MICROLOCAL ANALYSIS OF ASYMPTOTICALLY HYPERBOLIC AND KERR-DE SITTER SPACES

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WITH AN APPENDIX BY SEMYON DYATLOV

Abstract. In this paper we develop a general, systematic, microlocal framework for the Fredholm analysis of non-elliptic problems, including high energy (or semiclassical) estimates, which is stable under perturbations. This framework, described in Section 2, resides on a compact manifold without boundary, hence in the standard setting of microlocal analysis.

Many natural applications arise in the setting of non-Riemannian b-metrics in the context of Melrose's b-structures. These include asymptotically de Sitter-type metrics on a blow-up of the natural compactification, Kerr-de Sitter-type metrics, as well as asymptotically Minkowski metrics.

The simplest application is a new approach to analysis on Riemannian or Lorentzian (or indeed, possibly of other signature) conformally compact spaces (such as asymptotically hyperbolic or de Sitter spaces), including a new construction of the meromorphic extension of the resolvent of the Laplacian in the Riemannian case, as well as high energy estimates for the spectral parameter in strips of the complex plane. These results are also available in a follow-up paper which is more expository in nature, [54].

The appendix written by Dyatlov relates his analysis of resonances on exact Kerr-de Sitter space (which then was used to analyze the wave equation in that setting) to the more general method described here.

1. Introduction

In this paper we develop a general microlocal framework which in particular allows us to analyze the asymptotic behavior of solutions of the wave equation on asymptotically Kerr-de Sitter and Minkowski space-times, as well as the behavior of the analytic continuation of the resolvent of the Laplacian on so-called conformally compact spaces. This framework is non-perturbative, and works, in particular, for black holes, for relatively large angular momenta (the restrictions come *purely* from dynamics, and not from methods of analysis of PDE), and also for perturbations of Kerr-de Sitter space, where 'perturbation' is only relevant to the extent that it guarantees that the relevant structures are preserved. In the context of analysis on conformally compact spaces, our framework establishes a Riemannian-Lorentzian duality; in this duality the spaces of different signature are smooth continuations of each other across a boundary at which the differential operator we study has some radial points in the sense of microlocal analysis.

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Since it is particularly easy to state, and only involves Riemannian geometry, we start by giving a result on manifolds with *even* conformally compact metrics. These are Riemannian metrics g_0 on the interior of a compact manifold with boundary X_0 such that near the boundary Y, with a product decomposition nearby and a defining function x, they are of the form

$$g_0 = \frac{dx^2 + h}{x^2},$$

where h is a family of metrics on ∂X_0 depending on x in an even manner, i.e. only even powers of x show up in the Taylor series. (There is a much more natural way to phrase the evenness condition, see [28, Definition 1.2].) We also write $X_{0,\mathrm{even}}$ for the manifold X_0 when the smooth structure has been changed so that x^2 is a boundary defining function; thus, a smooth function on X_0 is even if and only if it is smooth when regarded as a function on $X_{0,\mathrm{even}}$. The analytic continuation of the resolvent in this category (but without the evenness condition) was obtained by Mazzeo and Melrose [37], with possibly some essential singularities at pure imaginary half-integers as noticed by Borthwick and Perry [6]. Using methods of Graham and Zworski [26], Guillarmou [28] showed that for even metrics the latter do not exist, but generically they do exist for non-even metrics. Further, if the manifold is actually asymptotic to hyperbolic space (note that hyperbolic space is of this form in view of the Poincaré model), Melrose, Sá Barreto and Vasy [41] showed high energy resolvent estimates in strips around the real axis via a parametrix construction; these are exactly the estimates that allow expansions for solutions of the wave equation in terms of resonances. Estimates just on the real axis were obtained by Cardoso and Vodev for more general conformal infinities [7, 58]. One implication of our methods is a generalization of these results.

Below $\dot{\mathcal{C}}^\infty(X_0)$ denotes 'Schwartz functions' on X_0, i.e. $\mathcal{C}^\infty$ functions vanishing with all derivatives at ∂X_0, and $\mathcal{C}^{-\infty}(X_0)$ is the dual space of 'tempered distributions' (these spaces are naturally identified for X_0 and $X_{0,\mathrm{even}}$), while $H^s(X_{0,\mathrm{even}})$ is the standard Sobolev space on $X_{0,\mathrm{even}}$ (corresponding to extension across the boundary, see e.g. [32, Appendix B], where these are denoted by $\bar{H}^s(X^\circ_{0,\mathrm{even}})$) and $H^s_h(X_{0,\mathrm{even}})$ is the standard semiclassical Sobolev space, so for $h > 0$ fixed this is the same as $H^s(X_{0,\mathrm{even}})$; see [17, 63].

Theorem. *(See Theorem 4.3 for the full statement.) Suppose that X_0 is an $(n-1)$-dimensional manifold with boundary Y with an even Riemannian conformally compact metric g_0. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-2}{2}\right)^2 - \sigma^2,$$

written as $\mathcal{R}(\sigma) : L^2 \to L^2$, has a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$,

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If, further, (X_0, g_0) is non-trapping, then non-trapping estimates hold in every strip $-C < \operatorname{Im}\sigma < C_+$, $|\operatorname{Re}\sigma| \gg 0$: for $s > \frac{1}{2} + C$,

$$\|x^{-(n-2)/2+\imath\sigma}\mathcal{R}(\sigma)f\|_{H^s_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+2)/2+\imath\sigma}f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}, \tag{1.1}$$

where $|\sigma|^{-1}$ is the semiclassical parameter. If f has compact support in X°_0, the $s-1$ norm on f can be replaced by the $s-2$ norm.

Further, as stated in Theorem 4.3, the resolvent is *semiclassically outgoing* with a loss of h^{-1}, in the sense of recent results of Datchev and Vasy [15] and [16]. This means that for mild trapping (where, in a strip near the spectrum, one has polynomially bounded resolvent for a compactly localized version of the trapped model) one obtains resolvent bounds of the same kind as for the above-mentioned trapped models, and lossless estimates microlocally away from the trapping. In particular, one obtains logarithmic losses compared to non-trapping on the spectrum for hyperbolic trapping in the sense of [61, Section 1.2], and polynomial losses in strips, since for the compactly localized model this was recently shown by Wunsch and Zworski [61].

For conformally compact spaces, without using wave propagation as motivation, our method is to change the smooth structure, replacing x by $\mu = x^2$, conjugate the operator by an appropriate weight as well as remove a vanishing factor of μ, and show that the new operator continues smoothly and non-degenerately (in an appropriate sense) across $\mu = 0$, i.e. Y, to a (non-elliptic) problem which we can analyze using by now almost standard tools of microlocal analysis. These steps are reflected in the form of the estimate (1.1); μ shows up in the evenness, conjugation due to the presence of $x^{-n/2+\imath\sigma}$, and the two halves of the vanishing factor of μ being removed in $x^{\pm 1}$ on the left and right hand sides. This approach is explained in full detail in the more expository and self-contained follow-up article, [54].

However, it is useful to think of a wave equation motivation — then $(n-1)$-dimensional hyperbolic space shows up[1] (essentially) as a model at infinity inside a backward light cone from a fixed point q_+ at future infinity on n-dimensional de Sitter space $\hat{M}$, see [53, Section 7], where this was used to construct the Poisson operator. More precisely, the light cone is singular at q_+, so to desingularize it, consider $[\hat{M}; \{q_+\}]$. After a Mellin transform in the defining function of the front face; the model continues smoothly across the light cone Y inside the front face of $[\hat{M}; \{q_+\}]$. The inside of the light cone corresponds to $(n-1)$-dimensional hyperbolic space (after conjugation, etc.) while the exterior is (essentially) $(n-1)$-dimensional de Sitter space; Y is the 'boundary' separating them. Here Y (or the whole light cone in $[\hat{M}; \{q_+\}]$) should be thought of as the event horizon in black hole terms (there is nothing more to event horizons in terms of local geometry!).

The resulting operator P_σ has radial points at the conormal bundle $N^*Y \setminus o$ of Y in the sense of microlocal analysis, i.e. the Hamilton vector field is radial at these points, i.e. is a multiple of the generator of dilations of the fibers of the cotangent bundle there. However, tools exist to deal with these, going back to Melrose's geometric treatment of scattering theory on asymptotically Euclidean spaces [39]. Note that $N^*Y \setminus o$ consists of two components, Λ_+, resp. Λ_-, and in $S^*X = (T^*X \setminus o)/\mathbb{R}^+$ the images, L_+, resp. L_-, of these are sinks, resp. sources, for the Hamilton flow. At $L_\pm$ one has choices regarding the direction one wants to propagate estimates (into or out of the radial points), which directly correspond to working with strong or weak Sobolev spaces. For the present problem, the relevant choice is propagating estimates *away from* the radial points, thus working with the 'good' Sobolev spaces (which can be taken to have as positive order as one wishes;

[1] General asymptotically hyperbolic spaces do not arise from a similar blow-up of a de Sitter-type space, rather could be thought of as a generalization of the blown-up n-dimensional de Sitter space, i.e. the generalization is *after* the blow-up. A different perspective, via an asymptotically Minkowski space, is briefly discussed in Section 5, and in more detail in [55].

there is a minimum amount of regularity imposed by our choice of propagation direction, cf. the requirement $s > \frac{1}{2} + C$ above (1.1)). All other points are either elliptic, or real principal type. It remains to either deal with the non-compactness of the 'far end' of the $(n-1)$-dimensional de Sitter space — or instead, as is indeed more convenient when one wants to deal with more singular geometries, adding complex absorbing 'potentials' (which are pseudodifferential operators here), in the spirit of works of Nonnenmacher and Zworski [44] and Wunsch and Zworski [61], and 'capping off' the manifold in the absorbing region to make it compact (e.g. by doubling, making the problem on the double elliptic by complex absorption). In fact, the complex absorption could be replaced by adding a space-like boundary, see Remark 2.6, but for many microlocal purposes complex absorption is more desirable, hence we follow the latter method. However, crucially, these complex absorbing techniques (or the addition of a space-like boundary) already enter in the non-semiclassical problem in our case, as we are in a non-elliptic setting.

One can reverse the direction of the argument and analyze the wave equation on an $(n-1)$-dimensional even asymptotically de Sitter space X'_0 by extending it across the boundary, much like the the Riemannian conformally compact space X_0 is extended in this approach. Then, performing microlocal propagation in the opposite direction, which amounts to working with the adjoint operators that we already need in order to prove existence of solutions for the Riemannian spaces[2], we obtain existence, uniqueness and structure results for asymptotically de Sitter spaces, recovering a large part[3] of the results of [53]. Here we only briefly indicate this method of analysis in Remark 4.6.

In other words, we establish a Riemannian-Lorentzian duality, that will have counterparts both in the pseudo-Riemannian setting of higher signature and in higher rank symmetric spaces, though in the latter the analysis might become more complicated. Note that asymptotically hyperbolic and de Sitter spaces are not connected by a 'complex rotation' (in the sense of an actual deformation); they are smooth continuations of each other in the sense we just discussed.

To emphasize the simplicity of our method, we list all of the microlocal techniques (which are relevant both in the classical and in the semiclassical setting) that we use on a *compact manifold without boundary*; in all cases *only microlocal Sobolev estimates* matter (not parametrices, etc.):

(i) Microlocal elliptic regularity.
(ii) Real principal type propagation of singularities.
(iii) *Rough* analysis at a Lagrangian invariant under the Hamilton flow which roughly behaves like a collection of radial points, though the internal structure does not matter, in the spirit of [39, Section 9].
(iv) Complex absorbing 'potentials' in the spirit of [44] and [61].

These are almost 'off the shelf' in terms of modern microlocal analysis, and thus our approach, from a microlocal perspective, is quite simple. We use these to show that on the continuation across the boundary of the conformally compact space we have a Fredholm problem, on a perhaps slightly exotic function space,

[2] This adjoint analysis also shows up for Minkowski space-time as the 'original' problem.

[3] Though not the parametrix construction for the Poisson operator, or for the forward fundamental solution of Baskin [1]; for these we would need a parametrix construction in the present compact boundaryless, but analytically non-trivial (for this purpose), setting.

which however is (perhaps apart from the complex absorption) the simplest possible coisotropic function space based on a Sobolev space, with order dictated by the radial points. Also, we propagate the estimates along bicharacteristics in different directions depending on the component $\Sigma_\pm$ of the characteristic set under consideration; correspondingly the sign of the complex absorbing 'potential' will vary with $\Sigma_\pm$, which is perhaps slightly unusual. However, this is completely parallel to solving the standard Cauchy, or forward, problem for the wave equation, where one propagates estimates in *opposite* directions relative to the Hamilton vector field in the two components.

The complex absorption we use modifies the operator P_σ outside $X_{0,\mathrm{even}}$. However, while $(P_\sigma - \imath Q_\sigma)^{-1}$ depends on Q_σ, its behavior on $X_{0,\mathrm{even}}$, and even near $X_{0,\mathrm{even}}$, is independent of this choice; see the proof of Proposition 4.2 for a detailed explanation. In particular, although $(P_\sigma - \imath Q_\sigma)^{-1}$ may have resonances other than those of $\mathcal{R}(\sigma)$, the (dual) resonant states of these additional resonances are supported outside $X_{0,\mathrm{even}}$, hence do not affect the singular behavior of the resolvent in $X_{0,\mathrm{even}}$. In the setting of Kerr-de Sitter space an analogous role is played by semiclassical versions of the standard energy estimate; this is stated in Subsection 3.3.

While the results are stated for the scalar equation, analogous results hold for operators on natural vector bundles, such as the Laplacian on differential forms. This is so because the results work if the principal symbol of the extended problem is scalar with the demanded properties, and the imaginary part of the subprincipal symbol is either scalar at the 'radial sets', or instead satisfies appropriate estimates (as an endomorphism of the pull-back of the vector bundle to the cotangent bundle) at this location; see Remark 2.1. The only change in terms of results on asymptotically hyperbolic spaces is that the threshold $(n-2)^2/4$ is shifted; in terms of the explicit conjugation of Subsection 4.9 this is so because of the change in the first order term in (4.30).

While here we mostly consider conformally compact Riemannian or Lorentzian spaces (such as hyperbolic space and de Sitter space) as appropriate boundary values (Mellin transform) of a blow-up of de Sitter space of one higher dimension, they also show up as a boundary value of Minkowski space. This is related to Wang's work on b-regularity [60], though Wang worked on a blown up version of Minkowski space-time; she also obtained her results for the (non-linear) Einstein equation there. It is also related to the work of Fefferman and Graham [22] on conformal invariants by extending an asymptotically hyperbolic manifold to Minkowski-type spaces of one higher dimension. We discuss asymptotically Minkowski spaces briefly in Section 5.

Apart from trapping — which is well away from the event horizons for black holes that do not rotate too fast — the microlocal structure on de Sitter space is *exactly* the same as on Kerr-de Sitter space, or indeed Kerr space near the event horizon. (Kerr space has a Minkowski-type end as well; although Minkowski space also fits into our framework, it does so a different way than Kerr at the event horizon, so the result there is not immediate; see the comments below.) This is to be understood as follows: from the perspective we present here (as opposed to the perspective of [53]), the tools that go into the analysis of de Sitter space-time suffice also for Kerr-de Sitter space, and indeed a much wider class, apart from the need to deal with trapping. The trapping itself was analyzed by Wunsch and Zworski [61]; their work fits immediately with our microlocal methods. Phenomena such as the ergosphere are shadows of a barely changed dynamics in the phase space, whose projection to the base space (physical space) undergoes serious changes. It is thus of great value to work microlocally, although it is certainly possible that for some non-linear purposes it is convenient to rely on physical space to the maximum possible extent, as was done in the recent (linear) works of Dafermos and Rodnianski [13, 14].

Below we state theorems for Kerr-de Sitter space time. However, it is important to note that all of these theorems have analogues in the general microlocal framework discussed in Section 2. In particular, analogous theorems hold on conjugated, re-weighted, and even versions of Laplacians on conformally compact spaces (of which one example was stated above as a theorem), and similar results apply on 'asymptotically Minkowski' spaces, with the slight twist that it is adjoints of operators considered here that play the direct role there.

We now turn to Kerr-de Sitter space-time and give some history. In exact Kerr-de Sitter space and for small angular momentum, Dyatlov [20, 19] has shown exponential decay to constants, even across the event horizon. This followed earlier work of Melrose, Sá Barreto and Vasy [40], where this was shown up to the event horizon in de Sitter-Schwarzschild space-times or spaces strongly asymptotic to these (in particular, no rotation of the black hole is allowed), and of Dafermos and Rodnianski in [11] who had shown polynomial decay in this setting. These in turn followed up pioneering work of Sá Barreto and Zworski [47] and Bony and Häfner [5] who studied resonances and decay away from the event horizon in these settings. (One can solve the wave equation explicitly on de Sitter space using special functions, see [45] and [62]; on asymptotically de Sitter spaces the forward fundamental solution was constructed as an appropriate Lagrangian distribution by Baskin [1].)

Also, polynomial decay on Kerr space was shown recently by Tataru and Tohaneanu [50, 49] and Dafermos and Rodnianski [13, 14], after pioneering work of Kay and Wald in [33] and [59] in the Schwarzschild setting. (There was also recent work by Marzuola, Metcalf, Tataru and Tohaneanu [36] on Strichartz estimates, and by Donninger, Schlag and Soffer [18] on L^∞ estimates on Schwarzschild black holes, following L^∞ estimates of Dafermos and Rodnianski [12, 10], of Blue and Soffer [4] on non-rotating charged black holes giving L^6 estimates, and Finster, Kamran, Smoller and Yau [23, 24] on Dirac waves on Kerr.) While some of these papers employ microlocal methods at the trapped set, they are mostly based on physical space where the phenomena are less clear than in phase space (unstable tools, such as separation of variables, are often used in phase space though). We remark that Kerr space is less amenable to immediate microlocal analysis to attack the decay of solutions of the wave equation due to the singular/degenerate behavior at zero frequency, which will be explained below briefly. This is closely related to the behavior of solutions of the wave equation on Minkowski space-times. Although our methods also deal with Minkowski space-times, this holds in a slightly different way than for de Sitter (or Kerr-de Sitter) type spaces at infinity, and combining the two ingredients requires some additional work. On perturbations of Minkowski space itself, the full non-linear analysis was done in the path-breaking work of Christodoulou and Klainerman [9], and Lindblad and Rodnianski simplified the analysis [34, 35], Bieri [2, 3] succeeded in relaxing the decay conditions, while Wang [60] obtained additional, b-type, regularity as already mentioned. Here we only give a linear result, but hopefully its simplicity will also shed new light on the non-linear problem.

As already mentioned, a microlocal study of the trapping in Kerr or Kerr-de Sitter was performed by Wunsch and Zworski in [61]. This is particularly important to us, as this is the only part of the phase space which does not fit directly into a relatively simple microlocal framework. Our general method is to use microlocal analysis to understand the rest of the phase space (with localization away from trapping realized via a complex absorbing potential), then use the gluing result of Datchev and Vasy [15] to obtain the full result.

Slightly more concretely, there is a partial compactification of space-time near the boundary of which the space-time has the form $X_\delta \times [0, \tau_0)_\tau$, where X_δ denotes an extension of the space-time across the event horizon. Thus, there is a manifold with boundary X_0, whose boundary Y is the event horizon, such that X_0 is embedded into X_δ, a (non-compact) manifold without boundary. We write $X_+ = X_0^\circ$ for 'our side' of the event horizon and $X_- = X_\delta \setminus X_0$ for the 'far side'. Over compact subsets of X_+, τ behaves like $e^{-\tilde{t}}$, where $\tilde{t}$ is the standard Kerr-de Sitter 'time'; see Section 6. Then the Kerr or Kerr-de Sitter d'Alembertians (or wave operators) are b-operators in the sense of Melrose [43] that extend smoothly across the event horizon Y. We further extend X_δ to a compact manifold without boundary (e.g. by doubling X_δ over its boundary) and extend the d'Alembertian as well in a somewhat arbitrary manner; the complex absorption we impose later serves to make the problem elliptic in this region, and the the particular choices we make do not affect the wave equation asymptotics in X_δ (we refer to Subsection 3.3 for details). Recall that in the Riemannian setting, b-metrics (whose Laplacians are then b-operators) are usually called 'cylindrical ends', see [43] for a general description; here the form of the d'Alembertian at the boundary (i.e. 'infinity') is similar, modulo ellipticity (which is lost). Our results hold for small smooth perturbations of Kerr-de Sitter space in this b-sense. Here the role of 'perturbations' is simply to ensure that the microlocal picture, in particular the dynamics, has not changed drastically. Although b-analysis is the right conceptual framework, we mostly work with the Mellin transform, hence on manifolds without boundary, so the reader need not be concerned about the lack of familiarity with b-methods. However, we briefly discuss the basics in Section 3.

We *immediately* Mellin transform in the defining function of the boundary (which is temporal infinity, though is not space-like everywhere) — in Kerr and Kerr-de Sitter spaces this is operation is 'exact', corresponding to $\tau\partial_\tau$ being a Killing vector field, i.e. is not merely at the level of normal operators, but this makes little difference (i.e. the general case is similarly treatable). After this transform we get a family of operators that e.g. in de Sitter space is elliptic on X_+, but in Kerr-de Sitter space (as well as in Kerr space in the analogous region) ellipticity is lost there. We consider the event horizon as a completely artificial boundary even in the de Sitter setting, i.e. work on a manifold that includes a neighborhood of $X_0 = \overline{X_+}$, hence a neighborhood of the event horizon Y.

As already mentioned, one feature of these space-times is some relatively mild trapping in X_+; this only plays a role in high energy (in the Mellin parameter, σ), or equivalently semiclassical (in $h = |\sigma|^{-1}$) estimates. We ignore a (semiclassical) microlocal neighborhood of the trapping for a moment; we place an absorbing 'potential' there. Another important feature of the space-times is that they are not naturally compact on the 'far side' of the event horizon (inside the black hole), i.e.

X_-, and bicharacteristics from the event horizon (classical or semiclassical) propagate into this region. However, we place an absorbing 'potential' (a second order operator) there to annihilate such phenomena which do not affect what happens on 'our side' of the event horizon, X_+, in view of the characteristic nature of the latter. This absorbing 'potential' could *easily* be replaced by a space-like boundary, in the spirit of introducing a boundary $t = t_1$, where $t_1 > t_0$, when one solves the Cauchy problem from t_0 for the standard wave equation; note that such a boundary does not affect the solution of the equation in $[t_0, t_1]_t$. Alternatively, if X_- has a well-behaved infinity, such as in de Sitter space, the analysis could be carried out more globally. However, as we wish to emphasize the microlocal simplicity of the problem, we do not touch on these issues.

All of our results are in a general setting of microlocal analysis explained in Section 2, with the Mellin transform and Lorentzian connection explained in Section 3. However, for the convenience of the reader here we state the results for perturbations of Kerr-de Sitter spaces. We refer to Section 6 for details. First, the general assumption is that

> P_σ, $\sigma \in \mathbb{C}$, is either the Mellin transform of the d'Alembertian $\Box_g$ for a Kerr-de Sitter spacetime, or more generally the Mellin transform of the normal operator of the d'Alembertian $\Box_g$ for a small perturbation, in the sense of b-metrics, of such a Kerr-de Sitter space-time;

see Section 3 for an explanation of these concepts. Note that for such perturbations the usual 'time' Killing vector field (denoted by $\partial_{\tilde{t}}$ in Section 6; this is indeed time-like in $X_+ \times [0, \epsilon)_{\tilde{t}}$ sufficiently far from ∂X_+) is no longer Killing. Our results on these space-times are proved by showing that the hypotheses of Section 2 are satisfied. We show this in general (under the conditions (6.2), which corresponds to $0 < \frac{9}{4}\Lambda r_s^2 < 1$ in de Sitter-Schwarzschild spaces, and (6.13), which corresponds to the lack of classical trapping in X_+; see Section 6), except where semiclassical dynamics matters. As in the analysis of Riemannian conformally compact spaces, we use a complex absorbing operator Q_σ; this means that its principal symbol in the relevant (classical, or semiclassical) sense has the correct sign on the characteristic set; see Section 2.

