I.1

1. Let $f_n \in L^p([0,1])$, where $1 < p < \infty$. Suppose that $\|f_n\|_p \leq 1$ and moreover that $f_n(x) \to 0$ for a.e. $x \in [0,1]$. Prove that $f_n \to 0$ weakly in $L^p$.

2. Let $v_1, \ldots, v_N$ be a finite sequence of unit vectors in a Hilbert space $\mathcal{H}$. Suppose that there exists a number $a \in (0, 1)$ such that

$$\langle v_i, v_j \rangle \leq -a, \forall i \neq j.$$

Find an upper bound for $N$ in terms of $a$.

3. Let $h \in L^2(S^1)$ and assume that $h(t) \neq 0$ for a.e. $t \in S^1$. Prove that the subspace

$$V = \{ P(t)h(t) : P \text{ a trigonometric polynomial} \} \subset L^2(S^1)$$

is dense.

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1. To show that $f_n \to 0$ weakly in $L^p$, it is enough to show that $\int_0^1 f_n g \to 0$ for each $g \in L^q([0,1])$, as $L^q$ is the dual of $L^p$.

Let $g \in L^q$ and fix $\epsilon > 0$. Recall from measure theory that there is some $\delta > 0$ such that if $A \subset [0,1]$ has $m(A) < \delta$, then

$$\left( \int_A |g|^q \right)^{1/q} < \frac{\epsilon}{2}.$$

$f_n \to 0$ almost everywhere, so by Lusin’s theorem there is a set $A \subset [0,1]$ with $m(A) < \delta$ such that $f_n \to 0$ uniformly on $[0,1] \setminus A$. In other words, there is some $N$ such that for all $n \geq N$ we have

$$|f_n(x)| \leq \frac{\epsilon}{2\|g\|_q} \text{ on } [0,1] \setminus A.$$

We then have that

$$|\langle f_n, g \rangle| \leq \int_A |f_n g| + \int_{[0,1]\setminus A} |f_n g|$$

$$\leq \|f_n\|_p \left( \int_A |g|^q \right)^{1/q} + \left( \int_{[0,1]\setminus A} |f_n|^p \right)^{1/p} \|g\|_q$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2\|g\|_q} \cdot \|g\|_q = \epsilon.$$
2. Consider the element \( v = \sum_{i=1}^{N} v_i \). We then have that
\[
0 \leq \langle v, v \rangle = \sum_{i=1}^{N} v_i + 2 \sum_{i \neq j} \langle v_i, v_j \rangle \leq N - 2 \binom{N}{2} a = N - N(N - 1)a,
\]
and so \( N(N - 1)a \leq N \). In other words, \( (N - 1)a \leq 1 \), or
\[
N \leq 1 + \frac{1}{a} = \frac{a + 1}{a}.
\]

3. To show that \( V \) is dense in \( L^2(S^1) \), we will use the fact that the trigonometric polynomials are dense here. We proceed by showing that if \( f \in L^2(S^1) \) is orthogonal to \( V \), then \( f = 0 \). Suppose that \( f \perp V \), so that
\[
\int_{S^1} P(x)h(x)\overline{f(x)}dx = 0
\]
for all trigonometric polynomials \( P \). In particular, we have that \( hf \) is orthogonal to the space of trigonometric polynomials. These are dense, so we must have that \( hf = 0 \). \( h(x) \neq 0 \) almost everywhere, so we can conclude that \( f = 0 \) almost everywhere, i.e. \( f = 0 \) in \( L^2(S^1) \).

\[ \blacksquare \]

I.2 Let \( \{f_k\} \) be a sequence of real-valued functions defined on \([-1, 1]\) such that
\[
|f_k(x) - f_k(y)| \leq \sqrt{|x - y|} + \frac{1}{k}
\]
for all \( k \geq 0 \) and \( x, y \in [-1, 1] \). Suppose also that each \( f_k(0) = 0 \). Prove that some subsequence of the \( f_k \) converges uniformly to a continuous function \( f \) on \([-1, 1]\).

