j \to 0$. C is closed, so $0 \in C$, and then 0 is the unique element of minimal norm. We may thus assume that $\eta > 0$.

We start by claiming that C contains an element of minimal norm. Indeed, take $z_j \in C$ such that $\|z_j\| \to \eta$. The convexity of C implies that $\frac{z_j + z_k}{2} \in C$, so that

$$\|z_j + z_k\|^2 = 4\|\frac{z_j + z_k}{2}\|^2 \geq 4\eta^2.$$

Recall now the parallelogram law for a Hilbert space:

$$\|x - y\|^2 + \|x + y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Applying this we have that

$$\begin{aligned}\|z_j - z_k\|^2 &= 2\left(\|z_k\|^2 + \|z_j\|^2\right) - \|z_j + z_k\|^2 \\ &\leq 2\left(\|z_k\|^2 + \|z_j\|^2\right) - 4\eta^2 \to 4\eta^2 - 4\eta^2 = 0\end{aligned}$$

as $j, k \to \infty$. Thus the z_j are a Cauchy sequence and so converge to some $z \in \mathcal{H}$. C is closed, so $z \in C$, and the norm function is continuous, so $\|z\| = \eta$, so that C contains an element of minimal norm.

The uniqueness of this element also follows from the parallelogram law and the convexity of C. If x, y are two such elements, then by the above we must have

$$\|x + y\|^2 \geq 4\eta^2,$$

and so the parallelogram law implies that

$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 \leq 4\eta^2 - 4\eta^2 = 0,$$

and so $x = y$. ■

II.1 Let $f \in C^0([\alpha, \beta])$, where $0 < \alpha < \beta < 1$. For each $n = 1, 2, \ldots$, define

$$P_n(x) = \frac{\int_\alpha^\beta f(u)[1 - (u - x)^2]^n du}{\int_{-1}^1 (1 - u^2)^n du}.$$

Show that $P_n(x)$ is a polynomial of degree at most $2n$ and that for any closed subinterval $[a, b] \subset (\alpha, \beta)$, $P_n \to f$ uniformly.

Consider first the polynomials

$$k_n(t) = \frac{(1-t^2)^n}{\int_{-1}^{1}(1-u^2)^n du}.$$

Observe that $\int_{-1}^{1} k_n(t)dt = 1$, and that $k_n(t) \geq 0$. We claim that for any $\delta > 0$, $k_n(t)$ converges uniformly to 0 on $[-1,1]\backslash(-\delta,\delta)$.

We'll start by computing a formula for

$$A_n = \int_{-1}^{1}(1-u^2)^n du.$$

Indeed, observe that by integrating by parts, we can see that

$$\begin{aligned}\int_{-1}^{1} t^2(1-t^2)^n dt &= -\frac{t}{2(n+1)}(1-t^2)^{n+1}|_{-1}^{1} + \frac{1}{2(n+1)}\int_{-1}^{1}(1-t^2)^{n+1}dt \\ &= 0 + \frac{A_{n+1}}{2(n+1)} = \frac{A_{n+1}}{2n+2}.\end{aligned}$$

We then have the recurrence relation

$$A_{n+1} = \int_{-1}^{1}(1-t^2)^{n+1} = A_n - \frac{1}{2(n+1)}A_{n+1},$$

i.e.

$$A_{n+1} = \frac{2n+2}{2n+3}A_n.$$

We can easily verify that

$$A_1 = \int_{-1}^{1}(1-t^2)dt = \frac{4}{3},$$

so we have

$$A_n = 2\prod_{j=1}^{n}\frac{2j}{2j+1}.$$

Now fix $\delta > 0$. If n is large enough, then $\frac{2n+2}{2n+3} \geq (1-\delta^2)^{1/2}$, and so

$$k_{n+1}(\delta) = \frac{(1-\delta^2)A_n}{A_{n+1}}k_n(\delta) \leq (1-\delta^2)^{1/2}k_n(\delta).$$

In particular, we have that if N is large, then for all $n \geq N$,

$$k_n(\delta) \leq (1-\delta^2)^{(n-N)/2}k_N(\delta).$$

$k_n(t)$ is decreasing for $t > 0$ and is also an even function, so this tells us that for all $x \in [-1,1]\backslash(-\delta,\delta)$,

$$k_n(x) \leq (1-\delta^2)^{(n-N)/2}k_N(\delta),$$

so we have shown the desired uniform convergence.