When semiclassical dynamics does matter, the *non-trapping assumption* with an absorbing operator Q_σ, $\sigma = h^{-1}z$, is

> in both the forward and backward directions, the bicharacteristics from any point in the semiclassical characteristic set of P_σ either enter the semiclassical elliptic set of Q_σ at some finite time, or tend to $L_\pm$;

see Definition 2.12. Here, as in the discussion above, $L_\pm$ are two components of the image of $N^*Y \setminus o$ in S^*X. (As L_+ is a sink while L_- is a source, even semiclassically, outside $L_\pm$ the 'tending' can only happen in the forward, resp. backward, directions.) Note that the semiclassical non-trapping assumption (in the precise sense used below) implies a classical non-trapping assumption, i.e. the analogous statement for classical bicharacteristics, i.e. those in S^*X. It is important to keep in mind that the classical non-trapping assumption can always be satisfied with Q_σ supported in X_-, far from Y.

In our first result in the Kerr-de Sitter type setting, to keep things simple, we ignore semiclassical trapping via the use of Q_σ; this means that Q_σ will have support in X_+. However, in X_+, Q_σ only matters in the semiclassical, or high energy, regime, and only for σ with bounded imaginary part. If the black hole is rotating relatively slowly, e.g. a satisfies the bound (6.27), the (semiclassical) trapping is always far from the event horizon, and one can make Q_σ supported away from there. Also, the Klein-Gordon parameter λ below is 'free' in the sense that it does not affect any of the relevant information in the analysis[4] (principal and subprincipal symbol; see below). *Thus, we drop it in the following theorems for simplicity.*

Theorem 1.1. *Let Q_σ be an absorbing operator such that the semiclassical non-trapping assumption holds. Let $\sigma_0 \in \mathbb{C}$, and*

$$\mathcal{X}^s = \{u \in H^s :\ (P_{\sigma_0} - \imath Q_{\sigma_0})u \in H^{s-1}\},\ \mathcal{Y}^s = H^{s-1},$$
$$\|u\|^2_{\mathcal{X}^s} = \|u\|^2_{H^s} + \|(P_{\sigma_0} - \imath Q_{\sigma_0})u\|^2_{H^{s-1}}.$$

Let $\beta_\pm > 0$ be given by the geometry at conormal bundle of the black hole $(-)$, resp. de Sitter $(+)$ event horizons, see Subsection 6.1, and in particular (6.10). For $s \in \mathbb{R}$, let[5] $\beta = \max(\beta_+, \beta_-)$ if $s \geq 1/2$, $\beta = \min(\beta_+, \beta_-)$ if $s < 1/2$. Then, for $\lambda \in \mathbb{C}$,

$$P_\sigma - \imath Q_\sigma - \lambda : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on

$$\mathbb{C}_s = \Big\{\sigma \in \mathbb{C} :\ \operatorname{Im}\sigma > \beta^{-1}\Big(\frac{1}{2} - s\Big)\Big\} \tag{1.2}$$

and has a meromorphic inverse,

$$R(\sigma) = (P_\sigma - \imath Q_\sigma - \lambda)^{-1},$$

which is holomorphic in an upper half plane, $\operatorname{Im}\sigma > C$. Moreover, given any $C' > 0$, there are only finitely many poles in $\operatorname{Im}\sigma > -C'$, and the resolvent satisfies non-trapping estimates there, which e.g. with $s = 1$ (which might need a reduction in $C' > 0$) take the form

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)f\|^2_{L^2} \leq C''|\sigma|^{-2}\|f\|^2_{L^2}.$$

The analogous result also holds on Kerr space-time if we suppress the Euclidean end by a complex absorption.

Dropping the semiclassical absorption in X_+, i.e. if we make Q_σ supported only in X_-, we have[6]

Theorem 1.2. *Let P_σ, β, $\mathbb{C}_s$ be as in Theorem 1.1, and let Q_σ be an absorbing operator supported in X_- which is classically non-trapping. Let $\sigma_0 \in \mathbb{C}$, $\mathcal{X}^s$ and $\mathcal{Y}^s$ as in Theorem 1.1. Then, $P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$ is an analytic family of Fredholm operators on $\mathbb{C}_s$, and has a meromorphic inverse,*

$$R(\sigma) = (P_\sigma - \imath Q_\sigma)^{-1},$$

[4] It does affect the *location* of the poles and corresponding resonant states of $(P_\sigma - \imath Q_\sigma)^{-1}$, hence the constant in Theorem 1.4 has to be replaced by the appropriate resonant state and exponential growth/decay, as in the second part of that theorem.

[5] This means that we require the stronger of $\operatorname{Im}\sigma > \beta_\pm^{-1}(1/2-s)$ to hold in (1.2). If we perturb Kerr-de Sitter space time, we need to increase the requirement on $\operatorname{Im}\sigma$ slightly, i.e. the size of the half space has to be slightly reduced.

[6] Since we are not making a statement for almost real σ, semiclassical trapping, discussed in the previous paragraph, does not matter.

which for any $\epsilon > 0$ is holomorphic in a translated sector in the upper half plane, $\operatorname{Im}\sigma > C + \epsilon|\operatorname{Re}\sigma|$*. The poles of the resolvent are called* resonances*. In addition,* $R(\sigma)$ *satisfies non-trapping estimates, e.g. with* $s = 1$*,*

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)f\|^2_{L^2} \leq C'|\sigma|^{-2}\|f\|^2_{L^2}$$

in such a translated sector.

It is in this setting that Q_σ could be replaced by working on a manifold with boundary, with the boundary being space-like, essentially as a time level set mentioned above, since it is supported in X_-.

Now we make the assumption that *the only semiclassical trapping is due to hyperbolic trapping with trapped set* Γ_z*,* $\sigma = h^{-1}z$, with hyperbolicity understood as in the 'Dynamical Hypotheses' part of [61, Section 1.2], i.e.

> in both the forward and backward directions, the bicharacteristics from any point in the semiclassical characteristic set of P_σ either enter the semiclassical elliptic set of Q_σ at some finite time, or tend to $L_\pm \cup \Gamma_z$.

We remark that just hyperbolicity of the trapped set suffices for the results of [61], see Section 1.2 of that paper; however, if one wants stability of the results under perturbations, one needs to assume that Γ_z is *normally hyperbolic*. We refer to [61, Section 1.2] for a discussion of these concepts. We show in Section 6 that for black holes satisfying (6.27) (so the angular momentum can be comparable to the mass) the operators Q_σ can be chosen so that they are supported in X_- (even quite far from Y) and the hyperbolicity requirement is satisfied. Further, we also show that for slowly rotating black holes the trapping is normally hyperbolic. Moreover, the statement of (normally) hyperbolic trapping is purely a statement in Hamiltonian dynamics, i.e. it is separate from the core subject of this paper (though of course it has implications for the subject of this paper). It might be known for an even larger range of rotation speeds, but the author is not aware of this.

Under this assumption, one can combine Theorem 1.1 with the results of Wunsch and Zworski [61] about hyperbolic trapping and the gluing results of Datchev and Vasy [15] to obtain a better result for the merely spatially localized (in the sense that Q_σ does not have support in X_+, unlike in Theorem 1.1) problem, Theorem 1.2:

Theorem 1.3. *Let P_σ, Q_σ, β, $\mathbb{C}_s$, $\mathcal{X}^s$ and $\mathcal{Y}^s$ be as in Theorem 1.2, and assume that the only semiclassical trapping is due to hyperbolic trapping. Then,*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on $\mathbb{C}_s$*, and has a meromorphic inverse,*

$$R(\sigma) = (P_\sigma - \imath Q_\sigma)^{-1},$$

which is holomorphic in an upper half plane, $\operatorname{Im}\sigma > C$*. Moreover, there exists* $C' > 0$ *such that there are only finitely many poles in* $\operatorname{Im}\sigma > -C'$*, and the resolvent satisfies polynomial estimates there as* $|\sigma| \to \infty$*,* $|\sigma|^\varkappa$*, for some* $\varkappa > 0$*, compared to the non-trapping case, with merely a logarithmic loss compared to non-trapping for real* σ*, e.g. with* $s = 1$*:*

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)f\|^2_{L^2} \leq C''|\sigma|^{-2}(\log|\sigma|)^2\|f\|^2_{L^2}.$$

Farther, there are approximate lattices of poles generated by the trapping, as studied by Sá Barreto and Zworski in [47], and further by Bony and Häfner in [5], in the exact De Sitter-Schwarzschild and Schwarzschild settings, and in ongoing work[7] by Dyatlov in the exact Kerr-de Sitter setting.

Theorem 1.3 immediately and directly gives the asymptotic behavior of solutions of the wave equation across the event horizon. Namely, the asymptotics of the wave equation depends on the finite number of resonances; their precise behavior depends on specifics of the space-time, i.e. on these resonances. This is true even in arbitrarily regular b-Sobolev spaces – in fact, the more decay we want to show, the higher Sobolev spaces we need to work in. Thus, a forteriori, this gives L^∞ estimates. We state this formally as a theorem in the simplest case of slow rotation; in the general case one needs to analyze the (finite!) set of resonances along the reals to obtain such a conclusion, and for the perturbation part also to show normal hyperbolicity (which we only show for slow rotation):

Theorem 1.4. *Let M_δ be the partial compactification of Kerr-de Sitter space as in Section 6, with τ the boundary defining function; thus, τ behaves like $e^{-\tilde{t}}$ over compact subsets of X_+, where $\tilde{t}$ is the standard Kerr-de Sitter 'time' variable. Suppose that g is either a slowly rotating Kerr-de Sitter metric, or a small perturbation as a symmetric bilinear form on ${}^{\mathrm{b}}TM_\delta$. Then there exist $C' > 0$, $\varkappa > 0$ such that for $0 < \epsilon < C'$ and $s > 1/2 + \beta\epsilon$ solutions of $\Box_g u = f$ with $f \in \tau^\epsilon H_{\mathrm{b}}^{s-1+\varkappa}(M_\delta)$ vanishing in $\tau > \tau_0$, and with u vanishing in $\tau > \tau_0$, satisfy that for some constant c_0,*

$$u - c_0 \in \tau^\epsilon H^s_{\mathrm{b,loc}}(M_\delta).$$

Here the norms of both c_0 in $\mathbb{C}$ and $u - c_0$ in $\tau^\epsilon H^s_{\mathrm{b,loc}}(M_{\delta'})$ are bounded by those of f in $\tau^\epsilon H_{\mathrm{b}}^{s-1+\varkappa}(M_\delta)$ for $\delta' < \delta$.

More generally, if g is a Kerr-de Sitter metric with hyperbolic trapping[8], then there exist $C' > 0$, $\varkappa > 0$ such that for $0 < \ell < C'$ and $s > 1/2 + \beta\ell$ solutions of $\Box_g u = f$ with $f \in \tau^\ell H_{\mathrm{b}}^{s-1+\varkappa}(M_\delta)$ vanishing in $\tau > \tau_0$, and with u vanishing in $\tau > \tau_0$, satisfy that for some $a_{j\kappa} \in \mathcal{C}^\infty(X_\delta)$ (which are resonant states) and $\sigma_j \in \mathbb{C}$ (which are the resonances),

$$u' = u - \sum_j \sum_{\kappa \le m_j} \tau^{\imath\sigma_j} (\log|\tau|)^\kappa a_{j\kappa} \in \tau^\ell H^s_{\mathrm{b,loc}}(M_\delta).$$

Here the (semi)norms of both $a_{j\kappa}$ in $\mathcal{C}^\infty(X_{\delta'})$ and u' in $\tau^\ell H^s_{\mathrm{b,loc}}(M_{\delta'})$ are bounded by that of f in $\tau^\ell H_{\mathrm{b}}^{s-1+\varkappa}(M_\delta)$ for $\delta' < \delta$. The same conclusion holds for sufficiently small perturbations of the metric as a symmetric bilinear form on ${}^{\mathrm{b}}TM_\delta$ provided the trapping is normally hyperbolic.

In special geometries (without the ability to add perturbations) such decay has been described by delicate separation of variables techniques, again see Bony-Häfner [5] in the De Sitter-Schwarzschild and Schwarzschild settings, but only away from the event horizons, and by Dyatlov [20, 19] in the Kerr-de Sitter setting. Thus, in these settings, we recover in a direct manner Dyatlov's result across the event horizon [19], modulo a knowledge of resonances near the origin contained in [20]. In fact, for small angular momenta one can use the results from de Sitter-Schwarzschild space directly to describe these finitely many resonances, as exposed in the works of Sá Barreto and Zworski [47], Bony and Häfner [5] and Melrose, Sá Barreto and Vasy [40], since 0 is an isolated resonance with multiplicity 1 and eigenfunction

[7] This has been completed after the original version of this manuscript, see [21].

[8] This is shown in Section 6 when (6.27) is satisfied.

1; this persists under small deformations, i.e. for small angular momenta. Thus, exponential decay to constants, Theorem 1.4, follows immediately.

One can also work with Kerr space-time, apart from issues of analytic continuation. By using weighted spaces and Melrose's results from [39] as well as those of Vasy and Zworski in the semiclassical setting [57], one easily gets an analogue of Theorem 1.2 in $\operatorname{Im}\sigma > 0$, with smoothness and the almost non-trapping estimates corresponding to those of Wunsch and Zworski [61] down to $\operatorname{Im}\sigma = 0$ for $|\operatorname{Re}\sigma|$ large. Since a proper treatment of this would exceed the bounds of this paper, we refrain from this here. Unfortunately, even if this analysis were carried out, low energy problems would still remain, so the result is not strong enough to deduce the wave expansion. As already alluded to, Kerr space-time has features of both Minkowski and de Sitter space-times; though both of these fit into our framework, they do so in different ways, so a better way of dealing with the Kerr space-time, namely adapting our methods to it, requires additional work.

While de Sitter-Schwarzschild space (the special case of Kerr-de Sitter space with vanishing rotation), via the same methods as those on de Sitter space which give rise to the hyperbolic Laplacian and its continuation across infinity, gives rise essentially to the Laplacian of a conformally compact metric, with similar structure but different curvature at the two ends (this was used by Melrose, Sá Barreto and Vasy [40] to do analysis up to the event horizon there), the analogous problem for Kerr-de Sitter is of edge-type in the sense of Mazzeo's edge calculus [38] apart from a degeneracy at the poles corresponding to the axis of rotation, though it is not Riemannian. Note that edge operators have global properties in the fibers; in this case these fibers are the orbits of rotation. A reasonable interpretation of the appearance of this class of operators is that the global properties in the fibers capture non-constant (or non-radial) bicharacteristics (in the classical sense) in the conormal bundle of the event horizon, and also possibly the (classical) bicharacteristics entering X_+. This suggests that the methods of Melrose, Sá Barreto and Vasy [40] would be much harder to apply in the presence of rotation.

It is important to point out that the results of this paper are stable under small $\mathcal{C}^\infty$ perturbations[9] of the Lorentzian metric on the b-cotangent bundle at the cost of changing the function spaces slightly; this follows from the estimates being stable in these circumstances. Note that the function spaces depend on the principal symbol of the operator under consideration, and the range of σ depends on the subprincipal symbol at the conormal bundle of the event horizon; under general small smooth perturbations, defining the spaces exactly as before, the results remain valid if the range of σ is slightly restricted.

In addition, the method is stable under gluing: already Kerr-de Sitter space behaves as two separate black holes (the Kerr and the de Sitter end), connected by semiclassical dynamics; since only one component (say $\Sigma_{\hbar,+}$) of the semiclassical characteristic set moves far into X_+, one can easily add as many Kerr black holes as one wishes by gluing beyond the reach of the other component, $\Sigma_{\hbar,-}$. Theorems 1.1 and 1.2 automatically remain valid (for the semiclassical characteristic set is then irrelevant), while Theorem 1.3 remains valid provided that the resulting dynamics only exhibits mild trapping (so that compactly localized models have at

[9]Certain kinds of perturbations conormal to the boundary, in particular polyhomogeneous ones, would only change the analysis and the conclusions slightly.

most polynomial resolvent growth), such as normal hyperbolicity, found in Kerr-de Sitter space.

Since the specifics of Kerr-de Sitter space-time are, as already mentioned, irrelevant in the microlocal approach we take, we start with the abstract microlocal discussion in Section 2, which is translated into the setting of the wave equation on manifolds with a Lorentzian b-metric in Section 3, followed by the description of de Sitter, Minkowski and Kerr-de Sitter space-times in Sections 4, 5 and 6. In order to streamline the arguments, we present results for $C_- < \operatorname{Im}\sigma < C_+$ in Sections 2 and 3; then in the final Section 7 we describe what happens when the upper bound is dropped on $\operatorname{Im}\sigma$. The more general result allowing $\operatorname{Im}\sigma$ large is convenient, and results in easier to state and somewhat stronger main theorems (though arguably the main point of all our results is what happens for $\operatorname{Im}\sigma$ in a bounded interval), but in order to minimize the additional complications it causes, it is is moved to its own separate section at the end of the paper. Theorems 1.1-1.4 are proved in Section 6 by showing that they fit into the abstract framework of Section 2; the approach is completely analogous to de Sitter and Minkowski spaces, where the fact that they fit into the abstract framework is shown in Sections 4 and 5. As another option, we encourage the reader to read the discussion of de Sitter space first, which also includes the discussion of conformally compact spaces, presented in Section 4, as well as Minkowski space-time presented in the section afterwards, to gain some geometric insight, then the general microlocal machinery, and finally the Kerr discussion to see how that space-time fits into our setting. Finally, if the reader is interested how conformally compact metrics fit into the framework and wants to jump to the relevant calculation, a reasonable place to start is Subsection 4.9. We emphasize that for the conformally compact Riemannian results, only Section 2 and Section 4.4-4.9, starting with the paragraph of (4.9), are strictly needed.

2. Microlocal framework

We now develop a setting which includes the geometry of the 'spatial' model of de Sitter space near its 'event horizon', as well as the model of Kerr and Kerr-de Sitter settings near the event horizon, and the model at infinity for Minkowski space-time near the light cone (corresponding to the adjoint of the problem described below in the last case). As a general reference for microlocal analysis, we refer to [32], while for semiclassical analysis, we refer to [17, 63]; see also [48] for the high-energy (or large parameter) point of view.

2.1. **Notation.** We recall the basic conversion between these frameworks. First, $S^k(\mathbb{R}^p;\mathbb{R}^\ell)$ is the set of $\mathcal{C}^\infty$ functions on $\mathbb{R}^p_z \times \mathbb{R}^\ell_\zeta$ satisfying uniform bounds

$$|D_z^\alpha D_\zeta^\beta a| \leq C_{\alpha\beta}\langle\zeta\rangle^{k-|\beta|},\ \alpha\in\mathbb{N}^p,\ \beta\in\mathbb{N}^\ell.$$

If $O \subset \mathbb{R}^p$ and $\Gamma \subset \mathbb{R}^\ell_\zeta$ are open, we define $S^k(O;\Gamma)$ by requiring[10] these estimates to hold only for $z \in O$ and $\zeta \in \Gamma$. We also let $S^{-\infty} = \cap_k S^k$; this is the same as a uniformly bounded (with all derivatives) z-dependent family of Schwartz functions in ζ (in the cone Γ). The class of classical (or one-step polyhomogeneous) symbols is

[10]Another possibility would be to require uniform estimates on compact subsets; this makes no difference here.

the subset $S^k_{\mathrm{cl}}(\mathbb{R}^p;\mathbb{R}^\ell)$ of $S^k(\mathbb{R}^p;\mathbb{R}^\ell)$ consisting of symbols possessing an asymptotic expansion

$$a(z,r\omega)\sim\sum a_j(z,\omega)r^{k-j},$$

where $a_j\in\mathcal{C}^\infty(\mathbb{R}^p\times\mathbb{S}^{\ell-1})$. Then on $\mathbb{R}^n_z$, pseudodifferential operators $A\in\Psi^k(\mathbb{R}^n)$ are of the form

$$A=\mathrm{Op}(a);\ \mathrm{Op}(a)u(z)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i(z-z')\cdot\zeta}a(z,\zeta)\,u(z')\,d\zeta\,dz',$$
$$u\in\mathcal{S}(\mathbb{R}^n),\ a\in S^k(\mathbb{R}^n;\mathbb{R}^n);$$

understood as an oscillatory integral. Classical pseudodifferential operators, $A\in\Psi^k_{\mathrm{cl}}(\mathbb{R}^n)$, form the subset where a is a classical symbol. The principal symbol $\sigma_k(A)$ of $A\in\Psi^k(\mathbb{R}^n)$ is the equivalence class $[a]$ of a in $S^k(\mathbb{R}^n;\mathbb{R}^n)/S^{k-1}(\mathbb{R}^n;\mathbb{R}^n)$. For classical a, one can instead consider $a_0(z,\omega)r^k$ as the principal symbol; it is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, which is homogeneous of degree k with respect to the $\mathbb{R}^+$-action given by dilations in the second factor, $\mathbb{R}^n\setminus\{0\}$. An operator $A\in\Psi^k(\mathbb{R}^n)$ is elliptic at $(z,\zeta)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ if (z,ζ) has a conic (i.e. invariant under dilations in the second factor) neighborhood $O\times\Gamma$ such that $((1-\chi)a^{-1})|_{O\times\Gamma}\in S^{-k}$ where χ has compact support, or equivalently a is invertible for large ζ in this cone with symbolic bounds in S^{-k} for this inverse there. This notion is unchanged if a symbol in S^{k-1} is added to a, i.e. is a property of the principal symbol. The set of elliptic points is denoted $\mathrm{ell}_k(A)$, or just $\mathrm{ell}(A)$ if the order is understood. For classical symbols a, this simply means $a_0(z,\omega)\neq 0$ where $\zeta=r\omega$, $r>0$. The wave front set $\mathrm{WF}'(A)$ is a closed conic (i.e. invariant under dilations in the second factor) subset of $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$; a point (z,ζ) is *not* in $\mathrm{WF}'(A)$ if it has a conic neighborhood $O\times\Gamma$, restricted to which a is in $S^{-\infty}$ (i.e. Schwartz). For classical symbols a this means that all the a_j (not merely a_0) vanish on such a neighborhood.

Differential operators on $\mathbb{R}^n$ form the subset of $\Psi(\mathbb{R}^n)$ in which a is polynomial in the second factor, $\mathbb{R}^n_\zeta$, so locally

$$A=\sum_{|\alpha|\leq k}a_\alpha(z)D_z^\alpha,\qquad\sigma_k(A)=\sum_{|\alpha|=k}a_\alpha(z)\zeta^\alpha.$$

If X is a manifold, one can transfer these definitions to X by localization and requiring that the Schwartz kernels are $\mathcal{C}^\infty$ right densities (i.e. densities in the right, or second, factor of $X\times X$) away from the diagonal in $X^2=X\times X$; then $\sigma_k(A)$ is in $S^k(T^*X)/S^{k-1}(T^*X)$, resp. $S^k_{\mathrm{hom}}(T^*X\setminus o)$ when $A\in\Psi^k(X)$, resp. $A\in\Psi^k_{\mathrm{cl}}(X)$; here o is the zero section, and hom stands for symbols homogeneous with respect to the $\mathbb{R}^+$ action, while $\mathrm{WF}'(A)$ is a conic subset of $T^*X\setminus o$, or equivalently a subset of $S^*X=(T^*X\setminus o)/\mathbb{R}^+$. If A is a differential operator, then the classical (or homogeneous) version of the principal symbol is a homogeneous polynomial in the fibers of the cotangent bundle of degree k. We can also work with operators depending on a parameter $\lambda\in O$ by replacing $a\in S^k(\mathbb{R}^n;\mathbb{R}^n)$ by $a\in S^k(\mathbb{R}^n\times O;\mathbb{R}^n)$, with $\mathrm{Op}(a_\lambda)\in\Psi^k(\mathbb{R}^n)$ smoothly dependent on $\lambda\in O$. For differential operators, a_α would simply depend smoothly on the parameter λ.