First note that the \( f_k \) are uniformly bounded by 2. We will construct a convergent subsequence of the \( f_k \) via a diagonalization process analogous to a hands-on proof of Arzela-Ascoli on an interval.

The sequence \( \{f_k(1)\} \) is bounded, so we may choose a convergent subsequence. In fact, we may choose a subsequence \( f_{k_0(j)} \) such that the values of \( f_{k_0(j)}(x) \) converge at the points \( x = -1, 0, 1 \). Now, given a subsequence \( f_{k_{a-1}(j)} \), we may choose a subsequence of this subsequence, call it \( f_{k_{a}(j)} \) such that its values at
\[
\left\{ \frac{j}{2^n} : j \in \{0, \pm 1, \ldots, \pm 2^n\} \right\}
\]
converge. Without loss of generality, we may assume that the \( r \)-th element of the \( r \)-th subsequence, \( f_{k_r(r)} \) satisfies
\[
|f_{k_r(r)}(\frac{j}{2^r}) - \text{its limit}| \leq \frac{1}{2^r}
\]
for all $j \in \{0, \pm 1, \ldots, \pm 2^n\}$. By a standard diagonalization argument, we obtain a subsequence (which, in an abuse of notation, we denote $f_k$) whose values converge at each dyadic rational. (To be more precise, the first element of our sequence is $f_{k_1(1)}$, and the $r$-th element is $f_{k_r(r)}$.)

We now claim that in fact $f_k$ must converge uniformly to a continuous function. To do this we will first show that $f_k$ is uniformly Cauchy to obtain a limiting function $f$. We will then show that this $f$ must be continuous.

Fix $\varepsilon > 0$. Let $N$ be such that $1/N < \varepsilon^2$ and $2^{(1-N)/2} < \varepsilon^2$. We claim that if $n, m \geq N$, then for all $x \in [-1, 1]$, $|f_n(x) - f_m(x)| < \varepsilon$. Indeed, there is an integer $j$ such that $|x - j/2^n| \leq 1/2^{n+1}$. We then have that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(j/2^n)| + |f_n(j/2^n) - f_m(j/2^n)| + |f_m(j/2^n) - f_m(x)|$$

$$\leq 2\sqrt{1/2^{n+1} + 2/N} + |f_n(j/2^n) - f_m(j/2^n)|$$

$$\leq 2^{-1/2} + 2/N + 2 \cdot 2^{1-N} \leq \varepsilon.$$ 

To rephrase the above in words, we may ensure that $f_k$ are uniformly Cauchy on the dyadic rationals, and then the assumption about the $f_k$ from the problem and the density of the dyadic rationals implies that the $f_k$ are uniformly Cauchy. This gives us a limiting function $f$.

We now must show that $f$ is continuous. Indeed, note that we must have

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)|$$

$$\leq \varepsilon + \sqrt{|x - y|} + 1/k + \varepsilon$$

$$\leq \sqrt{|x - y|} + 1/k + 2\varepsilon.$$ 

By taking limits, we see that $|f(x) - f(y)| \leq \sqrt{|x - y|}$ and so must be continuous. 

I.3

1. Construct a sequence $\{f_n\}$ of positive continuous functions on $\mathbb{R}$ such that $f_n(x)$ is bounded is $n \to \infty$ when $x \in \mathbb{Q}$, but $f_n(x)$ is unbounded for $x \in \mathbb{R}\setminus\mathbb{Q}$.

2. Prove that there is no sequence $\{g_n\}$ of positive continuous functions such that $g_n(x)$ is bounded when $x \in \mathbb{R}\setminus\mathbb{Q}$, but $g_n(x)$ is unbounded when $x \in \mathbb{Q}$.

1. The idea here is to construct a sequence of functions $f_n$ with very large spikes so that for any rational number $x$, $f_n(x)$ is eventually 0. A picture here is very helpful, I think.