Now that we know these properties of k_n, we wish to show that P_n converges uniformly to f. We can write

$$P_n(x) = \int_\alpha^\beta f(u)k_n(x-u)du = \int_{-1}^1 f(u)\chi(u)k_n(x-u)du = \int_{-1}^1 f(x-u)\chi(x-u)k_n(u)du,$$

where χ is the characteristic function of the interval $[\alpha, \beta]$.

Fix $\epsilon > 0$ and $[a, b] \subset (\alpha, \beta)$. Let $\delta > 0$ be such that $|b-\beta| < \delta$, $|a-\alpha| < \delta$, and that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $|x-y| < \delta$. Let N be such that for all $n \geq N$,

$$|k_n(x)| \leq \frac{\epsilon}{4\|f\|_{C^0}}$$

whenever $x \in [-1,1]\backslash(-\delta, \delta)$. We then use that $\int_{-1}^1 k_n = 1$ to write

$$\begin{aligned}
|P_n(x) - f(x)| &= \left|\int_{-1}^1 (f(x-u)\chi(x-u) - f(x)\chi(x))k_n(u)du\right| \\
&\leq \int_{[-1,1]\backslash(-\delta,\delta)} |f(x)\chi(x) - f(x-u)\chi(x-u)|k_n(u)du \\
&+ \int_{-\delta}^{\delta} |f(x-u)\chi(x-u) - f(x)\chi(x)|k_n(u)du \\
&\leq 2\|f\|_{C^0} \int_{[-1,1]\backslash(-\delta,\delta)} k_n(u)du + \frac{\epsilon}{2}\int_{-1}^1 k_n(u)du \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

because $|x - (x-u)| = |u| < \delta$.

That P_n are polynomials of degree at most $2n$ follows from the fact that we may write $k_n(x-u) = \sum_{i=0}^{2n} g_{n,i}(u)x^i$, where $g_{n,i}$ are polynomials in u, and then

$$P_n(x) = \sum_{i=0}^{n} x^i \int_\alpha^\beta f(u)g_{n,i}(u)du.$$

■

II.2 Let W be any vector space, and suppose that $u, v_1, \ldots, v_k$ are linear functionals on W. Endow W with the weakest topology such that the functionals $v_1, \ldots, v_k$ are continuous. Suppose that u is continuous in this topology. Prove that u is a linear combination of the v_j.

We first claim that

$$\bigcap_{j=1}^{k} \ker v_j \subset \ker u.$$

Indeed, we know that a basis for the topology of W at 0 consists of the sets

$$U_\epsilon = \{w \in W : |v_j(w)| < \epsilon \text{ for all } j = 1, \ldots, k\}.$$

u is continuous with respect to this topology, so there is some $\epsilon > 0$ such that

$$U_\epsilon \subset \{w \in W : |u(w)| < 1\},$$

and so for all $\delta > 0$,

$$U_{\delta\epsilon} \subset \{w \in W : |u(w)| < \delta\}.$$

We know that

$$\bigcap_{j=1}^{k} \ker v_j \subset U_\epsilon$$

for all ϵ, and so

$$\bigcap_{j=1}^{k} \ker v_j \subset \{w \in W : |u(w)| < \delta\}$$

for all δ. Thus if $x \in \cap \ker v_j$, then $u(x) = 0$, i.e. $x \in \ker u$.

Now we claim that if

$$\bigcap_{j=1}^{k} \ker v_j \subset \ker u,$$

then u must be a linear combination of the v_j. Without loss of generality, we may assume that the v_j are linearly independent. Consider the map $W \to \mathbb{R}^{k+1}$ given by

$$w \mapsto (v_1(w), \ldots, v_k(w), u(w)).$$

We know that $\cap \ker v_j \subset \ker u$, so $(0, \ldots, 0, 1)$ is not in the image of this map. There is thus some non-zero vector $(\beta_1, \ldots, \beta_k, \alpha)$ orthogonal (by the standard dot product on $\mathbb{R}^{k+1}$) to the image of this map. This tells us that for all $w \in W$,

$$\alpha u(w) + \sum_{j=1}^{k} \beta_j v_j(w) = 0.$$

In other words, $\alpha u + \sum \beta_j v_j = 0$. The v_j were linearly independent, so $\alpha \neq 0$ or else this would give a linear dependence among the v_j. Thus

$$u = -\frac{1}{\alpha} \sum_{j=1}^{k} \beta_j v_j,$$

so that u is a linear combination of the v_j. ■