It is often convenient to work with the fiber-radial compactification $\overline{T}^*X$ of T^*X, in particular when discussing semiclassical analysis; see for instance [39, Sections 1 and 5]. This is a ball-bundle over X, with fiber being the radial compactification of the vector space T^*_zX as a closed ball. Thus, S^*X should be considered as the boundary of $\overline{T}^*X$. When one is working with homogeneous objects, as is the case in classical microlocal analysis, one can think of S^*X as $(\overline{T}^*X \setminus o)/\mathbb{R}^+$, but this is not a useful point of view in semiclassical analysis[11]. Thus, if $\tilde{\rho}$ is a non-vanishing homogeneous degree -1 function on $T^*X \setminus o$, it is a defining function of S^*X in $\overline{T}^*X \setminus o$; if the homogeneity requirement is dropped it can be modified near the zero section to make it a defining function of S^*X in $\overline{T}^*X$. The principal symbol a of $A \in \Psi^k_{\mathrm{cl}}(X)$ is a homogeneous degree k function on $T^*X \setminus o$, so $\tilde{\rho}^k a$ is homogeneous degree 0 there, thus are smooth functions[12] on $\overline{T}^*X$ near its boundary, S^*X, and in particular on S^*X. Moreover, H_a is homogeneous degree $k-1$ on $T^*X \setminus o$, thus $\tilde{\rho}^{k-1}\mathsf{H}_a$ a smooth vector field tangent to the boundary on $\overline{T}^*X$ (defined near the boundary), and in particular induces a smooth vector field on S^*X.

The large parameter, or high energy, version of $\Psi^k(\mathbb{R}^n)$, with the large parameter denoted by σ, is that

$$A^{(\sigma)} = \mathrm{Op}^{(\sigma)}(a),\ \mathrm{Op}^{(\sigma)}(a)u(z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} a(z,\zeta,\sigma)\,u(z')\,d\zeta\,dz',$$
$$u \in \mathcal{S}(\mathbb{R}^n),\ a \in S^k(\mathbb{R}^n;\mathbb{R}^n_\zeta \times \Omega_\sigma),$$

where $\Omega \subset \mathbb{C}$, with $\mathbb{C}$ identified with $\mathbb{R}^2$; thus there are joint symbol estimates in ζ and σ. The high energy principal symbol now should be thought of as an equivalence class of functions on $\mathbb{R}^n_z \times \mathbb{R}^n_\zeta \times \Omega_\sigma$, or invariantly on $T^*X \times \Omega$. Differential operators with polynomial dependence on σ now take the form

$$A^{(\sigma)} = \sum_{|\alpha|+j\leq k} a_{\alpha,j}(z)\sigma^j D_z^\alpha, \qquad \sigma_k^{(\sigma)}(A) = \sum_{|\alpha|+j=k} a_{\alpha,j}(z)\sigma^j\zeta^\alpha. \tag{2.1}$$

Note that the principal symbol includes terms that would be subprincipal with $A^{(\sigma)}$ considered as a differential operator for a fixed value of σ.

The semiclassical operator algebra[13], $\Psi_\hbar(\mathbb{R}^n)$, is given by

$$A_h = \mathrm{Op}_\hbar(a);\ \mathrm{Op}_\hbar(a)u(z) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta/h} a(z,\zeta,h)\,u(z')\,d\zeta\,dz',$$
$$u \in \mathcal{S}(\mathbb{R}^n),\ a \in \mathcal{C}^\infty([0,1)_h; S^k(\mathbb{R}^n_z;\mathbb{R}^n_\zeta));$$

its classical subalgebra, $\Psi_{\hbar,\mathrm{cl}}(\mathbb{R}^n)$ corresponds to $a \in \mathcal{C}^\infty([0,1)_h; S^k_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{R}^n_\zeta))$. The semiclassical principal symbol is now $\sigma_{\hbar,k}(A) = a|_{h=0} \in S^k(T^*X)$; the 'standard' principal symbol is still the equivalence class of a in $\mathcal{C}^\infty([0,1)_h; S^k/S^{k-1})$, or an element of $\mathcal{C}^\infty([0,1)_h; S^k_{\mathrm{hom}})$ in the classical setting. There are natural extensions to manifolds X.

[11] In fact, even in classical microlocal analysis it is better to keep at least a 'shadow' of the interior of S^*X by working with $T^*X \setminus o$ considered as a half-line bundle over S^*X with homogeneous objects on it; this keeps the action of the Hamilton vector field on the fiber-radial variable, i.e. the defining function of S^*X in $\overline{T}^*X$, non-trivial, which is important at radial points, which in turn play a central role below.

[12] This depends on choices unless $k = 0$; they are naturally sections of a line bundle that encodes the differential of the boundary defining function at S^*X. However, the only relevant notion here is ellipticity, and later the Hamilton vector field up to multiplication by a positive function, which is independent of choices. In fact, we emphasize that all the requirements in Subsection 2.2 listed for p, q and later $p_{\hbar,z}$ and $q_{\hbar,z}$, except possibly (2.5)-(2.6), are also fulfilled if $P_\sigma - \imath Q_\sigma$ is replaced by *any* smooth positive multiple, so one may factor out positive factors at will. This is useful in the Kerr-de Sitter space discussion. For (2.5)-(2.6), see Footnote 19.

[13] We adopt the convention that $\hbar$ denotes semiclassical objects, while h is the actual semiclassical parameter.

Here it is particularly instructive to consider the compactified point of view. Thus, for $A \in \Psi^0_{\hbar,\mathrm{cl}}(X)$, the corresponding a is a smooth function on $[0,1)_h \times \overline{T}^* X$. The two principal symbols are just the restrictions of this function to the two boundary hypersurfaces, namely $\{0\} \times \overline{T}^* X$ and $[0,1)_h \times \partial\overline{T}^* X = [0,1)_h \times S^*X$; note the compatibility at the corner $\{0\} \times S^*X$. Ellipticity as well as the wave front set are then naturally defined on this fiber-compactified space, namely $\mathrm{ell}_\hbar(A)$ and $\mathrm{WF}'_\hbar(A)$ are subsets of $\partial([0,1)_h \times \overline{T}^* X) = \{0\} \times \overline{T}^* X \cup [0,1)_h \times S^*X$, in the case of $\mathrm{ell}_\hbar(A)$ given by the non-vanishing of $a|_{\partial([0,1)_h \times \overline{T}^* X)}$, while in the case of the *complement* of $\mathrm{WF}'_\hbar(A)$, there being a neighborhood in $[0,1)_h \times \overline{T}^* X$ restricted to which a vanishes to infinite order at *both* boundary hypersurfaces (away from the corner, only one of the hypersurfaces would intersect sufficiently small neighborhoods, so this is the only relevant information). If $A \in \Psi^k_{\hbar,\mathrm{cl}}(X)$, then with $\tilde\rho$ as above one can replace a by $\tilde\rho^k a$ for this discussion, see Footnote 12. Even if A is non-classical, restriction of a to $U \setminus [0,1)_h \times \partial\overline{T}^* X \subset [0,1)_h \times T^*X$, U a neighborhood of a point in question in $\partial([0,1)_h \times \overline{T}^* X)$, can be used to define $\mathrm{ell}_\hbar(A)$ and $\mathrm{WF}'_\hbar(A)$ in a completely analogous manner. One can also define the semiclassical wavefront set of a distribution relative to a Sobolev space $h^r H^s_\hbar(X)$, namely a point $\alpha \in \partial([0,1)_h \times \overline{T}^* X)$ is *not* in $\mathrm{WF}^{s,r}_\hbar(u_h)$ of a polynomially bounded family $\{u_h\}_{h\in(0,1)}$ (i.e. $h^N u_h \in H^{-N}_\hbar(X)$ for some N) if there exists $A \in \Psi^0_\hbar(X)$ elliptic at α with $\{A_h u_h\}_{h\in(0,1)} \in h^r H^s_\hbar(X)$.

We can again add an extra parameter $\lambda \in O$, so $a \in \mathcal{C}^\infty([0,1)_h; S^k(\mathbb{R}^n \times O; \mathbb{R}^n_\zeta))$; then in the invariant setting the principal symbol is $a|_{h=0} \in S^k(T^*X \times O)$. Note that if $A^{(\sigma)} = \mathrm{Op}^{(\sigma)}(a)$ is a classical operator with a large parameter, then for $\lambda \in O \subset \mathbb{C}$, $\overline{O}$ compact, $0 \notin \overline{O}$,

$$h^k \,\mathrm{Op}^{(h^{-1}\lambda)}(a) = \mathrm{Op}_\hbar(\tilde a),\ \tilde a(z,\zeta,h) = h^k a(z, h^{-1}\zeta, h^{-1}\lambda),$$

and $\tilde a \in \mathcal{C}^\infty([0,1)_h; S^k_{\mathrm{cl}}(\mathbb{R}^n \times O_\lambda; \mathbb{R}^n_\zeta))$. The converse is not quite true: roughly speaking, the semiclassical algebra is a blow-up[14] of the large parameter algebra; to obtain an equivalence, we would need to demand in the definition of the large parameter algebra merely that $a \in S^k(\mathbb{R}^n; \overline{[\mathbb{R}^n_\zeta \times \Omega_\sigma; \partial\mathbb{R}^n_\zeta \times \{0\}]})$, so in particular for bounded σ, a is merely a family of symbols depending smoothly on σ (not jointly symbolic); we do not discuss this here further. Note, however, that it is the (smaller, i.e. stronger) large parameter algebra that arises naturally when one Mellin transforms in the b-setting, see Subsection 3.1.

[14]If X is a manifold with corners and Z is a product-type, or p-, submanifold, i.e. there are local coordinates $x_1, \ldots, x_l$, $y_1, \ldots, y_{n-l}$ near $p \in Z$ in which X is given by $x_1 \geq 0, \ldots, x_l \geq 0$, and Z is given by the vanishing of some of the x_j and some of the y_i, then one can blow up Z in X to obtain a new manifold with corners, $[X; Z]$. This is a space that is identical to X away from Z and in which Z is replaced by the front face ff of the blow up, namely the inward pointing spherical normal bundle, S^+NZ, of Z in X. The space comes with a $\mathcal{C}^\infty$ blow-down map $\beta : [X; Z] \to X$ which is thus a diffeomorphism away from ff. Roughly, the blow-up amounts to introducing cylindrical coordinates around Z: directions tangent to Z are unaffected, but those normal to Z are, and one is distinguishing directions of approach to Z modulo TZ; recall that $N_pZ = T_pX/T_pZ$. One can easily write down projective local coordinates on this space. We refer to the Appendix of [39] for a concise but more detailed description, and to [43, Section 4] for a more leasurely discussion in a special case.

Differential operators now take the form

$$A_{h,\lambda} = \sum_{|\alpha|\leq k} a_\alpha(z,\lambda;h)(hD_z)^\alpha. \tag{2.2}$$

Such a family has two principal symbols, the standard one (but taking into account the semiclassical degeneration, i.e. based on $(hD_z)^\alpha$ rather than D_z^α), which depends on h and is homogeneous, and the semiclassical one, which is at $h = 0$, and is not homogeneous:

$$\sigma_k(A_{h,\lambda}) = \sum_{|\alpha|=k} a_\alpha(z,\lambda;h)\zeta^\alpha,$$

$$\sigma_\hbar(A_{h,\lambda}) = \sum_{|\alpha|\leq k} a_\alpha(z,\lambda;0)\zeta^\alpha.$$

However, the restriction of $\sigma_k(A_{h,\lambda})$ to $h = 0$ is the principal part of $\sigma_\hbar(A_{h,\lambda})$. In the special case in which $\sigma_k(A_{h,\lambda})$ is independent of h (which is true in the setting considered below), one can simply regard the usual principal symbol as the principal part of the semiclassical symbol. Note that for $A^{(\sigma)}$ as in (2.1),

$$h^k A^{(h^{-1}\lambda)} = \sum_{|\alpha|+j\leq k} h^{k-j-|\alpha|} a_{\alpha,j}(z)\lambda^j (hD_z)^\alpha,$$

which is indeed of the form (2.2), with polynomial dependence on both h and λ. Note that in this case the standard principal symbol is independent of h and λ.

2.2. **General assumptions.** Let X be a compact manifold and ν a smooth non-vanishing density on it; thus $L^2(X)$ is well-defined as a Hilbert space (and not only up to equivalence). We consider operators $P_\sigma \in \Psi^k_{\mathrm{cl}}(X)$ on X depending on a complex parameter σ, with the dependence being analytic (thus, if P_σ is a differential operator, the coefficients depend analytically on σ). We also consider a complex absorbing 'potential', $Q_\sigma \in \Psi^k_{\mathrm{cl}}(X)$, defined for $\sigma \in \Omega$, $\Omega \subset \mathbb{C}$ is open. It can be convenient to take Q_σ formally self-adjoint, which is possible when Q_σ is independent of σ, but this is inconvenient when one wants to study the large σ (i.e. semiclassical) behavior. The operators we study are $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma^*$; P_σ^* depends on the choice of the density ν.

Typically we shall be interested in P_σ on an open subset U of X, and have Q_σ supported in the complement of U, such that over some subset K of $X \setminus U$, Q_σ is elliptic on the characteristic set of P_σ. In the Kerr-de Sitter setting, we would have $\overline{X_+} \subset U$. However, *this is not part of the general set-up.*

Since there are a number of ingredients we need to describe, we start by giving an example for the reader to keep in mind; we state this in a slightly simpler form which means that it is only a valid example in strips $|\operatorname{Im}\sigma| < C$. This is a slight simplification of the model in the case of the extension of the 'conjugated' Laplacian on hyperbolic space across its boundary, see (4.9); indeed, it is essentially a special case of the generalization described in (4.24) up to changing σ by a constant factor, and removing σ^2, as one can do by an appropriate (smooth) conjugation in the strip $|\operatorname{Im}\sigma| < C$. Thus, on the space $(-1,1)_\mu \times \mathbb{T}_y^{n-2}$ (so the total dimension is $n-1$) consider

$$P_\sigma = D_\mu \mu D_\mu - \sigma D_\mu + \sum_{j=1}^{n-2} D_{y_j}^2,$$

which is formally self-adjoint relative to the density $|d\mu\, dy|$ when σ is real. Thus,

$$p_{\text{full}} = \mu\xi^2 - \sigma\xi + |\eta|_y^2.$$

Here $(-1,1)$ is non-compact, but may be replaced by a circle, and then adding complex absorption Q_σ near $\mu = \pm 1$ to make the problem elliptic there would place it in our framework completely. Notice that the σD_μ term does not affect the standard principal symbol, as it is subprincipal, but nonetheless plays a major role in the solvability of the PDE, in particular in the function spaces that must be used. Indeed, the distributions $(\mu \pm \imath 0)^{\imath\sigma}$ are annihilated by P_σ on $(-1,1)_\mu \times \mathbb{T}_y^{n-2}$, and the Sobolev space $H^{1/2-\operatorname{Im}\sigma}$ which barely fails to contain these distributions is borderline for solvability properties. This is a point we explain in Subsection 2.4, where the microlocal estimates at radial points are shown, and in Subsection 2.6, where these are used to obtain actual solvability results. Note that this operator is rather different from the Tricomi operator $D_\mu^2 + \mu D_y^2$. Tricomi operators do not have radial points, as is easily verified.

We now return to discussing the general setup. We assume that the principal symbol p, resp. q, of P_σ, resp. Q_σ, are real, are independent of σ, $p = 0$ implies $dp \neq 0$. We assume that the characteristic set of P_σ is of the form

$$\Sigma = \Sigma_+ \cup \Sigma_-,\ \Sigma_+ \cap \Sigma_- = \emptyset,$$

$\Sigma_\pm$ are[15] relatively open[16] in Σ, and

$$\mp q \geq 0 \text{ near } \Sigma_\pm.$$

We assume that there are conic submanifolds $\Lambda_\pm \subset \Sigma_\pm$ of $T^*X \setminus o$, outside which the Hamilton vector field H_p is not radial, and to which the Hamilton vector field H_p is tangent, and

$$\mathrm{WF}'(Q_\sigma) \cap \Lambda_\pm = \emptyset.$$

Here $\Lambda_\pm$ are typically Lagrangian, but this is not needed[17]. The properties we want at $\Lambda_\pm$ are (probably) not stable under general smooth perturbations; the perturbations need to have certain properties at $\Lambda_\pm$. However, the estimates we then derive *are stable* under such perturbations. First, we want that for a homogeneous degree -1 defining function $\tilde\rho$ of S^*X near $L_\pm$, the image of $\Lambda_\pm$ in S^*X,

$$\tilde\rho^{k-2}\mathsf{H}_p\tilde\rho|_{L_\pm} = \mp\beta_0,\ \beta_0 \in \mathcal{C}^\infty(L_\pm),\ \beta_0 > 0. \tag{2.3}$$

Next, we require the existence of a non-negative homogeneous degree zero quadratic defining function ρ_0 of $\Lambda_\pm$ within Σ (i.e. the restriction of ρ_0 to Σ vanishes quadratically at $\Lambda_\pm$, and is non-degenerate) and $\beta_1 > 0$ such that, restricted to Σ,

$$\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0 - \beta_1\rho_0 \tag{2.4}$$

is ≥ 0 modulo[18] cubic vanishing terms at $\Lambda_\pm$. Under these assumptions, L_- is a source and L_+ is a sink for the H_p-dynamics within Σ in the sense that nearby bicharacteristics tend to $L_\pm$ as the parameter along the bicharacteristic goes to

[15]Unfortunately the sign convention here is the opposite of that adopted in the more expository paper, [54].

[16]Thus, they are connected components in the extended sense that they may be empty.

[17]An extreme example would be $\Lambda_\pm = \Sigma_\pm$. Another extreme is if one or both are empty.

[18]The precise behavior of $\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0$, or of linear defining functions, is irrelevant, because we only need a relatively weak estimate. It would be relevant if one wanted to prove Lagrangian regularity.

$\pm\infty$. Finally, we assume that the imaginary part of the subprincipal symbol at $\Lambda_\pm$, which is the symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*) \in \Psi_{\mathrm{cl}}^{k-1}(X)$ as p is real, is[19]

$$\pm\tilde\beta\beta_0(\operatorname{Im}\sigma)\tilde\rho^{-k+1},\ \tilde\beta \in \mathcal{C}^\infty(L_\pm), \tag{2.5}$$

$\tilde\beta$ is positive along $L_\pm$, and write

$$\beta_{\sup} = \sup\tilde\beta,\ \beta_{\inf} = \inf\tilde\beta > 0. \tag{2.6}$$

If $\tilde\beta$ is a constant, we may write

$$\beta = \beta_{\inf} = \beta_{\sup}. \tag{2.7}$$

The results take a little nicer form in this case since depending on various signs, sometimes $\beta_{\inf}$ and sometimes $\beta_{\sup}$ is the relevant quantity.

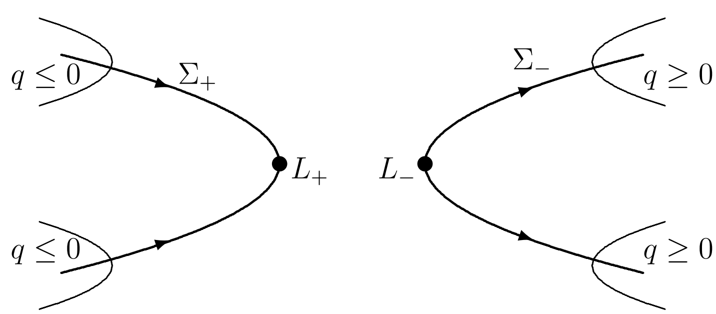

FIGURE 1. The components $\Sigma_\pm$ of the characteristic set in the cosphere bundle S^*X. The submanifolds $L_\pm$ are points here, with L_- a source, L_+ a sink. The thin lined parabolic regions near the edges show the absorbing region, i.e. the support of q. For $P_\sigma - \imath Q_\sigma$,the estimates are always propagated away from $L_\pm$ towards the support of q, so in the direction of the Hamilton flow in Σ_-, and in the direction opposite of the Hamilton flow in Σ_+; for $P_\sigma^* + \imath Q_\sigma^*$, the directions are reversed. See also Footnote 15.

We make the following *non-trapping* assumption. For $\alpha \in S^*X$, let $\gamma_+(\alpha)$, resp. $\gamma_-(\alpha)$ denote the image of the forward, resp. backward, half-bicharacteristic from α. We write $\gamma_\pm(\alpha) \to L_\pm$ (and say $\gamma_\pm(\alpha)$ tends to $L_\pm$) if given any neighborhood O of $L_\pm$, $\gamma_\pm(\alpha) \cap O \neq \emptyset$; by the source/sink property this implies that the points on the curve are in O for sufficiently large (in absolute value) parameter values. We

[19]If H_p is radial at $L_\pm$, this is independent of the choice of the density ν. Indeed, with respect to $f\nu$, the adjoint of P_σ is $f^{-1}P_\sigma^* f$, with P_σ^* denoting the adjoint with respect to ν. This is $P_\sigma^* + f^{-1}[P_\sigma^*, f]$, and the principal symbol of $f^{-1}[P_\sigma^*, f] \in \Psi_{\mathrm{cl}}^{k-1}(X)$ vanishes at $L_\pm$ as $\mathsf{H}_p f = 0$. In general, we can only change the density by factors f with $\mathsf{H}_p f|_{L_\pm} = 0$, which in Kerr-de Sitter space-times would mean factors independent of ϕ at the event horizon. A similar argument shows the independence of the condition from the choice of f when one replaces P_σ by fP_σ, under the same conditions: either radiality, or just $\mathsf{H}_p f|_{L_\pm} = 0$.

assume that, with $\mathrm{ell}(Q_\sigma)$ denoting the elliptic set of Q_σ,

(2.8)
$$\alpha\in\Sigma_-\setminus L_-\Rightarrow\big(\gamma_-(\alpha)\to L_-\ \text{or}\ \gamma_-(\alpha)\cap\mathrm{ell}(Q_\sigma)\neq\emptyset\big)\ \text{and}\ \gamma_+(\alpha)\cap\mathrm{ell}(Q_\sigma)\neq\emptyset,$$
$$\alpha\in\Sigma_+\setminus L_+\Rightarrow\big(\gamma_+(\alpha)\to L_+\ \text{or}\ \gamma_+(\alpha)\cap\mathrm{ell}(Q_\sigma)\neq\emptyset\big)\ \text{and}\ \gamma_-(\alpha)\cap\mathrm{ell}(Q_\sigma)\neq\emptyset.$$

That is, all forward and backward half-(null)bicharacteristics of P_σ either enter the elliptic set of Q_σ, or go to $\Lambda_\pm$, i.e. $L_\pm$ in S^*X. The point of the assumptions regarding Q_σ and the flow is that we are able to propagate estimates forward near where $q\geq 0$, backward near where $q\leq 0$, so by our hypotheses we can always propagate estimates for $P_\sigma-\imath Q_\sigma$ from $\Lambda_\pm$ towards the elliptic set of Q_σ, and also if both ends of a bicharacteristic go to the elliptic set of Q_σ then we can propagate the estimates from one of the directions. On the other hand, for $P_\sigma^*+\imath Q_\sigma^*$, we can propagate estimates from the elliptic set of Q_σ towards $\Lambda_\pm$, and again if both ends of a bicharacteristic go to the elliptic set of Q_σ then we can propagate the estimates from one of the directions. This behavior of $P_\sigma-\imath Q_\sigma$ vs. $P_\sigma^*+\imath Q_\sigma^*$ is important for duality reasons.