We start by letting

$$h(x) = \begin{cases} 2x & x \in [0, 1/2] \\ 2 - 2x & x \in [1/2, 1] \end{cases}.$$
extended to be periodic with period 1. Now, for $x \in \mathbb{R}$, we let

$$f_n(x) = (n + 1)h(n! \cdot x).$$

We then have that $f_n(x) = 0$ for all $x \in \mathbb{Q}$ such that $n! \cdot x \in \mathbb{Z}$. In particular, if $x \in \mathbb{Q}$, we have that $f_n(x) = 0$ for all large enough $x$, and so $f_n(x)$ is bounded for $x \in \mathbb{Q}$.

It remains to show only that $f_n(x)$ is unbounded when $x \notin \mathbb{Q}$. Indeed, consider the sets

$$B_M = \{ x : f_n(x) \text{ eventually bounded by } M \}$$

$$= \bigcup_k \bigcap_{n \geq k} \left( \bigcup_j \left[ j \cdot \frac{M}{j!} - \frac{M}{2(n+1)!} \cdot \frac{j}{n!} + \frac{M}{2(n+1)!} \cdot \frac{j}{n!} + \frac{M}{2(n+1)!} \right] \right)$$

$$= \bigcup_k S_{M,k}.$$

Observe that the sets $S_{M,k}$ are finite; indeed, it is easy to check that

$$S_{M,k} = \{ \frac{j}{k!} : j \in \mathbb{Z} \}.$$

This tells us that

$$B_M = \bigcup_k S_{M,k} = \mathbb{Q},$$

and so

$$\{ x : f_n(x) \text{ is bounded} \} = \bigcup_M B_M = \mathbb{Q}.$$

Thus $f_n(x)$ is unbounded for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

2. We just showed that the set of points where $g_n(x)$ is bounded is

$$\bigcup_{M=1}^{\infty} \bigcup_k \bigcap_{n \geq k} \{ x : g_n(x) \leq M \}.$$

$g_n(x)$ is continuous, so this is a countable union of closed sets, and so is a $F_\sigma$ set. $\mathbb{R} \setminus \mathbb{Q}$ is not an $F_\sigma$ set, so we cannot have such a sequence.

Here is a quick proof that $\mathbb{R} \setminus \mathbb{Q}$ is not an $F_\sigma$. If it were, then $\mathbb{Q}$ would be a $G_\delta$ set. Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ would then both be dense $G_\delta$ sets, so the Baire category theorem tells us that their intersection should be dense. But this is impossible because they have empty intersection.

I.5 Let $C$ be a closed convex set in a Hilbert space $\mathcal{H}$. Prove that $C$ contains a unique element of minimal norm.
Let 
\[ \eta = \inf_{z \in C} \|z\|. \]
If \( \eta = 0 \), then there is a sequence \( z_j \in C \), \( \|z_j\| \to 0 \), so \( z_j \to 0 \). \( C \) is closed, so \( 0 \in C \), and then \( 0 \) is the unique element of minimal norm. We may thus assume that \( \eta > 0 \).

We start by claiming that \( C \) contains an element of minimal norm. Indeed, take \( z_j \in C \) such that \( \|z_j\| \to \eta \). The convexity of \( C \) implies that \( \frac{z_j + z_k}{2} \in C \), so that
\[
\|z_j + z_k\|^2 = 4 \left\| \frac{z_j + z_k}{2} \right\|^2 \geq 4\eta^2.
\]
Recall now the parallelogram law for a Hilbert space:
\[
\|x - y\|^2 + \|x + y\|^2 = 2 \left( \|x\|^2 + \|y\|^2 \right).
\]
Applying this we have that
\[
\|z_j - z_k\|^2 = 2 \left( \|z_k\|^2 + \|z_j\|^2 \right) - \|z_j + z_k\|^2
\leq 2 \left( \|z_k\|^2 + \|z_j\|^2 \right) - 4\eta^2 \to 4\eta^2 - 4\eta^2 = 0
\]
as \( j, k \to \infty \). Thus the \( z_j \) are a Cauchy sequence and so converge to some \( z \in \mathcal{H} \). \( C \) is closed, so \( z \in C \), and the norm function is continuous, so \( \|z\| = \eta \), so that \( C \) contains an element of minimal norm.