Remark 2.1. For simplicity of notation we have not considered vector bundles on X. However, if E is a vector bundle on X with a positive definite inner product on the fibers and $P_\sigma,Q_\sigma\in\Psi^k_{\mathrm{cl}}(X;E)$ with scalar principal symbol p and q, and in case of P_σ the imaginary part of the subprincipal symbol is of the form (2.5) with $\tilde\beta$ a bundle-endomorphism satisfying an inequality in (2.6) as a bundle endomorphism, the arguments we present go through.

2.3. **Elliptic and real principal type points.** We now turn to analysis. First, by the usual elliptic theory, on the elliptic set of $P_\sigma-\imath Q_\sigma$, so both on the elliptic set of P_σ and on the elliptic set of Q_σ, one has elliptic estimates[20]: for all s and N, and for all $B,G\in\Psi^0(X)$ with G elliptic on $\mathrm{WF}'(B)$, $P_\sigma-\imath Q_\sigma$ elliptic on $\mathrm{WF}'(B)$,

$$\|Bu\|_{H^s}\leq C(\|G(P_\sigma-\imath Q_\sigma)u\|_{H^{s-k}}+\|u\|_{H^{-N}}), \tag{2.9}$$

with the estimate also holding[21] for $P_\sigma^*+\imath Q_\sigma^*$. By propagation of singularities[22], in $\Sigma\setminus(\mathrm{WF}'(Q_\sigma)\cup L_+\cup L_-)$, one can propagate regularity estimates either forward or

[20]Our convention in estimates such as (2.9) and (2.10) is that if one assumes that all the quantities on the right hand side are in the function spaces indicated by the norms then so is the quantity on the left hand side, and the estimate holds. As we see below, at $\Lambda_\pm$ not all relevant function space statements appear in the estimate, so we need to be more explicit there.

[21]These estimates follow immediately from the microlocal elliptic parametrix construction. Alternatively, they follow from microlocal elliptic regularity plus the closed graph theorem, as used below in the real principal type setting.

[22]See e.g. [32, Theorem 26.1.4] for the standard statement, and see the proof of [32, Theorem 26.1.6] for turning this into an estimate. Concretely, with the notation below, we may assume $-N<s$, and one has by the standard form of the theorem that if $u\in H^{-N}$, $Au\in H^s$, $GP_\sigma u\in H^{s-k+1}$, then $Bu\in H^s$. Let $\mathcal{Z}$ be the Hilbert space of distributions $u\in H^{-N}$ with $Au\in H^s$, $GP_\sigma u\in H^{s-k+1}$ with norm $\|u\|^2_{\mathcal{Z}}=\|u\|^2_{H^{-N}}+\|Au\|^2_{H^s}+\|GP_\sigma u\|^2$. Then $B:\mathcal{Z}\to H^{-N}$ is continuous, since it is already continuous $H^{-N}\to H^{-N}$, and it takes values in H^s by the standard version of the propagation of singularities. Thus, if $u_j\to u$ in $\mathcal{Z}$ and $Bu_j\to v$ in H^s, then $Bu_j\to Bu$ in H^{-N}, so $v=Bu$ and thus $v\in\operatorname{Ran}B$, so the graph of $B:\mathcal{Z}\to H^s$ is closed, so it is continuous, giving (2.10) below. However, this is a *rather* round about argument, since propagation of singularities is typically proved by positive commutator estimates (cf. the proofs of Propositions 2.3-2.4 below); these are microlocal so would need to pieced together to

backward along bicharacteristics, i.e. for all s and N, and for all $A, B, G \in \Psi^0(X)$ such that $\mathrm{WF}'(G) \cap \mathrm{WF}'(Q_\sigma) = \emptyset$, and forward (or backward) bicharacteristics from $\mathrm{WF}'(B)$ reach the elliptic set of A, while remaining in the elliptic set of G, one has estimates

$$\|Bu\|_{H^s} \leq C(\|GP_\sigma u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}}). \tag{2.10}$$

Here P_σ can be replaced by $P_\sigma - \imath Q_\sigma$ or $P_\sigma^* + \imath Q_\sigma^*$ by the condition on $\mathrm{WF}'(G)$; namely $GQ_\sigma \in \Psi^{-\infty}(X)$, and can thus be absorbed into the $\|u\|_{H^{-N}}$ term. As usual, there is a loss of one derivative compared to the elliptic estimate, i.e. the assumption on $P_\sigma u$ is H^{s-k+1}, not H^{s-k}, and one needs to make H^s assumptions on Au, i.e. regularity propagates.

Remark 2.2. We remark here that in the various estimates in Subsections 2.3-2.5, though the error terms are stated globally as $\|u\|_{H^{-N}}$, in fact can be localized to any neighborhood U of a set $K \subset X$ if $\operatorname{supp} G \subset K \times K$, with a similar property for the other pseudodifferential operators other than P_σ, Q_σ in the statement, while P_σ, Q_σ are supported sufficiently close to the diagonal (depending on U), or if, say, Q_σ is supported in $(X \setminus U) \times (X \setminus U)$. This observation is mostly of interest in the setting of the localized wave equation behavior discussed in Subsection 3.3.

2.4. **Analysis near $\Lambda_\pm$.** At $\Lambda_\pm$, for $s \geq m > (k-1)/2 - \beta \operatorname{Im}\sigma$, β given by the subprincipal symbol at $\Lambda_\pm$, we can propagate estimates *away* from $\Lambda_\pm$:

Proposition 2.3. *For $\operatorname{Im}\sigma \geq 0$, let[23] $\beta = \beta_{\mathrm{inf}}$, for $\operatorname{Im}\sigma < 0$, let $\beta = \beta_{\mathrm{sup}}$. For all N, for $s \geq m > (k-1)/2 - \beta \operatorname{Im}\sigma$, and for all $A, B, G \in \Psi^0(X)$ such that A, G are elliptic at $\Lambda_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}'(B)$ tend to $\Lambda_\pm$, with closure in the elliptic set of G, one has estimates*

$$\|Bu\|_{H^s} \leq C(\|GP_\sigma u\|_{H^{s-k+1}} + \|Au\|_{H^m} + \|u\|_{H^{-N}}), \tag{2.11}$$

in the sense that if $u \in H^{-N}$, $Au \in H^m$ and $GP_\sigma u \in H^{s-k+1}$, then $Bu \in H^s$, and (2.11) holds. In fact, Au can be dropped from the right hand side (but one must assume $Au \in H^m$):

$$Au \in H^m \Rightarrow \|Bu\|_{H^s} \leq C(\|GP_\sigma u\|_{H^{s-k+1}} + \|u\|_{H^{-N}}), \tag{2.12}$$

where $u \in H^{-N}$ and $GP_\sigma u \in H^{s-k+1}$ is considered implied by the right hand side. Note that Au does not appear on the right hand side, hence the display before the estimate.

This is completely analogous to Melrose's estimates in asymptotically Euclidean scattering theory at the radial sets [39, Section 9]. Note that the H^s regularity of Bu is 'free' in the sense that we do not need to impose H^s assumptions on u anywhere; merely H^m at $\Lambda_\pm$ does the job; of course, on $P_\sigma u$ one must make the H^{s-k+1} assumption, i.e. the loss of one derivative compared to the elliptic setting.

At the cost of changing regularity, one can propagate estimate *towards* $\Lambda_\pm$. Keeping in mind that for P_σ^* the subprincipal symbol becomes $\beta\overline{\sigma}$, we have the following:

prove (2.10); using the standard propagation of singularities avoids the explicit 'piecing together' at the cost of invoking, somewhat superfluously, the closed graph theorem.

[23] Note that this is consistent with (2.7).

Proposition 2.4. *For* $\operatorname{Im}\sigma > 0$, *let*[24] $\beta = \beta_{\sup}$, *for* $\operatorname{Im}\sigma \leq 0$, *let* $\beta = \beta_{\inf}$. *For* $s < (k-1)/2 + \beta \operatorname{Im}\sigma$, *for all* N, *and for all* $A, B, G \in \Psi^0(X)$ *such that* B, G *are elliptic at* $\Lambda_\pm$, *and forward (or backward) bicharacteristics from* $\mathrm{WF}'(B) \setminus \Lambda_\pm$ *reach* $\mathrm{ell}(A)$, *while remaining in the elliptic set of* G, *one has estimates*

$$\|Bu\|_{H^s} \leq C(\|GP_\sigma^* u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}}). \tag{2.13}$$

Both of Propositions 2.3-2.4 are stated globally in $\Lambda_\pm$ in view of the ellipticity requirements. In fact, after the original version of this paper became available, the work of Haber and Vasy [29] showed that one can localize even *within* $\Lambda_\pm$ if the latter is a Lagrangian manifold of radial points; we do not need this in the present paper. The paper [29, Section 3] treats the proof in great detail, and the arguments presented there are directly applicable in the current more general setting, with the microlocalizers used here, so it can be used as an additional reference.

Proof of Propositions 2.3-2.4. It suffices to prove that there exist O_j open with $L_\pm \subset O_{j+1} \subset O_j$, $\cap_{j=1}^\infty O_j = L_\pm$, and A_j, B_j, G_j with WF' in O_j, B_j elliptic on $L_\pm$, and in case of Proposition 2.4 such that $\mathrm{WF}'(A_j) \cap \Lambda_\pm = \emptyset$, such that the statements of the propositions hold. Indeed, in case of Proposition 2.3 the general case follows by taking j such that A, G are elliptic on O_j, use the estimate for A_j, B_j, G_j, where the right hand side then can be estimated by A and G, and then use microlocal ellipticity, propagation of singularities and a covering argument to prove the proposition. In case of Proposition 2.4, the general case follows by taking j such that G, B are elliptic on O_j, so all forward (or backward) bicharacteristics from $O_j \setminus \Lambda_\pm$ reach $\mathrm{ell}(A)$, thus microlocal ellipticity, propagation of singularities and a covering argument proves $\|A_j u\|_{H^s} \leq C(\|GP_\sigma^* u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}})$, and then the special case of the proposition for this O_j gives an estimate for $\|B_j u\|_{H^s}$ in terms of the same quantities. The full estimate for $\|Bu\|_{H^s}$ is then again a straightforward consequence of microlocal ellipticity, propagation of singularities and a covering argument.

We now consider commutants $C_\epsilon^* C_\epsilon$ with $C_\epsilon \in \Psi^{s-(k-1)/2-\delta}(X)$ for $\epsilon > 0$, uniformly bounded in $\Psi^{s-(k-1)/2}(X)$ as $\epsilon \to 0$; with the ϵ-dependence used to regularize the argument[25]. More precisely, let[26]

$$c = \phi(\rho_0)\phi_0(p_0)\tilde{\rho}^{-s+(k-1)/2}, \qquad c_\epsilon = c(1+\epsilon\tilde{\rho}^{-1})^{-\delta},\ p_0 = \tilde{\rho}^k p,$$

where $\phi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ identically 1 near 0, $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi' \leq 0$ on $[0,\infty)$ and ϕ is supported sufficiently close to 0 so that

$$\alpha \in \operatorname{supp} d(\phi \circ \rho_0) \cap \Sigma \Rightarrow \mp(\tilde{\rho}^{k-1}\mathsf{H}_p \rho_0)(\alpha) > 0; \tag{2.14}$$

such ϕ exists by (2.4). Note that the sign of $\mathsf{H}_p \tilde{\rho}^{-s+(k-1)/2}$ depends on the sign of $-s+(k-1)/2$ which explains the difference between $s > (k-1)/2$ and $s < (k-1)/2$

[24]Note the switch compared to Proposition 2.3! Also, β does not matter when $\operatorname{Im}\sigma = 0$; we define it here so that the two Propositions are consistent via dualization, which reverses the sign of the imaginary part.

[25]In particular that C_ϵ is not a continuous family with values in *classical* operators in $\Psi^{s-(k-1)/2}(X)$, so principal symbols for the family should be considered as representatives of equivalence classes modulo lower order symbols, $S^{s-(k-1)/2-1}$. The same applies to the commutators computed below.

[26]Strictly speaking, with the definitions we have adopted, one should multiply c by $1-\psi$, where ψ has compact support, identically 1 near the zero section. As this does not affect the behavior of c near fiber-infinity we do not do this explicitly here.

in Propositions 2.3-2.4 when there are no other contributions to the threshold value of s. The contribution of the subprincipal symbol, however, shifts the critical value $(k-1)/2$.

Now let $C \in \Psi^{s-(k-1)/2}(X)$ have principal symbol c, and have $\mathrm{WF}'(C) \subset \operatorname{supp} \phi \circ \rho_0$, and let $C_\epsilon = CS_\epsilon$, $S_\epsilon \in \Psi^{-\delta}(X)$ uniformly bounded in $\Psi^0(X)$ for $\epsilon > 0$, converging to Id in $\Psi^{\delta'}(X)$ for $\delta' > 0$ as $\epsilon \to 0$, with principal symbol $(1+\epsilon\tilde\rho^{-1})^{-\delta}$. Thus, the principal symbol of C_ϵ is c_ϵ.

First, consider (2.11). Then

$$\begin{aligned}\sigma_{2s}(\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma)) &= \sigma_{k-1}(\imath(P_\sigma^* - P_\sigma))c_\epsilon^2 + 2c_\epsilon \mathsf{H}_p c_\epsilon\\ &= \mp 2\Big(-\tilde\beta \operatorname{Im}\sigma \beta_0 \phi\phi_0 + \beta_0\Big(-s + \frac{k-1}{2}\Big)\phi\phi_0 \mp (\tilde\rho^{k-1}\mathsf{H}_p\rho_0)\phi'\phi_0\\ &\qquad + \delta\beta_0 \frac{\epsilon}{\tilde\rho+\epsilon}\phi\phi_0 + k\beta_0 p_0 \phi\phi_0'\Big)\phi\phi_0\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-2\delta},\end{aligned} \tag{2.15}$$

so

$$\begin{aligned}&\mp\sigma_{2s}(\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma))\\ &\quad \leq -2\beta_0\left(s - \frac{k-1}{2} + \tilde\beta \operatorname{Im}\sigma - \delta\right)\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-2\delta}\phi^2\phi_0^2\\ &\qquad + 2(\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0)\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-2\delta}\phi'\phi\phi_0^2 + k\beta_0 p_0 \tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-2\delta}\phi^2\phi_0'\phi_0.\end{aligned} \tag{2.16}$$

Here the first term on the right hand side is negative if $s-(k-1)/2+\beta\operatorname{Im}\sigma-\delta > 0$ (since $\tilde\beta \operatorname{Im}\sigma \geq \beta \operatorname{Im}\sigma$ by our definition of β), and this is the same sign as that of ϕ' term; the presence of δ (needed for the regularization) is the reason for the appearance of m in the estimate. One can either use the sharp Gårding inequality, or instead, as we do, choose ϕ so that $\sqrt{-\phi\phi'}$ is $\mathcal{C}^\infty$, and then

$$\mp\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma) = -S_\epsilon^*(B^*B + B_1^*B_1 + B_{2,\epsilon}^*B_{2,\epsilon})S_\epsilon + H_\epsilon P_\sigma + F_\epsilon,$$

with $B, B_1, B_{2,\epsilon} \in \Psi^s(X)$, $B_{2,\epsilon}$ uniformly bounded in $\Psi^s(X)$ as $\epsilon \to 0$, H_ϵ uniformly bounded in $\Psi^{2s-k}(X)$ (arising from the last term on the right hand side of (2.16) as well as from $\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0$ having the non-negative lower bound on Σ only), F_ϵ uniformly bounded in $\Psi^{2s-1}(X)$ (arising from the fact that the principal symbol of the sum of the remaining terms on the right hand side is the same as that of the left hand side by (2.16)), and $\sigma_s(B)$ an elliptic multiple of $\phi_0(p_0)\phi(\rho_0)\tilde\rho^{-s}$. Computing the pairing, using an extra regularization (insert a regularizer $\Lambda_r \in \Psi^{-1}(X)$, uniformly bounded in $\Psi^0(X)$, converging to Id in $\Psi^\delta(X)$ to justify integration by parts, and use that $[\Lambda_r, P_\sigma^*]$ is uniformly bounded in $\Psi^1(X)$, converging to 0 strongly, cf. [52, Lemma 17.1] and its use in [52, Lemma 17.2]) yields

$$\langle \imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma)u, u\rangle = \langle \imath C_\epsilon^* C_\epsilon u, P_\sigma u\rangle - \langle \imath P_\sigma u, C_\epsilon^* C_\epsilon u\rangle. \tag{2.17}$$

We use Cauchy-Schwartz on the right hand side, more precisely choosing an elliptic $T \in \Psi^{(k-1)/2}(X)$ with principal symbol $\tilde\rho^{-(k-1)/2}$, with parametrix R so $RT = \mathrm{Id} + E$, with $E \in \Psi^{-\infty}(X)$, we write

$$\langle C_\epsilon u, C_\epsilon P_\sigma u\rangle = \langle TC_\epsilon u, R^* C_\epsilon P_\sigma u\rangle + \langle EC_\epsilon, C_\epsilon P_\sigma u\rangle,$$

and write $2|\langle TC_\epsilon u, R^* C_\epsilon P_\sigma u\rangle| \leq \digamma^{-1}\|TC_\epsilon u\|^2 + \digamma\|R^* C_\epsilon P_\sigma u\|^2$, where we take $\digamma > 0$ large. We can then absorb $\digamma^{-1}\|TC_\epsilon u\|^2$ in $\frac{1}{2}\|BS_\epsilon u\|^2$; this is possible since TC_ϵ is an elliptic 0th order multiple of B (modulo lower order operators). All remaining terms on the right hand side of (2.17) are uniformly bounded as $\epsilon \to 0$

by the a priori assumptions provided u is microlocally in $H^{s-\delta}$ on $\mathrm{WF}'(C)$ (and $\delta \leq 1/2$). A standard functional analytic argument using the weak-* compactness of the unit ball in L^2 (see, for instance, Melrose [39, Proof of Proposition 7 and Section 9]) gives an estimate for Bu, showing u is in H^s on the elliptic set of B, provided u is microlocally in $H^{s-\delta}$ on $\mathrm{WF}'(B)$. A standard inductive argument, starting with $s-\delta = m$ and improving regularity by $\leq 1/2$ in each step proves (2.11).

For (2.13), the argument is similar, but we want to change the sign of the term in (2.15) corresponding to the first term on the right hand side of (2.16), i.e. we want it to be positive. Note that the regularization (the δ-term) now contributes a term of the right sign, so is not an issue. The desired positivity is thus obtained if $s-(k-1)/2+\beta\,\mathrm{Im}\,\sigma < 0$ (since $\tilde{\beta}\,\mathrm{Im}\,\sigma \leq \beta\,\mathrm{Im}\,\sigma$ by our definition of β in Proposition 2.4). On the other hand, ϕ' now has the wrong sign, so one needs to make an assumption on $\mathrm{supp}\, d\phi$, which is the Au term in (2.13). From this point on the argument is similar to that of (2.11). We again refer to [39, Section 9] and [29] for more detailed treatments in somewhat different settings. □

Remark 2.5. Fixing a ϕ, it follows from the proof that the same ϕ works for (small) smooth perturbations of P_σ with real principal symbol[27], even if those perturbations do not preserve the event horizon, namely even if (2.4) does not hold any more: only its implication, (2.14), on $\mathrm{supp}\, d\phi$ matters, which *is* stable under perturbations. Moreover, as the rescaled Hamilton vector field $\tilde{\rho}^{k-1}\mathsf{H}_p$ is a smooth vector field tangent to the boundary of the fiber-compactified cotangent bundle, i.e. a b-vector field, and as such depends smoothly on the principal symbol, and it is *non-degenerate* radially by (2.3), the weight, which provides the positivity at the radial points in the proof above, still gives a positive Hamilton derivative for small perturbations. Since this proposition thus holds for $\mathcal{C}^\infty$ perturbations of P_σ with real principal symbol, and this proposition is the only delicate estimate we use, and it is only marginally so, we deduce that all the other results below also hold in this generality.

2.5. **Complex absorption.** Finally, one has propagation estimates for complex absorbing operators, requiring a sign condition. We refer to, for instance, [44] and [15, Lemma 5.1] in the semiclassical setting; the changes are minor in the 'classical' setting. We also give a sketch of the main 'commutator' calculation below.

First, one can propagate regularity to $\mathrm{WF}'(Q_\sigma)$ (of course, in the elliptic set of Q_σ one has a priori regularity). Namely, for all s and N, and for all $A, B, G \in \Psi^0(X)$ such that $q \leq 0$, resp. $q \geq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach the elliptic set of A, while remaining in the elliptic set of G, one has the usual propagation estimates

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|Au\|_{H^s} + \|u\|_{H^{-N}}).$$

Thus, for $q \geq 0$ one can propagate regularity in the forward direction along the Hamilton flow, while for $q \leq 0$ one can do so in the backward direction.

On the other hand, one can propagate regularity away from the elliptic set of Q_σ. Namely, for all s and N, and for all $B, G \in \Psi^0(X)$ such that $q \leq 0$, resp. $q \geq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach

[27] Reality is needed to ensure that (2.15) holds.

the elliptic set of Q_σ, while remaining in the elliptic set of G, one has the usual propagation estimates

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|u\|_{H^{-N}}). \tag{2.18}$$

Again, for $q \geq 0$ one can propagate regularity in the forward direction along the Hamilton flow, while for $q \leq 0$ one can do so in the backward direction. At the cost of reversing the signs of q, this also gives that for all s and N, and for all $B, G \in \Psi^0(X)$ such that $q \geq 0$, resp. $q \leq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach the elliptic set of Q_σ, while remaining in the elliptic set of G, one has the usual propagation estimates

$$\|Bu\|_{H^s} \leq C(\|G(P_\sigma^* + \imath Q_\sigma^*)u\|_{H^{s-k+1}} + \|u\|_{H^{-N}}).$$

We remark that again, these estimates are stable under small perturbations in $\Psi^k(X)$ of P_σ and Q_σ provided the perturbed operators still have real principal symbols, and in the case of Q_σ, satisfy $q \geq 0$, since the geometric assumptions, namely $q \leq 0$, resp. $q \geq 0$, on $\mathrm{WF}'(G)$, and forward, resp. backward, bicharacteristics of P_σ from $\mathrm{WF}'(B)$ reach the elliptic set of A, while remaining in the elliptic set of G, are stable under these. However, since [15, Lemma 5.1] is stated in a somewhat different setting, we give a brief sketch, which in particular shows this stability. We follow the proof of [15, Lemma 5.1]; the role of the absorbing potential $W \geq 0$ there is played by the formally self-adjoint operator $\tilde{Q}_\sigma = \frac{1}{2}(Q_\sigma + Q_\sigma^*)$ with principal symbol q here. Although there W is a function on X (rather than a general pseudodifferential operator), the only properties that matter in the present notation are that the principal symbols are real and $q \geq 0$. Indeed, in this case, writing C (analogously to the proof of Propositions 2.3-2.4 here) instead of Q for the commutant of [15, Lemma 5.1] to avoid confusion, and denoting its (real) principal symbol by c, and letting $\tilde{P}_\sigma = P_\sigma + \frac{1}{2\imath}(Q_\sigma - Q_\sigma^*)$, so $P_\sigma - \imath Q_\sigma = \tilde{P}_\sigma - \imath\tilde{Q}_\sigma$, and the principal symbol of the formally self-adjoint operator $\tilde{P}_\sigma$ is p, we have

$$\langle u, -\imath[C^*C, \tilde{P}_\sigma]u\rangle = -2\,\mathrm{Re}\langle u, \imath C^*C(P_\sigma - \imath Q_\sigma)u\rangle - 2\,\mathrm{Re}\langle u, C^*C\tilde{Q}_\sigma u\rangle. \tag{2.19}$$

The operator on the left hand side has principal symbol $\mathsf{H}_p c^2$, and will preserve its signs under sufficiently small perturbations of p using the same construction of c as in [15, Lemma 5.1] (which is just a real-principal type construction), much as in the radial point setting discussed in the previous subsection. On the other hand, the second term on the right hand side can be rewritten as

$$2\,\mathrm{Re}\langle u, C^*C\tilde{Q}_\sigma u\rangle = 2\,\mathrm{Re}\langle u, C^*\tilde{Q}_\sigma Cu\rangle + 2\,\mathrm{Re}\langle u, C^*[C, \tilde{Q}_\sigma]u\rangle,$$

where the second term is $\langle u, [C, [C, \tilde{Q}_\sigma]]u\rangle$ plus similar pairings involving $(C^* - C)[C, \tilde{Q}_\sigma]$, etc., which are all lower order than the operator on the left hand side of (2.19) due to the real principal symbol of C and the presence of a commutator, or to the presence of the double commutator. The first term, on the other hand, is non-negative modulo terms that can be absorbed into the left hand side of (2.19), since by the sharp Gårding inequality[28], $\langle u, C^*\tilde{Q}_\sigma Cu\rangle \geq -\langle u, C^*R_\sigma Cu\rangle$ where R_σ is one order lower than Q_σ, i.e. is in $\Psi^{k-1}(X)$, and as the principal symbol of $C^*R_\sigma C$ does not contain derivatives of c, an appropriate choice of C lets one use the $\mathsf{H}_p c^2$ term, i.e. the principal symbol of the left hand side of (2.19), to dominate this, as

[28]If one assumes that q is microlocally the square of a symbol, one need not use the sharp Gårding inequality. Since q is a tool we use in the problems we study, not given to us by the problem, one may make this choice if one wishes to do so.

usual in real principal type estimates when subprincipal terms are dominated (c.f. the treatment of the $\operatorname{Im}\lambda$ term in [15, Lemma 5.1]). Thus, (2.19) implies (2.18).

Remark 2.6. As mentioned in the introduction, these complex absorption methods could be replaced in specific cases, including all the specific examples we discuss here, by adding a boundary $\tilde{Y}$ instead, provided that the Hamilton flow is well-behaved relative to the base space, namely inside the characteristic set H_p is not tangent to $T^*_{\tilde{Y}}X$ with orbits crossing $T^*_{\tilde{Y}}X$ in the opposite directions in $\Sigma_\pm$ in the following way. If $\tilde{Y}$ is defined by $\tilde{y}$ which is positive on 'our side' U with U as discussed at the beginning of Subsection 2.2, we need $\pm\mathsf{H}_p\tilde{y}|_{\tilde{Y}} > 0$ on $\Sigma_\pm$. Then the functional analysis described in [32, Proof of Theorem 23.2.2], see also [56, Proof of Lemma 4.14], can be used to prove analogues of the results we give below on $X_+ = \{\tilde{y} \geq 0\}$. For instance, if one has a Lorentzian metric on X near $\tilde{Y}$, and $\tilde{Y}$ is space-like, then (up to the sign) this statement holds with $\Sigma_\pm$ being the two components of the characteristic set. However, in the author's opinion, this detracts from the clarity of the microlocal analysis by introducing projection to physical space in an essential way.

2.6. **Global estimates.** Recall now that $q \geq 0$ near Σ_-, and $q \leq 0$ on Σ_+, and recall our non-trapping assumptions, i.e. (2.8). Thus, we can piece together the estimates described earlier (elliptic, real principal type, radial points, complex absorption) to propagate estimates forward in Σ_- and backward in Σ_+, thus away from $\Lambda_\pm$ (as well as from one end of a bicharacteristic which intersects the elliptic set of q in both directions). This yields that for any N, and for any $s \geq m > (k-1)/2 - \beta\operatorname{Im}\sigma$, and for any $A \in \Psi^0(X)$ elliptic at $\Lambda_+ \cup \Lambda_-$,

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|Au\|_{H^m} + \|u\|_{H^{-N}}).$$

This implies that for any $s > m > (k-1)/2 - \beta\operatorname{Im}\sigma$,

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \imath Q_\sigma)u\|_{H^{s-k+1}} + \|u\|_{H^m}). \tag{2.20}$$

On the other hand, recalling that the adjoint switches the sign of the imaginary part of the principal symbol and also that of the subprincipal symbol at the radial sets, propagating the estimates in the other direction, i.e. backward in Σ_- and forward in Σ_+, thus towards $\Lambda_\pm$, from the elliptic set of q, we deduce that for any N (which we take to satisfy $s' > -N$) and for any $s' < (k-1)/2 + \beta\operatorname{Im}\sigma$,

$$\|u\|_{H^{s'}} \leq C(\|(P_\sigma^* + \imath Q_\sigma^*)u\|_{H^{s'-k+1}} + \|u\|_{H^{-N}}). \tag{2.21}$$

Note that the dual of H^s, $s > (k-1)/2 - \beta\operatorname{Im}\sigma$, is $H^{-s} = H^{s'-k+1}$, $s' = k-1-s$, so $s' < (k-1)/2 + \beta\operatorname{Im}\sigma$, while the dual of H^{s-k+1}, $s > (k-1)/2 - \beta\operatorname{Im}\sigma$, is $H^{k-1-s} = H^{s'}$, with $s' = k-1-s < (k-1)/2 + \beta\operatorname{Im}\sigma$ again. Thus, the spaces (apart from the residual spaces, into which the inclusion is compact) in the left, resp. right, side of (2.21), are exactly the duals of those on the right, resp. left, side of (2.20). Thus, by a standard functional analytic argument, see e.g. [32, Proof of Theorem 26.1.7] or indeed [54, Section 4.3] in the present context, namely dualization and using the compactness of the inclusion $H^{s'} \to H^{-N}$ for $s' > -N$, this gives the solvability of

$$(P_\sigma - \imath Q_\sigma)u = f,\ s > (k-1)/2 - \beta\operatorname{Im}\sigma,$$

for f in the annihilator in H^{s-k+1} (via the duality between H^{s-k+1} and H^{-s+k-1} induced by the L^2-pairing) of the finite dimensional subspace $\operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*)$ of

$H^{-s+k-1} = H^{s'}$, and indeed elements of this finite dimensional subspace have wave front set[29] in $\Lambda_+ \cup \Lambda_-$ and lie in $\cap_{s'<(k-1)/2+\beta \operatorname{Im}\sigma} H^{s'}$. Thus, there is the usual real principal type loss of one derivative relative to the elliptic problem, and in addition, there are restrictions on the orders for which is valid.

In addition, one also has almost uniqueness by a standard compactness argument (using the compactness of the inclusion of H^s into H^m for $s > m$), by (2.20), namely not only is the space of f in the space as above is finite codimensional, but the nullspace of $P_\sigma - \imath Q_\sigma$ on H^s, $s > (k-1)/2 - \beta \operatorname{Im}\sigma$, is also finite dimensional, and its elements are in $\mathcal{C}^\infty(X)$; again, see [54, Section 4.3] for details in this setup.

In order to analyze the σ-dependence of solvability of the PDE, we reformulate our problem as a more conventional Fredholm problem. Thus, let $\tilde{P}$ be any operator with principal symbol $p - \imath q$; e.g. $\tilde{P}$ is $P_{\sigma_0} - \imath Q_{\sigma_0}$ for some σ_0. Then consider

$$\mathcal{X}^s = \{u \in H^s : \ \tilde{P}u \in H^{s-k+1}\}, \ \mathcal{Y}^s = H^{s-k+1}, \tag{2.22}$$

with

$$\|u\|^2_{\mathcal{X}^s} = \|u\|^2_{H^s} + \|\tilde{P}u\|^2_{H^{s-k+1}}.$$

Note that $\mathcal{X}^s$ only depends on the principal symbol of $\tilde{P}$. Moreover, $\mathcal{C}^\infty(X)$ is dense in $\mathcal{X}^s$; this follows by considering $R_\epsilon \in \Psi^{-\infty}(X)$, $\epsilon > 0$, such that $R_\epsilon \to \mathrm{Id}$ in $\Psi^\delta(X)$ for $\delta > 0$, R_ϵ uniformly bounded in $\Psi^0(X)$; thus $R_\epsilon \to \mathrm{Id}$ strongly (but not in the operator norm topology) on H^s and H^{s-k+1}. Then for $u \in \mathcal{X}^s$, $R_\epsilon u \in \mathcal{C}^\infty(X)$ for $\epsilon > 0$, $R_\epsilon u \to u$ in H^s and $\tilde{P} R_\epsilon u = R_\epsilon \tilde{P} u + [\tilde{P}, R_\epsilon]u$, so the first term on the right converges to $\tilde{P}u$ in H^{s-k+1}, while $[\tilde{P}, R_\epsilon]$ is uniformly bounded in $\Psi^{k-1}(X)$, converging to 0 in $\Psi^{k-1+\delta}(X)$ for $\delta > 0$, so converging to 0 strongly as a map $H^s \to H^{s-k+1}$. Thus, $[\tilde{P}, R_\epsilon]u \to 0$ in H^{s-k+1}, and we conclude that $R_\epsilon u \to u$ in $\mathcal{X}^s$. (In fact, $\mathcal{X}^s$ is a first-order coisotropic space, more general function spaces of this nature are discussed by Melrose, Vasy and Wunsch in [42, Appendix A].)

With these preliminaries, for each σ with $s \geq m > (k-1)/2 - \beta \operatorname{Im}\sigma$,

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is Fredholm; these form an analytic family of bounded operators in this half-plane.

Theorem 2.7. *Let P_σ, Q_σ be as above, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.22)*. If $k-1-2s > 0$, let $\beta = \beta_{\inf}$, if $k-1-2s < 0$, let $\beta = \beta_{\sup}$. Then*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on[30] $\mathbb{C}_s \cap \Omega$*, where*

$$\mathbb{C}_s = \Big\{\sigma \in \mathbb{C} : \ \operatorname{Im}\sigma > \beta^{-1}\Big(\frac{k-1}{2} - s\Big)\Big\}. \tag{2.23}$$

Thus, analytic Fredholm theory applies, giving the meromorphy of the inverse provided the inverse exists for a particular value of σ.

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[29] Since the original version of this paper, the work of Haber and Vasy [29] showed that elements of this kernel are in fact Lagrangian distributions, i.e. they possess iterative regularity under the module of first order pseudodifferential operators with principal symbol vanishing on the Lagrangian, under additional assumptions on the structure at $\Lambda_\pm$, namely that $\Lambda_\pm$ consists of radial points for H_p. The latter holds, for instance, in de Sitter and de Sitter-Schwarzschild spaces, as well as on Minkowski space, but not on Kerr-de Sitter space.

[30] Recall that Ω is the domain of σ for which Q_σ is defined.

Remark 2.8. Note that the Fredholm property means that $P_\sigma^*+\imath Q_\sigma^*$ is also Fredholm on the dual spaces; this can also be seen directly from the estimates; rather than being a holomorphic family, it is an anti-holomorphic family. The analogue of this remark also applies to the semiclassical discussion below.

Remark 2.9. Note that if $s'>s\geq m>(k-1)/2-\beta\operatorname{Im}\sigma$ and if $P_\sigma-\imath Q_\sigma:\mathcal{X}^s\to\mathcal{Y}^s$ and $P_\sigma-\imath Q_\sigma:\mathcal{X}^{s'}\to\mathcal{Y}^{s'}$ are both invertible, then, as $\mathcal{X}^{s'}\subset\mathcal{X}^s$ and $\mathcal{Y}^{s'}\subset\mathcal{Y}^s$, $(P_\sigma-\imath Q_\sigma)^{-1}|_{\mathcal{Y}^{s'}}$ agrees with $(P_\sigma-\imath Q_\sigma)^{-1}:\mathcal{Y}^{s'}\to\mathcal{X}^{s'}$. Moreover, as $\mathcal{Y}^{s'}=H^{s'-k+1}$ is dense in $\mathcal{Y}^s$, $(P_\sigma-\imath Q_\sigma)^{-1}:\mathcal{Y}^{s'}\to\mathcal{X}^{s'}$ determines $(P_\sigma-\imath Q_\sigma)^{-1}:\mathcal{Y}^s\to\mathcal{X}^s$, i.e. if $A:\mathcal{Y}^s\to\mathcal{X}^s$ is continuous and $A|_{\mathcal{Y}^{s'}}$ is $(P_\sigma-\imath Q_\sigma)^{-1}:\mathcal{Y}^{s'}\to\mathcal{X}^{s'}$ then A is $(P_\sigma-\imath Q_\sigma)^{-1}:\mathcal{Y}^s\to\mathcal{X}^s$. Thus, in this sense, $(P_\sigma-\imath Q_\sigma)^{-1}$ is independent of s (satisfying $s\geq m>(k-1)/2-\beta\operatorname{Im}\sigma$).

2.7. **Stability.** We also want to understand the behavior of $P_\sigma-\imath Q_\sigma$ under perturbations. To do so, assume that $P_\sigma=P_\sigma(w)$, $Q_\sigma=Q_\sigma(w)$ depend continuously on a parameter $w\in\mathbb{R}^l$, with values in (analytic functions of σ with values in) $\Psi^k(X)$ and the principal symbols of $P_\sigma(w)$ and $Q_\sigma(w)$ are real and independent of σ with that of $\mp Q_\sigma(w)$ being non-negative near $\Sigma_\pm$. We *do not* assume that the principal symbols are independent of w, in fact, fixing some w_0, we do not even assume that for $w\neq w_0$ the other assumptions on $P_\sigma(w)-\imath Q_\sigma(w)$ are satisfied for $w\neq w_0$. (So, for instance, as already mentioned in Remark 2.5, the structure of the radial set at w_0 may drastically change for $w\neq w_0$.) However, see Remark 2.5 for the most delicate part, our estimates at w_0 are stable just under the assumption of continuous dependence with values $\Psi^k(X)$, thus there exists $\delta_0>0$ such that for $|w-w_0|<\delta_0$, we have uniform versions of the estimates (2.20)-(2.21), i.e. the constant C and the orders m and N can be taken to be uniform in these (independent of w), so e.g.

$$\|u\|_{H^s}\leq C(\|(P_\sigma(w)-\imath Q_\sigma(w))u\|_{H^{s-k+1}}+\|u\|_{H^m}). \tag{2.24}$$

Thus, $P_\sigma(w)-\imath Q_\sigma(w):\mathcal{X}^s(w)\to\mathcal{Y}^s(w)$ is Fredholm, depending analytically on σ, for each w with $|w-w_0|<\delta_0$, $\mathcal{Y}^s(w)=\mathcal{Y}^s=H^{s-k+1}$ is independent of w (and of σ), but $\mathcal{X}^s(w)=\{u\in H^s:\ P_\sigma(w)u\in H^{s-k+1}\}\subset H^s$ does depend on w (but not on σ). We claim, however, that, assuming that $(P_\sigma(w_0)-\imath Q_\sigma(w_0))^{-1}$ is meromorphic in σ (i.e. the inverse exists at least at one point σ), $(P_\sigma(w)-\imath Q_\sigma(w))^{-1}$ is also meromorphic in σ for w close to w_0, and it depends continuously on w in the weak operator topology of $\mathcal{L}(\mathcal{Y}^s,H^s)$, and thus in the norm topology of $\mathcal{L}(H^{s-k+1+\epsilon},H^{s-\epsilon})$ for $\epsilon>0$.

To see this, note first that if $P_{\sigma_0}(w_0)-\imath Q_{\sigma_0}(w_0):\mathcal{X}^s(w_0)\to\mathcal{Y}^s$ is invertible, then so is $P_\sigma(w)-\imath Q_\sigma(w):\mathcal{X}^s(w)\to\mathcal{Y}^s$ for w near w_0 and σ near σ_0. Once this is shown, the meromorphy of $(P_\sigma(w)-\imath Q_\sigma(w))^{-1}$ follows when w is close to w_0, with this operator being the inverse of an analytic Fredholm family which is invertible at a point. To see the invertibility of $P_\sigma(w)-\imath Q_\sigma(w)$ for w near w_0 and σ near σ_0, first suppose there exist sequences $w_j\to w_0$ and $\sigma_j\to\sigma_0$ such that $P_{\sigma_j}(w_j)-\imath Q_{\sigma_j}(w_j)$ is not invertible, so either $\operatorname{Ker}(P_{\sigma_j}(w_j)-\imath Q_{\sigma_j}(w_j))$ on H^s or $\operatorname{Ker}(P_{\sigma_j}(w_j)^*+\imath Q_{\sigma_j}(w_j)^*)$ on $(H^{s-k+1})^*$ is non-trivial in view of the preceding Fredholm discussion. By passing to a subsequence, we may assume that the same one of these two possibilities holds for all j, and as the case of the adjoint is completely analogous, we may also assume that $\operatorname{Ker}(P_{\sigma_j}(w_j)-\imath Q_{\sigma_j}(w_j))$ on H^s is non-trivial for all j. Now, if $u_j\in H^s$, $\|u_j\|_{H^s}=1$, and $(P_{\sigma_j}(w_j)-\imath Q_{\sigma_j}(w_j))u_j=0$ then (2.24) gives $1\leq C\|u_j\|_{H^m}$. Now, u_j has a weakly convergent subsequence in H^s to some $u_0 \in H^s$, which is thus norm-convergent in H^m; so $(P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))u_0 = 0$. Since $1 \leq C\|u_j\|_{H^m}$, and the subsequence is norm-convergent in H^m, $u_0 \neq 0$, and thus $\mathrm{Ker}(P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))$ on H^s is non-trivial, so $P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0)$ is not invertible, proving our claim.

So suppose now that $f_j \in H^{s-k+1}$ and $\|f_j\|_{H^{s-k+1}} \leq 1$. Let $w_j \to w_0$, $\sigma_j \to \sigma_0$ (with w_j sufficiently close to w_0, σ_j sufficiently close to σ_0 for invertibility), and let $u_j = (P_{\sigma_j}(w_j) - \imath Q_{\sigma_j}(w_j))^{-1} f_j$. Suppose first that u_j is not bounded in H^s, and let $v_j = \frac{u_j}{\|u_j\|_{H^s}}$. Then by (2.24), $1 \leq C(\|u_j\|^{-1}_{H^s} + \|v_j\|_{H^m})$, so for j sufficiently large, $\|v_j\|_{H^m} \geq \frac{1}{2C}$. On the other hand, a subsequence v_{j_r} of v_j converges weakly to some v_0 in H^s, and $\frac{f_{j_r}}{\|u_{j_r}\|_{H^s}} = (P_{\sigma_{j_r}}(w_{j_r}) - \imath Q_{\sigma_{j_r}}(w_{j_r}))v_{j_r} \to (P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))v_0$ weakly in H^{s-k}, so as the left hand side converges to 0 in H^{s-k+1}, $(P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))v_0 = 0$. As $v_{j_r} \to v_0$ in norm in H^m, we deduce that $v_0 \neq 0$, contradicting the invertibility of $P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0)$. Thus, u_j is uniformly bounded in H^s.

Next, suppose that $f_j \to f$ in H^{s-k+1}, so u_j is bounded in H^s by what we just showed. Then any subsequence of u_j has a weakly convergent subsequence u_{j_r} with some limit $u_0 \in H^s$. Then $f_{j_r} = (P_{\sigma_{j_r}}(w_{j_r}) - \imath Q_{\sigma_{j_r}}(w_{j_r}))u_{j_r} \to (P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))u_0$ weakly in H^{s-k}, so $(P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))u_0 = f$. By the injectivity of $P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0)$, u_0 is thus independent of the subsequence of u_j, i.e. every subsequence of u_j has a subsequence converging weakly to u_0, and thus u_j converges weakly to u_0 in H^s. This gives the convergence of $(P_\sigma(w) - \imath Q_\sigma(w))^{-1}$ to $(P_{\sigma_0}(w_0) - \imath Q_{\sigma_0}(w_0))^{-1}$ in the weak operator topology on $\mathcal{L}(\mathcal{Y}^s, H^s)$ as $\sigma \to \sigma_0$ and $w \to w_0$, and thus in the norm topology on $\mathcal{L}(\mathcal{Y}^{s+\epsilon}, H^{s-\epsilon})$ for $\epsilon > 0$.

2.8. **Semiclassical estimates.** For reasons of showing meromorphy of the inverse, and also for wave propagation, we also want to know the $|\sigma| \to \infty$ asymptotics of $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma^*$; here P_σ, Q_σ are operators with a large parameter. As discussed earlier, this can be turned into a semiclassical problem; one obtains families of operators $P_{h,z}$, with $h = |\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. As usual, we multiply through by h^k for convenient notation when we define $P_{h,z}$:

$$P_{h,z} = h^k P_{h^{-1}z} \in \Psi^k_{\hbar,\mathrm{cl}}(X).$$

Here we obtain uniform estimates in strips $|\operatorname{Im}\sigma| < C''$, which amounts to $|\operatorname{Im} z| < C'h$. Later, in Section 7, we extend these results to more general regions. While these strengthen the results, for our main results these extensions are much less significant than the treatment of strips. Since there are technical complications, we postpone their treatment to the end of the paper; see also Remarks 3.3 and 7.4.

From now on, we merely require $P_{h,z}, Q_{h,z} \in \Psi^k_{\hbar,\mathrm{cl}}(X)$ (rather than the large parameter statement). Here z satisfies $|\operatorname{Im} z| < C'h$, and thus $P_{h,z} - P_{h,\operatorname{Re} z} \in h\Psi^k_{\hbar,\mathrm{cl}}(X)$; similarly with $Q_{h,z}$. Thus, *for semiclassical principal symbol considerations, we regard $p_{\hbar,z}$ as defined for z real*: the semiclassical principal symbol $p_{\hbar,z}$, $z \in O \subset \mathbb{R}$, $0 \notin \overline{O}$ compact, is a function on T^*X, has limit p at infinity in the fibers of the cotangent bundle, so is in particular real in the limit. More precisely, as in the classical setting, but now $\tilde\rho$ made smooth at the zero section as well (so is not homogeneous there), we consider

$$\tilde\rho^k p_{\hbar,z} \in \mathcal{C}^\infty(\overline{T}^* X \times O);$$

then $\tilde{\rho}^k p_{\hbar,z}|_{S^*X\times O} = \tilde{\rho}^k p$, where $S^*X = \partial\overline{T}^*X$. We assume that $p_{\hbar,z}$ and $q_{\hbar,z}$ themselves are real[31]. Thus, $P_{\hbar,z} - P^*_{\hbar,z} \in h\Psi^{k-1}_\hbar(X)$.