The uniqueness of this element also follows from the parallelogram law and the convexity of \( C \). If \( x, y \) are two such elements, then by the above we must have
\[
\|x + y\|^2 \geq 4\eta^2,
\]
and so the parallelogram law implies that
\[
\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 \leq 4\eta^2 - 4\eta^2 = 0,
\]
and so \( x = y \).

\[ \blacksquare \]

**II.1** Let \( f \in C^0([\alpha, \beta]) \), where \( 0 < \alpha < \beta < 1 \). For each \( n = 1, 2, \ldots \), define
\[
P_n(x) = \frac{\int_{\alpha}^{\beta} f(u) \left[ 1 - (u - x)^2 \right]^n du}{\int_{-1}^{1} (1 - u^2)^n du}.
\]
Show that \( P_n(x) \) is a polynomial of degree at most \( 2n \) and that for any closed subinterval \( [a, b] \subset (\alpha, \beta) \), \( P_n \to f \) uniformly.
Consider first the polynomials
\[
k_n(t) = \frac{(1-t^2)^n}{\int_{-1}^{1} (1-u^2)^n du}.
\]
Observe that \(\int_{-1}^{1} k_n(t)dt = 1\), and that \(k_n(t) \geq 0\). We claim that for any \(\delta > 0\), \(k_n(t)\) converges uniformly to 0 on \([-1, 1]\setminus(-\delta, \delta)\).

We’ll start by computing a formula for
\[
A_n = \int_{-1}^{1} (1-u^2)^n du.
\]
Indeed, observe that by integrating by parts, we can see that
\[
\int_{-1}^{1} t^2 (1-t^2)^n dt = -\frac{t}{2(n+1)}(1-t^2)^{n+1}|_{-1}^{1} + \frac{1}{2(n+1)} \int_{-1}^{1} (1-t^2)^{n+1} dt
\]
\[
= 0 + \frac{A_{n+1}}{2(n+1)} = \frac{A_{n+1}}{2n+2}.
\]
We then have the recurrence relation
\[
A_{n+1} = \int_{-1}^{1} (1-t^2)^{n+1} = A_n - \frac{1}{2(n+1)} A_{n+1},
\]
i.e.
\[
A_{n+1} = \frac{2n+2}{2n+3} A_n.
\]
We can easily verify that
\[
A_1 = \int_{-1}^{1} (1-t^2) dt = \frac{4}{3},
\]
so we have
\[
A_n = 2 \prod_{j=1}^{n} \frac{2j}{2j+1}.
\]
Now fix \(\delta > 0\). If \(n\) is large enough, then \(\frac{2n+2}{2n+3} \geq (1-\delta^2)^{1/2}\), and so
\[
k_{n+1}(\delta) = \frac{(1-\delta^2)A_n}{A_{n+1}} k_n(\delta) \leq (1-\delta^2)^{1/2} k_n(\delta).
\]
In particular, we have that if \(N\) is large, then for all \(n \geq N\),
\[
k_n(\delta) \leq (1-\delta^2)^{(n-N)/2} k_N(\delta).
\]
\(k_n(t)\) is decreasing for \(t > 0\) and is also an even function, so this tells us that for all \(x \in [-1, 1]\setminus(-\delta, \delta)\),
\[
k_n(x) \leq (1-\delta^2)^{(n-N)/2} k_N(\delta),
\]
so we have shown the desired uniform convergence.

Now that we know these properties of $k_n$, we wish to show that $P_n$ converges uniformly to $f$. We can write

$$P_n(x) = \int_\alpha^\beta f(u)k_n(x-u)du = \int_{-1}^1 f(u)\chi(u)k_n(x-u)du = \int_{-1}^1 f(x-u)\chi(x-u)k_n(u)du,$$

where $\chi$ is the characteristic function of the interval $[\alpha, \beta]$.