We write the semiclassical characteristic set of $p_{\hbar,z}$ as $\Sigma_{\hbar,z}$, and sometimes drop the z dependence and write $\Sigma_\hbar$ simply; assume that

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-},\ \Sigma_{\hbar,+} \cap \Sigma_{\hbar,-} = \emptyset,$$

$\Sigma_{\hbar,\pm}$ are relatively open in $\Sigma_\hbar$, and

$$\mp q_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm}.$$

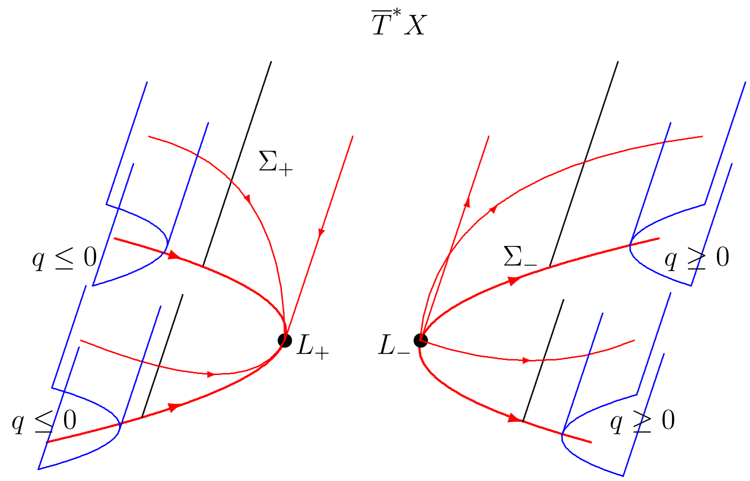

FIGURE 2. The components $\Sigma_{\hbar,\pm}$ of the semiclassical characteristic set in $\overline{T}^*X$, which are now two-dimensional in the figure. The cosphere bundle is the horizontal plane at the bottom of the picture; the intersection of this figure with the cosphere bundle is what is shown on Figure 1. The submanifolds $L_\pm$ are still points, with L_- a source, L_+ a sink. The red lines are bicharacteristics, with the thick ones inside $S^*X = \partial\overline{T}^*X$. The blue regions near the edges show the absorbing region, i.e. the support of q. For $P_{h,z} - \imath Q_{h,z}$, the estimates are always propagated away from $L_\pm$ towards the support of q, so in the direction of the Hamilton flow in $\Sigma_{\hbar,-}$, and in the direction opposite of the Hamilton flow in $\Sigma_{\hbar,+}$; for $P^*_{h,z} + \imath Q^*_{h,z}$, the directions are reversed.

Microlocal results analogous to the classical results also exist in the semiclassical setting. In the interior of $\overline{T}^*X$, i.e. in T^*X, only the microlocal elliptic, real principal type and complex absorption (in which one considers the bicharacteristics of $\operatorname{Re} p_{\hbar,z}$) estimates are relevant. At $L_\pm \subset S^*X$ we in addition need the analogue

[31]If we do not restrict $|\operatorname{Im} z| < C'h$, we would only assume this when z is real. We discuss this in Section 7.

of Propositions 2.3-2.4. As these are the only non-standard estimates[32], though they are very similar to estimates of [57], where, however, only global estimates were stated, we explicitly state these here and indicate the very minor changes needed in the proof compared to Propositions 2.3-2.4.

Proposition 2.10. *For all N, for $s \geq m > (k-1)/2 - \beta \operatorname{Im}\sigma$, $\sigma = h^{-1}z$, and for all $A, B, G \in \Psi_{\hbar}^0(X)$ such that $\mathrm{WF}_h'(G) \cap \mathrm{WF}_h'(Q_\sigma) = \emptyset$, A elliptic at $L_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}_h'(B)$ tend to $L_\pm$, with closure in the elliptic set of G, one has estimates*[33]

$$Au \in H_h^m \Rightarrow \|Bu\|_{H_h^s} \leq C(h^{-1}\|GP_{h,z}u\|_{H_h^{s-k+1}} + h\|u\|_{H_h^{-N}}), \tag{2.25}$$

where, as usual, $GP_{h,z}u \in H_h^{s-k+1}$ and $u \in H_h^{-N}$ are assumptions implied by the right hand side.

Proposition 2.11. *For $s < (k-1)/2 + \beta \operatorname{Im}\sigma$, for all N, $\sigma = h^{-1}z$, and for all $A, B, G \in \Psi_{\hbar}^0(X)$ such that $\mathrm{WF}_h'(G) \cap \mathrm{WF}_h'(Q_\sigma) = \emptyset$, B, G elliptic at $L_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}_h'(B) \setminus L_\pm$ reach $\mathrm{WF}_h'(A)$, while remaining in the elliptic set of G, one has estimates*

$$\|Bu\|_{H_h^s} \leq C(h^{-1}\|GP_{h,z}^* u\|_{H_h^{s-k+1}} + \|Au\|_{H_h^s} + h\|u\|_{H_h^{-N}}). \tag{2.26}$$

Proof. We just need to localize in $\tilde\rho$ in addition to ρ_0; such a localization in the classical setting is implied by working on S^*X or with homogeneous symbols. We achieve this by modifying the localizer ϕ in the commutant constructed in the proof of Propositions 2.3-2.4. As already remarked, the proof is much like at radial points in semiclassical scattering on asymptotically Euclidean spaces, studied by Vasy and Zworski [57], but we need to be more careful about localization in ρ_0 and $\tilde\rho$ as we are assuming less about the structure.

First, note that $L_\pm$ is defined by $\tilde\rho = 0$, $\rho_0 = 0$ within $\Sigma_\hbar$, so for $\digamma > 0$ to be determined, $\digamma\tilde\rho^2 + \rho_0$ is a quadratic defining function of $L_\pm$. Note that on $\Sigma_{\hbar,z}$,

$$\mp\tilde\rho^{k-1}(\mathsf{H}_{p_{\hbar,z}}\rho_0 + 2\tilde\rho\mathsf{H}_{p_{\hbar,z}}\tilde\rho) \geq \beta_1\rho_0 + 2\digamma\beta_0\tilde\rho^2 - \mathcal{O}(\tilde\rho\rho_0^{1/2}) - \mathcal{O}((\tilde\rho^2+\rho_0)^{3/2}),$$

where the first $\mathcal{O}$ term is independent of $\digamma$, and is due to $\tilde\rho^{k-1}\mathsf{H}_{p_{\hbar,z}}$ restricting to $\tilde\rho^{k-1}\mathsf{H}_p$ at S^*X (so the difference between these two, using say the homogeneous extension of the latter, is $\mathcal{O}(\tilde\rho)$) plus the non-degenerate quadratic vanishing of ρ_0 (so its differential can be estimated by its square root): these mean that modulo $\mathcal{O}(\tilde\rho\rho_0^{1/2})$, $\mp\tilde\rho^{k-1}\mathsf{H}_{p_{\hbar,z}}\rho_0$ is bounded below by $\mp\tilde\rho^{k-1}\mathsf{H}_p\rho_0$. Given $K > 0$ taking $\digamma >$

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[32] As an example, the standard semiclassical propagation of singularities result is that $\mathrm{WF}_h^{s,r}(u)$, $u = \{u_h\}_{h\in(0,1)}$ a polynomially bounded family, is a union of maximally extended bicharacteristics of $p_{\hbar,z}$ in $\Sigma_{\hbar,z} \setminus \mathrm{WF}_h^{s-1,r+1}(P_{\hbar,z}u)$; this is the Sobolev version of [63, Theorem 12.5]. Note that the choice of $r = -1$ is most usual, but h commutes with $P_{\hbar,z}$, so the orders may be adjusted easily. This can be translated into a uniform estimate as $h \to 0$ using the uniform boundedness principle; see [54, Section 4.4], though as in the case of the classical estimates and the application of the closed graph theorem there, this is somewhat round about: see Footnote 22. Concretely, in this case, one has the estimate

$$\|Bu\|_{H_h^s} \leq C(h^{-1}\|GP_{h,z}u\|_{H_h^{s-k+1}} + \|Au\|_{H_h^s} + h\|u\|_{H_h^{-N}}),$$

in the notation corresponding to (2.10), with bicharacteristics understood as integral curves of $\mathsf{H}_{p_{\hbar,z}}$ both at $h = 0$ and as at fiber-infinity, S^*X.

[33] Here and below the error term $h\|u\|_{H_h^{-N}}$ can be changed to $h^N\|u\|_{H_h^{-N}}$ by iterating the estimates, as required anyway for the proof of the $\|u\|_{H_h^{-N}}$-type error.

0 sufficiently large, $\beta_1\rho_0 + 2F\beta_0\tilde{\rho}^2 - K\tilde{\rho}\rho_0^{1/2}$ becomes a positive definite quadratic form in $(\tilde{\rho}, \rho_0^{1/2})$. Correspondingly, there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near 0, $\phi' \leq 0$ and ϕ supported sufficiently close to 0 so that

$$\alpha \in \operatorname{supp} d(\phi \circ (\tilde{\rho}^2 + \rho_0)) \cap \Sigma_\hbar \Rightarrow \mp\tilde{\rho}^{k-1}(\mathsf{H}_p\rho_0 + 2\tilde{\rho}\mathsf{H}_p\tilde{\rho})(\alpha) > 0$$

and

$$\alpha \in \operatorname{supp} d(\phi \circ (\tilde{\rho}^2 + \rho_0)) \cap \Sigma_\hbar \Rightarrow \mp\tilde{\rho}^{k-2}\mathsf{H}_p\tilde{\rho}(\alpha) > 0.$$

Then let c be given by

$$c = \phi(\rho_0 + F\tilde{\rho}^2)\phi_0(p_0)\tilde{\rho}^{-s+(k-1)/2}, \qquad c_\epsilon = c(1 + \epsilon\tilde{\rho}^{-1})^{-\delta},\ p_0 = \tilde{\rho}^k p_{\hbar,z}.$$

The rest of the proof proceeds similarly to Propositions 2.3-2.4; one computes principal symbols now in $h\Psi_\hbar^{k-1}(X)$. □

In order to have global estimates, we need to make a non-trapping assumption, which we turn into a definition:

Definition 2.12. We say that $p_{\hbar,z} - \imath q_{\hbar,z}$ is *semiclassically non-trapping* if the bicharacteristics of $p_{\hbar,z}$ from any point in $\Sigma_\hbar \setminus (L_+ \cup L_-)$ flow to $\operatorname{ell}(q_{\hbar,z}) \cup L_+$ (i.e. either enter $\operatorname{ell}(q_{\hbar,z})$ at some finite time, or tend to L_+) in the forward direction, and to $\operatorname{ell}(q_{\hbar,z}) \cup L_-$ in the backward direction.

Remark 2.13. The part of the semiclassically non-trapping property on S^*X is just the classical non-trapping property; thus, the point is its extension into to the interior T^*X of $\overline{T}^*X$.

Let H_h^s denote the usual semiclassical function spaces. The semiclassical version of the classical estimates, stated above, are then applicable, and one obtains on the one hand that for any $s \geq m > (k-1)/2 - \beta \operatorname{Im} z/h$, $h < h_0$,

$$\|u\|_{H_h^s} \leq Ch^{-1}(\|(P_{h,z} - \imath Q_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^m}), \tag{2.27}$$

On the other hand, for any N and for any $s < (k-1)/2 + \beta \operatorname{Im} z/h$, $h < h_0$,

$$\|u\|_{H_h^s} \leq Ch^{-1}(\|(P_{h,z}^* + \imath Q_{h,z}^*)u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^{-N}}), \tag{2.28}$$

The h^2 term can be absorbed in the left hand side for sufficiently small h, so we automatically obtain invertibility of $P_{h,z} - \imath Q_{h,z}$.

We now translate this into the classical setting, where in particular, this gives the meromorphy of $P_\sigma - \imath Q_\sigma$. Note also that for instance

$$\|u\|^2_{H^1_{|\sigma|^{-1}}} = \|u\|^2_{L^2} + |\sigma|^{-2}\|du\|^2_{L^2}, \qquad \|u\|_{H^0_{|\sigma|^{-1}}} = \|u\|_{L^2},$$

(with the norms with respect to any positive definite inner product). We thus have

Theorem 2.14. *Let P_σ, Q_σ, $\mathbb{C}_s$, β be as above (see* (2.23) *for the definition of $\mathbb{C}_s$), in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.22)*. Let $C_- > \beta^{-1}(\frac{k-1}{2} - s)$, $C_+ > 0$. Then there exists σ_0 such that*

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is holomorphic in $\{\sigma \in \Omega :\ C_- < \operatorname{Im}\sigma < C_+,\ |\operatorname{Re}\sigma| > \sigma_0\}$, assumed to be a subset of $\mathbb{C}_s$, and non-trapping estimates

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-k+1}\|f\|_{H^{s-k+1}_{|\sigma|^{-1}}}$$

hold. For $s=1$, $k=2$ this states that for $|\operatorname{Re}\sigma|>\sigma_0$, $C_-<\operatorname{Im}\sigma<C_+$,

$$\|R(\sigma)f\|^2_{L^2}+|\sigma|^{-2}\|dR(\sigma)\|^2_{L^2}\leq C''|\sigma|^{-2}\|f\|^2_{L^2}.$$

Further, $R(\sigma):\mathcal{Y}^s\to\mathcal{X}^s$ is meromorphic in $\{\sigma\in\Omega:\ C_-<\operatorname{Im}\sigma<C_+\}\subset\mathbb{C}_s$.

While we only stated the global results here, one has microlocal estimates for the solution. In particular we have the following, stated in the semiclassical language, as immediate from the estimates used to derive from the Fredholm property:

Theorem 2.15. *Let P_σ, Q_σ, β be as above, in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in (2.22).*

For $\operatorname{Re}z>0$ and $s'>s$, the resolvent $R_{h,z}$ is semiclassically outgoing with a loss of h^{-1} *in the sense that if $\alpha\in\overline{T}^*X\cap\Sigma_{\hbar,\pm}$, and if for the forward (+), resp. backward (−), bicharacteristic $\gamma_\pm$, from α, $\mathrm{WF}_h^{s'-k+1}(f)\cap\overline{\gamma_\pm}=\emptyset$ then $\alpha\notin\mathrm{WF}_h^{s'}(hR_{h,z}f)$.*

In fact, for any $s'\in\mathbb{R}$, the resolvent $R_{h,z}$ extends to $f\in H_h^{s'}(X)$, with non-trapping bounds, provided that $\mathrm{WF}_h^s(f)\cap(L_+\cup L_-)=\emptyset$. The semiclassically outgoing with a loss of h^{-1} result holds for such f and s' as well.

Proof. The only part that is not immediate by what has been discussed is the last claim. This follows, however, from microlocal solvability in arbitrary ordered Sobolev spaces away from the radial points (i.e. solvability modulo $\mathcal{C}^\infty$, with semiclassical estimates), combined with our preceding results to deal with this smooth remainder plus the contribution near $L_+\cup L_-$, which are assumed to be in $H_h^s(X)$. □

This result is needed for gluing constructions as in [15], namely polynomially bounded trapping with appropriate microlocal geometry can be glued to our resolvent. Furthermore, it gives non-trapping estimates microlocally away from the trapped set provided the overall (trapped) resolvent is polynomially bounded as shown by Datchev and Vasy [16].

Definition 2.16. Suppose $K_\pm\subset T^*X$ is compact, and $O_\pm$ is a neighborhood of K with compact closure and $O_\pm\cap\Sigma_\hbar\subset\Sigma_{\hbar,\pm}$. We say that $p_{\hbar,z}$ is *semiclassically locally mildly trapping of order $\varkappa$ in a C_0-strip* if

(i) there is a function[34] $F\in\mathcal{C}^\infty(T^*X)$, $F\geq 2$ on $K_\pm$, $F\leq 1$ on $T^*X\setminus O_\pm$, and for $\alpha\in(O_\pm\setminus K_\pm)\cap\Sigma_{\hbar,\pm}$, $(\mathsf{H}_{p_{\hbar,z}}F)(\alpha)=0$ implies $(\mathsf{H}^2_{p_{\hbar,z}}F)(\alpha)<0$; and

(ii) there exists $\tilde{Q}_{h,z}\in\Psi_\hbar(X)$ with $\mathrm{WF}'_h(\tilde{Q}_{h,z})\cap K_\pm=\emptyset$, $\mp\tilde{q}_{\hbar,z}\geq 0$ near $\Sigma_{\hbar,\pm}$, $\tilde{q}_{\hbar,z}$ elliptic on $\Sigma_\hbar\setminus(O_+\cup O_-)$ and $h_0>0$ such that if $\operatorname{Im}z>-C_0h$ and $h<h_0$ then

$$\|(P_{h,z}-\imath\tilde{Q}_{h,z})^{-1}f\|_{H_h^s}\leq Ch^{-\varkappa-1}\|f\|_{H_h^{s-k+1}},\ f\in H_h^{s-k+1}. \tag{2.29}$$

We say that $p_{\hbar,z}-\imath q_{\hbar,z}$ is *semiclassically mildly trapping of order $\varkappa$ in a C_0-strip* if it is semiclassically locally mildly trapping of order $\varkappa$ in a C_0-strip and if the bicharacteristics from any point in $\Sigma_{\hbar,+}\setminus(L_+\cup K_+)$ flow to $\{q_{\hbar,z}<0\}\cup O_+$ in the

[34] For $\epsilon>0$, such a function F provides an escape function, $\tilde{F}=e^{-CF}\mathsf{H}_{p_{\hbar,z}}F$ on the set where $1+\epsilon\leq F\leq 2-\epsilon$. Namely, by taking $C>0$ sufficiently large, $\mathsf{H}_{p_{\hbar,z}}\tilde{F}<0$ there; thus, every bicharacteristic must leave the compact set $F^{-1}([1+\epsilon,2-\epsilon])$ in finite time. However, the existence of such an F is a stronger statement than that of an escape function: a bicharacteristic segment cannot leave $F^{-1}([1+\epsilon,2-\epsilon])$ via the boundary $F=2-\epsilon$ in both directions since F cannot have a local minimum. This is exactly the way this condition is used in [15].

backward direction and to $\{q_{\hbar,z} < 0\} \cup O_+ \cup L_+$ in the forward direction, while the bicharacteristics from any point in $\Sigma_{\hbar,-} \setminus (L_- \cup K_-)$ flow to $\{q_{\hbar,z} > 0\} \cup O_- \cup L_-$ in the backward direction and to $\{q_{\hbar,z} > 0\} \cup O_-$ in the forward direction.

An example[35] of locally mild trapping is hyperbolic trapping, studied by Wunsch and Zworski [61], which is of order $\varkappa$ for some $\varkappa > 0$. Note that (i) states that the sets $K_c = \{F \geq c\}$, $1 < c < 2$, are bicharacteristically convex in $O_\pm$, for by (i) any critical points of F along a bicharacteristic are strict local maxima.

As a corollary, we have:

Theorem 2.17. *Let P_σ, Q_σ, $\mathbb{C}_s$, β be as above (see* (2.23) *for the definition of $\mathbb{C}_s$), satisfying mild trapping assumptions with order $\varkappa$ estimates in a C_0-strip, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.22)*, C_+, C_- as in Theorem 2.14. Then*

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is meromorphic in $\mathbb{C}_s$ and there exists σ_0 such that

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is holomorphic in $\{\sigma \in \Omega : \max(-C_0, C_-) < \operatorname{Im}\sigma < C_+,\ |\operatorname{Re}\sigma| > \sigma_0\}$, and

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{\varkappa-k+1}\|f\|_{H^{s-k+1}_{|\sigma|^{-1}}}. \tag{2.30}$$

Further, if one has logarithmic loss in (2.29)*, i.e. if $h^{-\varkappa}$ can be replaced by $\log(h^{-1})$, for $\sigma \in \mathbb{R}$,* (2.30) *also holds with a logarithmic loss, i.e. $|\sigma|^\varkappa$ can be replaced by $\log|\sigma|$ for σ real.*

Proof. This is an almost immediate consequence of [15]. To get into that setting, we replace $Q_{h,z}$ by $Q'_{h,z}$ with $\mathrm{WF}'_h(Q_{h,z} - Q'_{h,z}) \subset O_+ \cup O_-$ and $Q'_{h,z}$ elliptic on $K_+ \cup K_-$, with $\mp q'_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. Then $P_{h,z} - \imath Q'_{h,z}$ is semiclassically non-trapping in the sense discussed earlier, so all of our estimates apply. With the polynomial resolvent bound assumption on $P_{h,z} - \imath\tilde{Q}_{h,z}$, and the function F in place of x used in [15], the results of [15] apply, taking into account Theorem 2.15 and [15, Lemma 5.1]. Note that the results of [15] are stated in a slightly different context for convenience, namely the function x is defined on the manifold X and not on T^*X, but this is a minor issue: the results and proofs apply verbatim in our setting. □

3. Mellin transform and Lorentzian b-metrics

3.1. **The Mellin transform.** In this section we discuss the basics of Melrose's b-analysis on an n-dimensional manifold with boundary $\bar{M}$, where the boundary is denoted by X. We refer to [43] as a general reference. In the main cases of interest here, the b-geometry is trivial, and $\bar{M} = X \times [0,\infty)_\tau$ with respect to some (almost) canonical (to the problem) product decomposition. Thus, the reader should feel comfortable in trivializing all the statements below with respect to

[35]Condition (i) follows by letting $\tilde{F} = \varphi_+^{2\kappa} + \varphi_-^{2\kappa}$ with the notation of [61, Lemma 4.1]; so

$$\mathsf{H}_p^2\tilde{F} = 4\kappa^2\big((c_+^4 - \kappa^{-1}c_+\mathsf{H}_pc_+)\varphi_+^{2\kappa} + 4\kappa^2\big((c_-^4 + \kappa^{-1}c_-\mathsf{H}_pc_-)\varphi_-^{2\kappa}$$

near the trapped set, $\varphi_+ = 0 = \varphi_-$. Thus, for sufficiently large κ, $\mathsf{H}_p\tilde{F} > 0$ outside $\tilde{F} = 0$. Since $\tilde{F} = 0$ defines the trapped set, in order to satisfy Definition 2.16, writing K and O instead of $K_\pm$ and $O_\pm$, one lets $K = \{\tilde{F} \leq \alpha\}$, $O = \{\tilde{F} < \beta\}$ for suitable (small) α and β, $\alpha < \beta$, and takes $F = G \circ \tilde{F}$ with G strictly decreasing, $G|_{[0,\alpha]} > 2$, $G|_{[\beta,\infty)} < 1$.

this decomposition. In this trivial case, the main result on the Mellin transform, Lemma 3.1 is fairly standard, with possibly different notation of the function spaces; we include it here for completeness.

First, recall that the Lie algebra of b-vector fields, $\mathcal{V}_{\mathrm{b}}(\bar{M})$ consists of $\mathcal{C}^\infty$ vector fields on $\bar{M}$ tangent to the boundary. In local coordinates (τ, y), such that τ is a boundary defining function, they are of the form $a_n\tau\partial_\tau + \sum_{j=1}^{n-1} a_j\partial_{y_j}$, with a_j arbitrary $\mathcal{C}^\infty$ functions. Correspondingly, they are the set of all smooth sections of a $\mathcal{C}^\infty$ vector bundle, ${}^{\mathrm{b}}T\bar{M}$, with local basis $\tau\partial_\tau, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$. The dual bundle, ${}^{\mathrm{b}}T^*\bar{M}$, thus has a local basis given by $\frac{d\tau}{\tau}, dy_1, \ldots, dy_{n-1}$. All tensorial constructions, such as form and density bundles, go through as usual.

We remark here that with $\tau = e^{-t}$, $t \geq 0$, say, the vector fields take the form $a_n\partial_t + \sum_{j=1}^{n-1} a_j\partial_{y_j}$, where a_j are smooth functions of e^{-t} and y. Thus, these vector fields, and the corresponding differential operators, are modelled on allowing variable coefficients on t-translation invariant vector fields, but require them to stabilize exponentially fast at infinity (in the precise sense of the stated smooth dependence). However, it is easier to keep track of the structure from the b-perspective, so we do not pursue this point of view.

The natural bundles related to the boundary are reversed in the b-setting. Thus, the b-normal bundle of the boundary X is well-defined as the span of $\tau\partial_\tau$ defined using any coordinates, or better yet, as the kernel of the natural map $\iota : {}^{\mathrm{b}}T_m\bar{M} \to T_m\bar{M}$, $m \in X$, induced via the inclusion $\mathcal{V}_{\mathrm{b}}(\bar{M}) \to \mathcal{V}(\bar{M})$, so

$$a_n\tau\partial_\tau + \sum_{j=1}^{n-1} a_j\partial_{y_j} \mapsto \sum_{j=1}^{n-1} a_j\partial_{y_j}, \qquad a_j \in \mathbb{R}.$$

Its annihilator in ${}^{\mathrm{b}}T^*_m\bar{M}$ is called the b-cotangent bundle of the boundary; in local coordinates (τ, y) it is spanned by $dy_1, \ldots, dy_{n-1}$. Invariantly, it is the image of $T^*_m\bar{M}$ in ${}^{\mathrm{b}}T^*_m\bar{M}$ under the adjoint of the tangent bundle map ι; as this has kernel $N^*_m X$, ${}^{\mathrm{b}}T^*_m\bar{M}$ is naturally identified with $T^*_m X = T^*_m\bar{M}/N^*_m X$.

The algebra of differential operators generated by $\mathcal{V}_{\mathrm{b}}(\bar{M})$ over $\mathcal{C}^\infty(\bar{M})$ is denoted $\mathrm{Diff}_{\mathrm{b}}(\bar{M})$; in local coordinates as above, elements of $\mathrm{Diff}^k_{\mathrm{b}}(\bar{M})$ are of the form

$$\mathcal{P} = \sum_{j+|\alpha|\leq k} a_{j\alpha}(\tau D_\tau)^j D^\alpha_y$$

in the usual multiindex notation, $\alpha \in \mathbb{N}^{n-1}$, with $a_{j\alpha} \in \mathcal{C}^\infty(\bar{M})$. Writing b-covectors as

$$\sigma\,\frac{d\tau}{\tau} + \sum_{j=1}^{n-1} \eta_j\, dy_j,$$

we obtain canonically dual coordinates to (τ, y), namely (τ, y, σ, η) are local coordinates on ${}^{\mathrm{b}}T^*\bar{M}$. The principal symbol of $\mathcal{P}$ is

$$\tilde{p} = \sigma_{\mathrm{b},k}(\mathcal{P}) = \sum_{j+|\alpha|=k} a_{j\alpha}\sigma^j\eta^\alpha; \tag{3.1}$$

it is a $\mathcal{C}^\infty$ function, which is a homogeneous polynomial of degree k in the fibers, on ${}^{\mathrm{b}}T^*\bar{M}$. Its Hamilton vector field, $\mathsf{H}_{\tilde{p}}$, is a $\mathcal{C}^\infty$ vector field, which is just the extension of the standard Hamilton vector field from $\bar{M}^\circ$, is homogeneous of degree $k-1$, on ${}^{\mathrm{b}}T^*\bar{M}$, and it is tangent to ${}^{\mathrm{b}}T^*_X\bar{M}$. Explicitly, as a change of variables shows, in local coordinates,

$$\mathsf{H}_{\tilde{p}} = (\partial_\sigma \tilde{p})(\tau\partial_\tau) + \sum_j (\partial_{\eta_j}\tilde{p})\partial_{y_j} - (\tau\partial_\tau\tilde{p})\partial_\sigma - \sum_j (\partial_{y_j}\tilde{p})\partial_{\eta_j}, \tag{3.2}$$

so the restriction of $\mathsf{H}_{\tilde{p}}$ to $\tau = 0$ is

$$\mathsf{H}_{\tilde{p}}|_{{}^{\mathrm{b}}T^*_X\bar{M}} = \sum_j (\partial_{\eta_j}\tilde{p})\partial_{y_j} - \sum_j (\partial_{y_j}\tilde{p})\partial_{\eta_j}, \tag{3.3}$$

and is thus tangent to the fibers (identified with T^*X) of ${}^{\mathrm{b}}T^*_X M$ over ${}^{\mathrm{b}}T^*_X M/T^*X$ (identified with $\mathbb{R}_\sigma$).

We next want to define normal operator[36] of $\mathcal{P} \in \mathrm{Diff}^k_{\mathrm{b}}(\bar{M})$, obtained by freezing coefficients at $X = \partial\bar{M}$. To do this naturally, we want to extend the 'frozen operator' to one invariant under dilations in the fibers of the inward pointing normal bundle ${}_+N(X)$ of X; see [43, Equation (4.91)]. The latter can always be trivialized by the choice of an inward-pointing vector field V, which in turn fixes the differential of a boundary defining function τ at X by $V\tau|_X = 1$; given such a choice we can identify ${}_+N(X)$ with a product

$$\bar{M}_\infty = X \times [0,\infty)_\tau,$$

with the normal operator being invariant under dilations in τ. Then for $m = (x,\tau)$, ${}^{\mathrm{b}}T_m\bar{M}_\infty$ is identified with ${}^{\mathrm{b}}T_{(x,0)}\bar{M}$.

On $\bar{M}_\infty$ operators of the form

$$\sum_{j+|\alpha|\leq k} a_{j\alpha}(y)(\tau D_\tau)^j D_y^\alpha,$$

i.e. $a_{j\alpha} \in \mathcal{C}^\infty(X)$, are invariant under the $\mathbb{R}^+$-action on $[0,\infty)_\tau$; its elements are denoted by $\mathrm{Diff}_{\mathrm{b},I}(\bar{M}_\infty)$. The *normal operator* of $\mathcal{P} \in \mathrm{Diff}^k_{\mathrm{b}}(\bar{M})$ is given by freezing the coefficients at X:

$$N(\mathcal{P}) = \sum_{j+|\alpha|\leq k} a_{j\alpha}(0,y)(\tau D_\tau)^j D_y^\alpha \in \mathrm{Diff}^k_{\mathrm{b},I}(\bar{M}_\infty).$$

The *normal operator family*, which is a large parameter (in σ) family of differential operators on X, is then defined as

$$\hat{N}(\mathcal{P})(\sigma) = P_\sigma = \sum_{j+|\alpha|\leq k} a_{j\alpha}(0,y)\sigma^j D_y^\alpha \in \mathrm{Diff}^k(X).$$

Note also that we can identify a neighborhood of X in $\bar{M}$ with a neighborhood of $X \times \{0\}$ in $\bar{M}_\infty$ (this depends on choices), and then transfer $\mathcal{P}$ to an operator (still denoted by $\mathcal{P}$) on $\bar{M}_\infty$, extended in an arbitrary smooth manner; then $\mathcal{P} - N(\mathcal{P}) \in \tau\mathrm{Diff}^k_{\mathrm{b}}(\bar{M}_\infty)$.

The principal symbol p of the normal operator family, including in the high energy (or, after rescaling, semiclassical) sense, is given by $\sigma_{\mathrm{b},k}(\mathcal{P})|_{{}^{\mathrm{b}}T^*_X\bar{M}}$. Correspondingly, the Hamilton vector field, including in the high-energy sense, of p is given by $\mathsf{H}_{\sigma_{\mathrm{b},k}(\mathcal{P})}|_{{}^{\mathrm{b}}T^*_X\bar{M}}$; see (3.3). It is useful to note that via this restriction we drop information about $\mathsf{H}_{\sigma_{\mathrm{b},k}(\mathcal{P})}$ as a b-vector field, namely the $\tau\partial_\tau$ component is neglected. Correspondingly, the dynamics (including at high energies) for the normal operator family is the same at radial points of the Hamilton flow regardless of the behavior of the $\tau\partial_\tau$ component, thus whether on ${}^{\mathrm{b}}S^*\bar{M} = ({}^{\mathrm{b}}T^*\bar{M} \setminus o)/\mathbb{R}^+$,

[36]In fact, $\mathcal{P} \in \Psi^k_{\mathrm{b}}(\bar{M})$ works similarly.

with the τ variable included, we have a source/sink, or a saddle point, with the other (stable/unstable) direction being transversal to the boundary. This is reflected by the same normal operator family showing up in both de Sitter space and in Minkowski space, even though in de Sitter space (and also in Kerr-de Sitter space) in the full b-sense the radial points are saddle points, while in Minkowski space they are sources/sinks (with a neutral direction along the conormal bundle of the event horizon/light cone inside the boundary in both cases).

We now translate our results to solutions of $(\mathcal{P}-\imath\mathcal{Q})u=f$ when $P_\sigma-\imath Q_\sigma$ is the normal operator family of the b-operator $\mathcal{P}-\imath\mathcal{Q}$, with $\mathcal{P},\mathcal{Q}\in\Psi^k_{\mathrm{b}}(\bar{M})$. A typical application is when $\mathcal{P}=\Box_g$ is the d'Alembertian of a Lorentzian b-metric on $\bar{M}$, discussed in Subsection 3.2.

Thus, consider the Mellin transform in τ, i.e. consider the map

$$\mathcal{M}:u\mapsto\hat{u}(\sigma,.)=\int_0^\infty \tau^{-\imath\sigma}u(\tau,.)\,\frac{d\tau}{\tau}, \tag{3.4}$$

with inverse transform

$$\mathcal{M}^{-1}:v\mapsto\check{v}(\tau,.)=\frac{1}{2\pi}\int_{\mathbb{R}+\imath\alpha}\tau^{\imath\sigma}v(\sigma,.)\,d\sigma, \tag{3.5}$$

with α chosen in the region of holomorphy. Note that for polynomially bounded (in τ) u (with values in a space, such as $\mathcal{C}^\infty(X)$, $L^2(X)$, $\mathcal{C}^{-\infty}(X)$), for u supported near $\tau=0$, $\mathcal{M}u$ is holomorphic in $\operatorname{Im}\sigma>C$, $C>0$ sufficiently large, with values in the same space (such as $\mathcal{C}^\infty(X)$, etc). We discuss more precise statements below. The Mellin transform is described in detail in [43, Section 5], but it is also merely a renormalized Fourier transform (corresponding to the exponential change of variables, $\tau=e^{-t}$, mentioned above), so the results below are simply those for the Fourier transform (often of Paley-Wiener type) after suitable renormalization.

First, Plancherel's theorem is that if ν is a smooth non-degenerate density on X and r_c denotes restriction to the line $\operatorname{Im}\sigma=c$, then

$$r_{-\alpha}\circ\mathcal{M}:\tau^\alpha L^2(X\times[0,\infty);\frac{|d\tau|}{\tau}\nu)\to L^2(\mathbb{R};L^2(X;\nu)) \tag{3.6}$$

is an isomorphism. We are interested in functions u supported near $\tau=0$, in which case, with $r_{(c_1,c_2)}$ denoting restriction to the strip $c_1<\operatorname{Im}\sigma<c_2$, for $N>0$,

$$\begin{aligned} &r_{-\alpha,-\alpha+N}\circ\mathcal{M}:\tau^\alpha(1+\tau)^{-N}L^2(X\times[0,\infty);\frac{|d\tau|}{\tau}\nu)\\ &\qquad\to\Big\{v:\mathbb{R}\times\imath(-\alpha,-\alpha+N)\ni\sigma\to v(\sigma)\in L^2(X;\nu);\\ &\qquad\qquad v \text{ is holomorphic in } \sigma \text{ and } \sup_{-\alpha<r<-\alpha+N}\|v(.+\imath r,.)\|_{L^2(\mathbb{R};L^2(X;\nu))}<\infty\Big\}, \end{aligned} \tag{3.7}$$

see [43, Lemma 5.18]. Note that in accordance with (3.6), v in (3.7) extends continuously to the boundary values, $r=-\alpha$ and $r=-\alpha-N$, with values in the same space as for holomorphy. Moreover, for functions supported in, say, $\tau<1$, one can take N arbitrary.

Analogous results also hold for the b-Sobolev spaces $H_{\mathrm{b}}^s(X\times[0,\infty))$. For $s\geq 0$, these can be defined as in [43, Equation (5.41)]:

$$\begin{aligned}&r_{-\alpha}\circ\mathcal{M}:\tau^\alpha H_{\mathrm{b}}^s(X\times[0,\infty);\frac{|d\tau|}{\tau}\nu)\\&\to\Big\{v\in L^2(\mathbb{R};H^s(X;\nu)):\ (1+|\sigma|^2)^{s/2}v\in L^2(\mathbb{R};L^2(X;\nu))\Big\},\end{aligned}\tag{3.8}$$

with the analogue of (3.7) also holding; for $s<0$ one needs to use the appropriate dual statements. See also [43, Equations (5.41)-(5.42)] for differential versions for integer order spaces. Note that the right hand side of (3.8) is equivalent to

$$\langle|\sigma|\rangle^s v\in L^2(\mathbb{R};H^s_{\langle|\sigma|\rangle^{-1}}(X;\nu)),\tag{3.9}$$

where the space on the right hand side is the standard semiclassical Sobolev space and $\langle|\sigma|\rangle=(1+|\sigma|^2)^{1/2}$; indeed, for $s\geq 0$ integer both are equivalent to the statement that for all α with $|\alpha|\leq s$, $\langle|\sigma|\rangle^{s-|\alpha|}D_y^\alpha v\in L^2(\mathbb{R};L^2(X;\nu))$. Here by equivalence we mean not only the membership of a set, but also that of the standard norms[37] corresponding to these spaces. Note that by dualization, (3.9) holds for all $s\in\mathbb{R}$.

If $\mathcal{P}-\imath\mathcal{Q}$ is invariant under dilations in τ on $\bar{M}_\infty=X\times[0,\infty)$ then $N(\mathcal{P}-\imath\mathcal{Q})$ can be identified with $\mathcal{P}-\imath\mathcal{Q}$ and we have the following simple lemma:

Lemma 3.1. *Suppose $\mathcal{P}-\imath\mathcal{Q}$ is invariant under dilations in τ for functions supported near $\tau=0$, and the normal operator family $\hat{N}(\mathcal{P}-\imath\mathcal{Q})$ is of the form $P_\sigma-\imath Q_\sigma$ satisfying the conditions of Section 2 and Section 7, including semiclassical non-trapping. Let σ_j be the poles of the meromorphic family $(P_\sigma-\imath Q_\sigma)^{-1}$. Then for $\ell<\beta^{-1}(s-(k-1)/2)$, β as in Proposition 2.3, $\ell\neq-\operatorname{Im}\sigma_j$ for any j, $(\mathcal{P}-\imath\mathcal{Q})u=f$, u tempered, supported near $\tau=0$, $f\in\tau^\ell H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, u has an asymptotic expansion*

$$u=\sum_j\sum_{\kappa\leq m_j}\tau^{\imath\sigma_j}(\log|\tau|)^\kappa a_{j\kappa}+u'\tag{3.10}$$

with $a_{j\kappa}\in\mathcal{C}^\infty(X)$ and $u'\in\tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$, and with the σ_j being the finite number of poles of the normal operator in $-\ell<\operatorname{Im}\sigma<-\alpha$, where α is such that $u\in\tau^{\alpha'}H_{\mathrm{b}}^r(\bar{M}_\infty)$ for some r and $\alpha'>\alpha$, $\alpha'<\beta^{-1}(r-(k-1)/2)$.

If instead $N(\mathcal{P}-\imath\mathcal{Q})$ is semiclassically mildly trapping of order $\varkappa$ in a C_0-strip then for $\ell<C_0$ (still with $\ell<\beta^{-1}(s-(k-1)/2)$, $\ell\neq-\operatorname{Im}\sigma_j$ for any j) and $f\in\tau^\ell H_{\mathrm{b}}^{s-k+1+\varkappa}(\bar{M}_\infty)$ one has

$$u=\sum_j\sum_{\kappa\leq m_j}\tau^{\imath\sigma_j}(\log|\tau|)^\kappa a_{j\kappa}+u'\tag{3.11}$$

with $a_{j\kappa}\in\mathcal{C}^\infty(X)$ and $u'\in\tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$.

Conversely, given f in the indicated spaces, with f supported near $\tau=0$, a solution u of $(\mathcal{P}-\imath\mathcal{Q})u=f$ of the form (3.10), resp. (3.11), supported near $\tau=0$ exists.

In either case, the coefficients $a_{j\kappa}$ are given by the Laurent coefficients of $(\mathcal{P}-\imath\mathcal{Q})^{-1}$ at the poles σ_j applied to f, with simple poles corresponding to $m_j=0$.

If $f=\sum_j\sum_{\kappa\leq m_j'}\tau^{\alpha_j}(\log|\tau|)^\kappa b_{j\kappa}+f'$, with f' in the spaces indicated above for f, and $b_{j\kappa}\in H^{s-k+1}(X)$, analogous results hold when the expansion of f is added

[37] Standard up to equivalence, such as $\Big(\sum_{|\alpha|\leq s}\int_{\operatorname{Im}\sigma=-\alpha}\langle|\sigma|\rangle^{2(s-|\alpha|)}\|D_y^\alpha v\|^2_{L^2(X;\nu)}\,d\sigma\Big)^{1/2}$.

to the form of (3.10) *and* (3.11)*, in the sense of the extended union of index sets, see* [43, Section 5.18]*.*

Further, the result is stable under sufficiently small dilation-invariant perturbations in the b-sense, i.e. if $\mathcal{P}'$ *and* $\mathcal{Q}'$ *are sufficiently close to* $\mathcal{P}$ *and* $\mathcal{Q}$ *in* $\Psi^k_{\mathrm{b}}(\bar{M}_\infty)$ *with* P'_σ *and* Q'_σ *possessing real principal symbols, and that of* Q'_σ *is non-negative, then there is a similar expansion for solutions of* $(\mathcal{P}' - \imath\mathcal{Q}')u = f$*.*

For $\mathcal{P}^* + \imath\mathcal{Q}^*$ *in place of* $\mathcal{P} - \imath\mathcal{Q}$*, analogous results apply, but we need* $\ell < -\beta^{-1}(s - (k-1)/2)$*, and the* $a_{j\kappa}$ *are not smooth, rather have wave front set*[38] *in the Lagrangians* $\Lambda_\pm$*.*

Remark 3.2. Thus, for $\mathcal{P} - \imath\mathcal{Q}$, the more terms we wish to obtain in an expansion, the better Sobolev space we need to work in. For $\mathcal{P}^* + \imath\mathcal{Q}^*$, dually, we need to be in a weaker Sobolev space under the same circumstances. However, these spaces only need to be worse at the radial points, so under better regularity assumptions on f we still get the expansion in better Sobolev spaces away from the radial points — in particular in elliptic regions. This is relevant in our description of Minkowski space.

Remark 3.3. If the large $\operatorname{Im}\sigma$, i.e. $\operatorname{Im} z \neq 0$, assumptions in Section 7 are not satisfied (but the assumptions corresponding to a strip, i.e. roughly real z, still are), the proof of this lemma still goes through apart from the support conclusion in the existence part. For the existence argument, one can then pick any $\alpha < \ell$ with $(P_\sigma - \imath Q_\sigma)^{-1}$ having no poles on the line $\operatorname{Im}\sigma = -\alpha$; different choices of α result in different solutions. See also Remark 7.4.

Proof. First consider the expansion. Suppose $\alpha, r \in \mathbb{R}$ are such that $\alpha < \beta^{-1}(r - (k-1)/2)$, $u \in \tau^\alpha H^r_{\mathrm{b}}(\bar{M}_\infty)$ and $(P_\sigma - \imath Q_\sigma)^{-1}$ has no poles on the line $\operatorname{Im}\sigma = -\alpha$; note that the vanishing of u for $\tau > 1$ and the absence of poles of $(P_\sigma - \imath Q_\sigma)^{-1}$ near infinity inside strips (by the semiclassical non-trapping/mildly trapping assumptions) means that this can be arranged, and then also $u \in \tau^\alpha (1+\tau)^{-N} H^r_{\mathrm{b}}(\bar{M}_\infty)$ for all N. The Mellin transform of the PDE, a priori on $\operatorname{Im}\sigma = -\alpha$, is $(P_\sigma - \imath Q_\sigma)\mathcal{M}u = \mathcal{M}f$. Thus,

$$\mathcal{M}u = (P_\sigma - \imath Q_\sigma)^{-1}\mathcal{M}f \tag{3.12}$$

there. If $f \in \tau^\ell H^{s-k+1}_{\mathrm{b}}(\bar{M}_\infty)$, then shifting the contour of integration in the inverse Mellin transform, (3.5), to $\operatorname{Im}\sigma = -\ell$, we obtain contributions from the poles of $(P_\sigma - \imath Q_\sigma)^{-1}$, giving the expansion in (3.10) and (3.11) by Cauchy's theorem. The shift of the contour is justified by Theorem 2.14 or Theorem 2.17, depending on non-trapping or mild trapping assumptions, giving the high energy estimates controlling the integral as $|\operatorname{Re}\sigma| \to \infty$. The error term u' is what one obtains by integrating along the new contour in view of the high energy bounds on $(P_\sigma - \imath Q_\sigma)^{-1}$ (which differ as one changes one's assumption from non-trapping to mild trapping), and the assumptions on f.

Conversely, to obtain existence, let $\alpha < \min(\ell, -\sup \operatorname{Im}\sigma_j)$ and define $u \in \tau^\alpha H^s_{\mathrm{b}}(\bar{M}_\infty)$ by (3.12) using the inverse Mellin transform with $\operatorname{Im}\sigma = -\alpha$. Then u solves the PDE, hence the expansion follows by the first part of the argument. The

[38] See the discussion after (2.21). Also note that while for $\mathcal{P} - \imath\mathcal{Q}$ the coefficients $a_{j\kappa}$ are smooth, the operator mapping f to $a_{j\kappa}$ is not smoothing exactly because of the non-smoothness of the expansion for the adjoint operator: it maps *sufficiently regular* f to smooth functions, but cannot be applied to *all* distributional f.

support property of u follows from Paley-Wiener, taking into account holomorphy in $\operatorname{Im}\sigma > -\alpha$, and the estimates on $\mathcal{M}f$ and $(P_\sigma - \imath Q_\sigma)^{-1}$ there.

Finally, stability of the expansion follows from Subsection 2.7 since the meromorphy and the large σ estimates are stable under such a perturbation. Note that the condition on the principal symbol of P'_σ and Q'_σ to be independent of σ is automatically satisfied, for this is just the principal symbol in $\Psi^k_{\mathrm{b}}(\bar{M}_\infty)$ (which stands for *one-step*, or *classical* b-pseudodifferential operators) of $\mathcal{P}'$ and $\mathcal{Q}'$ evaluated at $\sigma = 0$ (or any other finite constant), cf. the large parameter discussion at the end of Subsection 2.1. □

Remark 3.4. One actually gets estimates for the coefficients $a_{j\kappa}$ and u' in (3.10)-(3.11). Indeed, in view of the isomorphism (3.9) and the contour deformation after (3.12), u' is bounded in $\tau^\ell H^s_{\mathrm{b}}(\bar{M}_\infty)$ by f in $\tau^\ell H^{s-k+1}_{\mathrm{b}}(\bar{M}_\infty)$ in the non-trapping case, with the $\varkappa$ shift in the mildly trapping case. Then, in the nontrapping case, the norm of f in $\tau^\ell H^{s-k+1}_{\mathrm{b}}(\bar{M}_\infty)$ gives a bound for the norm of $\mathcal{M}f$ on the line $\operatorname{Im}\sigma = -\ell$ in $\langle|\sigma|\rangle^{-(s-k+1)}L^2(\mathbb{R}; H^{s-k+1}_{\langle|\sigma|\rangle^{-1}}(X))$; now the non-trapping bounds for $(P_\sigma - \imath Q_\sigma)^{-1}$ imply that $(P_\sigma - \imath Q_\sigma)^{-1}\mathcal{M}f$ is bounded in $\langle|\sigma|\rangle^{-s}L^2(\mathbb{R}; H^s_{\langle|\sigma|\rangle^{-1}}(X))$, and thus u' is bounded in $\tau^\ell H^s_{\mathrm{b}}(\bar{M}_\infty)$. One gets similar bounds for the $a_{j\kappa}$; indeed only the norm of f in slightly less weighted spaces (i.e. with a weight $\ell' < \ell$) corresponding to the location of the poles of $(P_\sigma - \imath Q_\sigma)^{-1}$ is needed to estimate these. In the case of mild trapping, one needs stronger norms on f corresponding to $\varkappa$ in view of the bounds on $(P_\sigma - \imath Q_\sigma)^{-1}$ then.