Fix $\epsilon > 0$ and $[a, b] \subset (\alpha, \beta)$. Let $\delta > 0$ be such that $|b - \beta| < \delta$, $|a - \alpha| < \delta$, and that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $|x - y| < \delta$. Let $N$ be such that for all $n \geq N$,

$$|k_n(x)| \leq \frac{\epsilon}{4\|f\|_{C^0}}$$

whenever $x \in [-1, 1] \setminus (-\delta, \delta)$. We then use that $\int_{-1}^1 k_n = 1$ to write

$$|P_n(x) - f(x)| = \left|\int_{-1}^1 (f(x-u)\chi(x-u) - f(x)\chi(x))k_n(u)du\right|$$

$$\leq \int_{[-1,1]\setminus(-\delta,\delta)} |f(x)\chi(x) - f(x-u)\chi(x-u)|k_n(u)du$$

$$+ \int_{-\delta}^\delta |f(x-u)\chi(x-u) - f(x)\chi(x)|k_n(u)du$$

$$\leq 2\|f\|_{C^0} \int_{[-1,1]\setminus(-\delta,\delta)} k_n(u)du + \frac{\epsilon}{2} \int_{-1}^1 k_n(u)du$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

because $|x - (x - u)| = |u| < \delta$.

That $P_n$ are polynomials of degree at most $2n$ follows from the fact that we may write $k_n(x-u) = \sum_{i=0}^{2n} g_{n,i}(u)x^i$, where $g_{n,i}$ are polynomials in $u$, and then

$$P_n(x) = \sum_{i=0}^{n} x^i \int_\alpha^\beta f(u)g_{n,i}(u)du.$$ 

II.2 Let $W$ be any vector space, and suppose that $u, v_1, \ldots, v_k$ are linear functionals on $W$. Endow $W$ with the weakest topology such that the functionals $v_1, \ldots, v_k$ are continuous. Suppose that $u$ is continuous in this topology. Prove that $u$ is a linear combination of the $v_j$.

We first claim that

$$\bigcap_{j=1}^k \ker v_j \subset \ker u.$$
Indeed, we know that a basis for the topology of $W$ at 0 consists of the sets

$$U_\epsilon = \{ w \in W : |v_j(w)| < \epsilon \text{ for all } j = 1, \ldots, k \}.$$  

$u$ is continuous with respect to this topology, so there is some $\epsilon > 0$ such that

$$U_\epsilon \subset \{ w \in W : |u(w)| < 1 \},$$

and so for all $\delta > 0$,

$$U_{\delta\epsilon} \subset \{ w \in W : |u(w)| < \delta \}.$$

We know that

$$\bigcap_{j=1}^{k} \ker v_j \subset U_\epsilon$$

for all $\epsilon$, and so

$$\bigcap_{j=1}^{k} \ker v_j \subset \{ w \in W : |u(w)| < \delta \}$$

for all $\delta$. Thus if $x \in \cap \ker v_j$, then $u(x) = 0$, i.e. $x \in \ker u$.

Now we claim that if

$$\bigcap_{j=1}^{k} \ker v_j \subset \ker u,$$

then $u$ must be a linear combination of the $v_j$. Without loss of generality, we may assume that the $v_j$ are linearly independent. Consider the map $W \to \mathbb{R}^{k+1}$ given by

$$w \mapsto (v_1(w), \ldots, v_k(w), u(w)).$$

We know that $\cap \ker v_j \subset \ker u$, so $(0, \ldots, 0, 1)$ is not in the image of this map. There is thus some non-zero vector $(\beta_1, \ldots, \beta_k, \alpha)$ orthogonal (by the standard dot product on $\mathbb{R}^{k+1}$) to the image of this map. This tells us that for all $w \in W$,

$$\alpha u(w) + \sum_{j=1}^{k} \beta_j v_j(w) = 0.$$  

In other words, $\alpha u + \sum \beta_j v_j = 0$. The $v_j$ were linearly independent, so $\alpha \neq 0$ or else this would give a linear dependence among the $v_j$. Thus

$$u = -\frac{1}{\alpha} \sum_{j=1}^{k} \beta_j v_j,$$

so that $u$ is a linear combination of the $v_j$.  

\[ \blacksquare \]