One can iterate this to obtain a full expansion even when $\mathcal{P} - \imath\mathcal{Q}$ is not dilation invariant. Note that in most cases considered below, Lemma 3.1 suffices; the exception is if we allow general, non-stationary, b-perturbations of Kerr-de Sitter or Minkowski metrics.

Proposition 3.5. *Suppose $(\mathcal{P} - \imath\mathcal{Q})u = f$, and the normal operator family $\hat{N}(\mathcal{P} - \imath\mathcal{Q})$ is of the form $P_\sigma - \imath Q_\sigma$ satisfying the conditions of Sections 2 and*[39] *7, including semiclassical non-trapping. Then for $\ell < \beta^{-1}(\min(s, r-1) - |\ell - \alpha| - (k-1)/2)$, $\ell \notin -\operatorname{Im}\sigma_j + \mathbb{N}$ for any j, $u \in \tau^\alpha H^r_{\mathrm{b}}(\bar{M}_\infty)$ supported near 0, $(\mathcal{P} - \imath\mathcal{Q})u = f$, $f \in \tau^\ell H^{s-k+1}_{\mathrm{b}}(\bar{M}_\infty)$, u has an asymptotic expansion*

$$u = \sum_j \sum_l \sum_{\kappa \le m_{jl}} \tau^{\imath\sigma_j + l}(\log|\tau|)^\kappa a_{j\kappa l} + u' \tag{3.13}$$

with $a_{j\kappa} \in \mathcal{C}^\infty(X)$ and $u' \in \tau^\ell H^{\min(s,r-1)-[\ell-\alpha]}_{\mathrm{b}}(\bar{M}_\infty)$, $[\ell - \alpha]$ being the integer part of $\ell - \alpha$.

If instead $N(\mathcal{P} - \imath\mathcal{Q})$ is semiclassically mildly trapping of order $\varkappa$ in a C_0-strip then for $\ell < C_0$ and $f \in \tau^\ell H^{s-k+1+\varkappa}_{\mathrm{b}}(\bar{M}_\infty)$ one has

$$u = \sum_j \sum_l \sum_{\kappa \le m_{jl}} \tau^{\imath\sigma_j + l}(\log|\tau|)^\kappa a_{j\kappa l} + u' \tag{3.14}$$

with $a_{j\kappa l} \in \mathcal{C}^\infty(X)$ and $u' \in \tau^\ell H^{\min(s,r-1)-[\ell-\alpha]}_{\mathrm{b}}(\bar{M}_\infty)$.

If $f = \sum_j \sum_{\kappa \le m'_j} \tau^{\alpha_j}(\log|\tau|)^\kappa b_{j\kappa} + f'$, with f' in the spaces indicated above for f, and $b_{j\kappa} \in H^{s-k+1}(X)$, analogous results hold when the expansion of f is added

[39] See Remark 3.3.

to the form of (3.13) *and* (3.14) *in the sense of the extended union of index sets, see* [43, Section 5.18].

If $\sigma_{\mathrm{b},k}(\mathcal{P}-\imath\mathcal{Q})$ vanishes on the characteristic set of $N(\mathcal{P}-\imath\mathcal{Q})$ to infinite order in Taylor series at $\tau=0$, then there are no losses in the order of u', i.e. one can replace $u'\in\tau^\ell H_{\mathrm{b}}^{\min(s,r-1)-[\ell-\alpha]}(\bar{M}_\infty)$ by $u'\in\tau^\ell H_{\mathrm{b}}^{\min(s,r)}(\bar{M}_\infty)$, and $\ell<\beta^{-1}(\min(s,r-1)-|\ell-\alpha|-(k-1)/2)$ by $\ell<\beta^{-1}(\min(s,r)-(k-1)/2)$, giving the same form as in Lemma 3.1 apart from the presence of the a priori regularity r.

Conversely, under the characteristic assumption in the previous paragraph, given f in the indicated spaces, with f supported near $\tau=0$, a solution u of $(\mathcal{P}-\imath\mathcal{Q})u=f+f^\sharp$ of the form (3.10)*, resp.* (3.11)*, $f^\sharp\in\tau^\infty H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, resp. $H_{\mathrm{b}}^{s-k+1+\varkappa}(\bar{M}_\infty)$, supported near $\tau=0$, exists.*

Again, the result is stable under sufficiently small perturbations, in the b-sense, of $\mathcal{P}$ and $\mathcal{Q}$, in the same sense, apart from dilation invariance, as stated in Lemma 3.1.

Remark 3.6. The losses in the regularity of u' without further assumptions are natural due to the lack of ellipticity. Specifically, if, for instance, u is conormal to a hypersurface S transversal to X, as is the case in many interesting examples, the orbits of the $\mathbb{R}^+$-action on $\bar{M}_\infty$ must be tangent to S to avoid losses of regularity in the Taylor series expansion.

In particular, there are no losses if $(\mathcal{P}-\imath\mathcal{Q})-N(\mathcal{P}-\imath\mathcal{Q})\in\tau\mathrm{Diff}_{\mathrm{b}}^{k-1}(\bar{M}_\infty)$, rather than merely in $\tau\mathrm{Diff}_{\mathrm{b}}^{k}(\bar{M}_\infty)$.

We only stated the converse result under the extra characteristic assumption to avoid complications with the Sobolev orders. Global solvability depends on more than the normal operator, which is why we do not state such a result here.

Proof. One proceeds as in Lemma 3.1, Mellin transforming the problem, but replacing $\mathcal{P}-\imath\mathcal{Q}$ by $N(\mathcal{P}-\imath\mathcal{Q})$. Note that $(\mathcal{P}-\imath\mathcal{Q})-N(\mathcal{P}-\imath\mathcal{Q})\in\tau\mathrm{Diff}_{\mathrm{b}}^{k}(\bar{M}_\infty)$. We treat

$$\tilde{f}=((\mathcal{P}-\imath\mathcal{Q})-N(\mathcal{P}-\imath\mathcal{Q}))u$$

as part of the right hand side, subtracting it from f, so

$$N(\mathcal{P}-\imath\mathcal{Q})u=f-\tilde{f}.$$

If $u\in\tau^\alpha H_{\mathrm{b}}^r(\bar{M}_\infty)$ is supported near 0, then $\tilde{f}\in\tau^{\alpha+1}H_{\mathrm{b}}^{r-k}(\bar{M}_\infty)$ (or, under the characteristic assumption on $\mathcal{P}-\imath\mathcal{Q}$, $\tilde{f}\in\tau^{\alpha+1}H_{\mathrm{b}}^{r-k+1}(\bar{M}_\infty)$), so Lemma 3.1 is applicable with ℓ replaced by $\min(\ell,\alpha+1)$. If $\ell\leq\alpha+1$, we are done, otherwise we repeat the argument. Indeed, we now know that u is given by an expansion giving rise to poles of $\mathcal{M}u$ in $\mathrm{Im}\,\sigma>\alpha+1$ plus an element of $\tau^{\alpha+1}H_{\mathrm{b}}^{\min(s,r-1)}(\bar{M}_\infty)$, so we also have better information on $\tilde{f}$, namely it is also given by a partial expansion, plus an element of $\tau^{\alpha+2}H_{\mathrm{b}}^{\min(s,r-1)-k}(\bar{M}_\infty)$, or indeed $\tau^{\alpha+2}H_{\mathrm{b}}^{\min(s,r)-k+1}(\bar{M}_\infty)$ under the characteristic assumption on $\mathcal{P}-\imath\mathcal{Q}$. Using the f with a partial expansion part of Lemma 3.1 to absorb the $\tilde{u}$ terms, we can work with ℓ replaced by $\min(\ell,\alpha+2)$. It is this step that starts generating the sum over l in (3.13) and (3.14). The iteration stops in a finite number of steps, completing the proof.

For the existence, define a zeroth approximation u_0 to u using $N(\mathcal{P}-\imath\mathcal{Q})$ in Lemma 3.1, and iterate away the error $\tilde{f}=((\mathcal{P}-\imath\mathcal{Q})-N(\mathcal{P}-\imath\mathcal{Q}))u_0-f$ in Taylor series. □

3.2. **Lorentzian metrics.** We now review common properties of Lorentzian b-metrics g on $\bar{M}$. Lorentzian b-metrics are symmetric non-degenerate bilinear forms on ${}^{\mathrm{b}}T_m\bar{M}$, $m\in\bar{M}$, of signature $(1,n-1)$, i.e. the maximal dimension of a subspace on which g is positive definite is *exactly* 1, which depend smoothly on m. In other words, they are symmetric sections of ${}^{\mathrm{b}}T^*\bar{M}\otimes{}^{\mathrm{b}}T^*\bar{M}$ which are in addition non-degenerate of Lorentzian signature. Usually it is more convenient to work with the dual metric G, which is then a symmetric section of ${}^{\mathrm{b}}T\bar{M}\otimes{}^{\mathrm{b}}T\bar{M}$ which is in addition non-degenerate of Lorentzian signature.

By non-degeneracy there is a nowhere vanishing b-density associated to the metric, $|dg|$, which in local coordinates (τ,y) is given by $\sqrt{|\det g|}\frac{|d\tau|}{\tau}\,|dy|$, and which gives rise to a Hermitian (positive definite!) inner product on functions. There is also a non-degenerate, but not positive definite, inner product on the fibers of the b-form bundle, ${}^{\mathrm{b}}\Lambda\bar{M}$, and thus, when combined with the aforementioned Hermitian inner product on functions, an inner product on differential forms which is not positive definite only due to the lack of definiteness of the fiber inner product. Thus, A^* is defined, as a formal adjoint, for any differential operator $A\in\mathrm{Diff}^k_{\mathrm{b}}(\bar{M};{}^{\mathrm{b}}\Lambda\bar{M})$ acting on sections of the b-form bundle, such as the exterior derivative, d. Thus, g gives rise to the d'Alembertian,

$$\Box_g=d^*d+dd^*\in\mathrm{Diff}^2_{\mathrm{b}}(\bar{M};\Lambda\bar{M}),$$

which preserves form degrees. The d'Alembertian on functions is also denoted by $\Box_g$. The principal symbol of $\Box_g$ is

$$\sigma_{\mathrm{b},2}(\Box_g)=G.$$

As discussed above, the normal operator of $\Box_g$ on $\bar{M}$ is $N(\Box_g)\in\mathrm{Diff}_{\mathrm{b},I}(\bar{M}_\infty)$, $\bar{M}_\infty=X\times[0,\infty)_\tau$. If $\bar{M}=\bar{M}_\infty$ (i.e. it is a product space to start with) and if $\Box_g$ already has this invariance property under a product decomposition, then the normal operator can be identified with $\Box_g$ itself. Taking the Mellin transform in τ, we obtain a family of operators, P_σ, on X, depending analytically on σ, the b-dual variable of τ. For $z\in\mathbb{C}$ not necessarily real the semiclassical principal symbol of $P_{h,z}=h^2P_{h^{-1}z}$ is just the dual metric G on the complexified cotangent bundle ${}^{\mathrm{b},\mathbb{C}}T^*_m\bar{M}$, $m=(x,\tau)$, evaluated on covectors $\varpi+z\,\frac{d\tau}{\tau}$, where ϖ is in the (real) span Π of the 'spatial variables' T^*_xX; thus Π and $\frac{d\tau}{\tau}$ are linearly independent. Although for now we are interested in z real mostly, we consider z complex for the next paragraphs since complex values of z motivate the choice of the function used to divide up the characteristic set in (3.16). Thus, for z complex,

$$\begin{aligned}&\langle\varpi+z\,\frac{d\tau}{\tau},\varpi+z\,\frac{d\tau}{\tau}\rangle_G\\&=\langle\varpi+\operatorname{Re}z\,\frac{d\tau}{\tau},\varpi+\operatorname{Re}z\,\frac{d\tau}{\tau}\rangle_G-(\operatorname{Im}z)^2\langle\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle_G\\&\quad+2\imath\operatorname{Im}z\langle\varpi+\operatorname{Re}z\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle_G.\end{aligned}\tag{3.15}$$

For $\operatorname{Im}z\neq0$, the vanishing of the imaginary part states that $\langle\varpi+\operatorname{Re}z\,\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle_G=0$; the real part is the first two terms on the right hand side of (3.15).

Although we do not need it for our considerations in $|\operatorname{Im}\sigma|<C$, when working with a larger half-plane it is very useful to assume, in view of (3.15)*, that $\frac{d\tau}{\tau}$ is time-like for G; see Section 7.*

Furthermore, for $z\in\mathbb{C}$ non-zero, motivated by (3.15), we consider the hypersurface $\langle\varpi+\operatorname{Re}z\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle_G=0$. The characteristic set of $p_{\hbar,z}$ cannot intersect this hypersurface, for G is negative definite on covectors satisfying this equality, so if the intersection were non-empty, $\varpi + \operatorname{Re} z\frac{d\tau}{\tau}$ would vanish there, as would $\operatorname{Im} z$ in view of the second term of (3.15), which cannot happen for $\varpi \in \Pi$ since $z \neq 0$ by assumption. Correspondingly, we can divide the semiclassical characteristic set in two parts by

$$\Sigma_{\hbar,\pm} \cap T^*X = \{\varpi \in \Sigma_\hbar \cap T^*X : \pm\langle \varpi + \operatorname{Re} z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G > 0\}; \tag{3.16}$$

note that by the definiteness of the quadratic form on this hypersurface, in fact this separation holds on the fiber-compactified bundle, $\overline{T}^*X$. In general, one of the 'components' $\Sigma_{\hbar,\pm}$ may be empty.

From now on in this Subsection, we take z real unless otherwise specified. Moreover, for $m \in \bar{M}$, and with Π denoting the 'spatial' hyperplane in the real cotangent bundle, ${}^{\mathrm{b}}T^*_m\bar{M}$, the Lorentzian nature of G means that for z real and non-zero, the intersection of $\Pi + z\frac{d\tau}{\tau}$ with the zero-set of G in ${}^{\mathrm{b}}T^*_q\bar{M}$, i.e. the characteristic set, has two components if $G|_\Pi$ is Lorentzian, and one component if it is negative definite (i.e. Riemannian, up to the sign). Further, in the second case, on the only component $\langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$ and $\langle z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$ have the same sign, so only $\Sigma_{\hbar,\operatorname{sgn} z}$ can enter the elliptic region.

We also need information about $p_{\hbar,z} - \imath q_{\hbar,z}$, i.e. when the complex absorption has been added, with $q_{\hbar,z}$ defined for z in an open set $\tilde{\Omega} \subset \mathbb{C} \setminus \{0\}$. Since for the semiclassical principal symbols only z real matters, here we need to choose $q_{\hbar,z}$ in such a way as to ensure that for real $z \neq 0$, $q_{\hbar,z}$ is real and $\mp q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$ (as well as an classical and semiclassical ellipticity condition in the region in which $q_{\hbar,z}$ is to provide absorption; we discuss this below (3.17) in terms of $\chi > 0$). In order to arrange this, we take[40], with f_z a first order symbol elliptic in the classical sense,

$$q_{\hbar,z} = -\chi f_z \langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G, \qquad \begin{aligned} &z \in \mathbb{R} \Rightarrow f_z \text{ is real}, \\ &\chi \geq 0, \text{ independent of } z; \end{aligned} \tag{3.17}$$

note that if in addition f_z is bounded away from 0 when z is bounded away from 0 in $\mathbb{R}$, then the above conditions for real z are then automatically satisfied in view of (3.16). In addition, at points where $\chi > 0$, $p_{\hbar,z} - \imath q_{\hbar,z}$ does not vanish for z real, since the imaginary part, $q_{\hbar,z}$, is non-zero except when $\langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = 0$, but in that case the real part satisfies $p_{\hbar,z} < 0$ by (3.16). Further, $p_{\hbar,z} - \imath q_{\hbar,z}$ is elliptic in the classical sense where $\chi > 0$ for the same reason.

Since $\langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$ is holomorphic in z, we actually obtain a holomorphic family of operators if we choose f_z to be such.

In fact, typically $p_{\hbar,z}$ itself is not globally defined, so we need to extend it beyond the domain where it *is* defined. Typically one has a function μ on X with $d\mu$ timelike[41] for μ near $\mu_1 \in \mathbb{R}$, and $p_{\hbar,z}$ is given by a Lorentzian b-metric in $\mu \geq \mu_1$, but we need to extend $p_{\hbar,z}$ to $\mu < \mu_1$. For this purpose, we first let $\tilde{H}$ to be a Riemannian metric on X, and then for some $j \geq 1$ integer we let

$$\hat{p}_{\hbar,z} = \left(\|\varpi\|^{2j}_{\tilde{H}} + z^{2j}\right)^{1/j};$$

[40]For now z real being the only case, but in Section 7 z complex is allowed in the same expression.

[41]This actually does not matter for the discussion below, but due to Subsection 3.3 it ensures that the choice of the extension is irrelevant.

thus the case of $j=1$ actually corresponds to a Riemannian b-metric in which $\frac{d\tau}{\tau}$ and $\varpi\in T^*_xX$ are orthogonal, unlike in the case of G. Here we choose the branch of the jth root function which is positive along the positive reals and has a branch cut along the negative reals, and take as the domain of $\hat p_{\hbar,z}$ the values of ϖ and z for which $\|\varpi\|^{2j}_{\tilde H}+z^{2j}$ lies outside the branch cut. Thus, the complement $\mathcal{D}_j$ of the rays $z=re^{i\pi(2k+1)/(2j)}$, k an integer, $r\geq 0$, is always in the domain of $\hat p_{\hbar,z}$, and thus as j varies, these domains cover any compact set of $\mathbb{C}$ disjoint from 0. Now let $\mu_0>\mu_0'>\mu_1'>\mu_1$, and choose a partition of unity $\chi_1+\chi_2=1$, $\chi_j\geq 0$, with $\operatorname{supp}\chi_1\subset(\mu_1',+\infty)$, $\operatorname{supp}\chi_2\subset(-\infty,\mu_0')$. Further, with $\digamma>0$ a constant to be chosen, let $\chi\geq 0$ be identically $\digamma$ on $[\mu_1',\mu_0']$ and supported in (μ_1,μ_0), let

$$f_z=\Big(\|\varpi\|^{2j}_{\tilde H}+z^{2j}\Big)^{1/2j},$$

where again the branch cut for the $2j$th root is along the negative reals, so $\hat p_{\hbar,z}=f_z^2$. In particular, note that $\operatorname{Re} f_z>0$ with this choice, and even $\operatorname{Re} f_z^2>0$ if $j\geq 2$. Now, let

$$\tilde p_{\hbar,z}=\chi_1 p_{\hbar,z}-\chi_2\hat p_{\hbar,z}, \tag{3.18}$$

and let $q_{\hbar,z}$ be defined by (3.17). We already know from the discussion after (3.17) that where $\chi_1=1$, $\tilde p_{\hbar,z}-\imath q_{\hbar,z}$ satisfies our requirements. We claim that where either $\chi_2>0$ or $\chi>0$, $\tilde p_{\hbar,z}-\imath q_{\hbar,z}$ is actually elliptic; where $\chi_2=0$ but $\chi>0$, this was checked above. Note that as $\|\varpi\|_{\tilde H}\to\infty$ and z in a compact set, semiclassical ellipticity becomes a statement (namely, that of ellipticity) for the standard principal symbol, $\tilde p-\imath q$, which is easy to check as

$$\tilde p-\imath q=\chi_1\langle\varpi,\varpi\rangle_G-\chi_2\|\varpi\|^2_{\tilde H}+\imath\chi\|\varpi\|_{\tilde H}\langle\varpi,\frac{d\tau}{\tau}\rangle,$$

homogeneous of degree 2 in ϖ. Indeed, the imaginary part of $\tilde p-\imath q$ only vanishes (for $\varpi\neq 0$) when either χ or $\langle\varpi,\frac{d\tau}{\tau}\rangle$ does. In the former case, $\chi_2=1$ and $\chi_1=0$, so $\tilde p-\imath q<0$. In the latter case, $\langle\varpi,\varpi\rangle_G$ is negative definite, so the real part does not vanish as $\chi_1+\chi_2=1$. Hence, for sufficiently large ϖ, $\tilde p_{\hbar,z}-\imath q_{\hbar,z}$ is elliptic in this region; we only need to consider whether it vanishes for finite ϖ. Next, if z is real, $z\neq 0$, then $f_z>0$,

$$\tilde p_{\hbar,z}-\imath q_{\hbar,z}=\chi_1\langle\varpi+z\frac{d\tau}{\tau},\varpi+z\frac{d\tau}{\tau}\rangle_G-\chi_2 f_z^2+\imath\chi f_z\langle\varpi+z\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle, \tag{3.19}$$

so again the imaginary part only vanishes if either χ or $\langle\varpi+z\frac{d\tau}{\tau},\frac{d\tau}{\tau}\rangle$ does. In the former case, $\chi_2=1$ and $\chi_1=0$ so $\tilde p_{\hbar,z}-\imath q_{\hbar,z}<0$. In the latter case, $\langle\varpi+z\frac{d\tau}{\tau},\varpi+z\frac{d\tau}{\tau}\rangle_G$ is negative definite, so the real part of $\tilde p_{\hbar,z}-\imath q_{\hbar,z}$ is negative.

Thus, if $q_{\hbar,z}$ is given by (3.17) and if we extend $p_{\hbar,z}$ to a new symbol, $\tilde p_{\hbar,z}$ across a hypersurface, $\mu=\mu_1$, in the manner (3.18), then with χ, χ_1 and χ_2 as discussed there, $\tilde p_{\hbar,z}-\imath q_{\hbar,z}$ satisfies the requirements for $p_{\hbar,z}-\imath q_{\hbar,z}$, and in addition it is elliptic in the extended part of the domain. We usually write $p_{\hbar,z}-\imath q_{\hbar,z}$ for this extension. Thus, these properties need not be checked individually in specific cases.

We remark that f_z as above arises from the standard quantization F_σ of

$$\Big(\|\varpi\|^{2j}_{\tilde H}+\sigma^{2j}+C^{2j}\Big)^{1/2j},$$

for $C>0$ arbitrarily chosen; the large-parameter rescaling $hF_{h^{-1}z}$ of this has the semiclassical principal symbol f_z. Then for the induced operators $P_\sigma-\imath Q_\sigma$, the operators are holomorphic in the domain $\Omega_{j,C}$ given with $\mathbb{C}$ with the half-lines $e^{i\pi(2k+1)/(2j)}[C,+\infty)$, k an integer, removed, and thus as j and C vary, these domains cover $\mathbb{C}$, and they all include strips $|\operatorname{Im}\sigma| < C'$ for sufficiently small $C' > 0$.

It is useful to note the following explicit calculation regarding the time-like character of $\frac{d\tau}{\tau}$ if we are given a Lorentzian b-metric g that, with respect to some local boundary defining function $\tilde\tau$ and local product decomposition $U \times [0,\delta)_{\tilde\tau}$ of $\bar M$ near $U \subset X$ open, is of the form $G = (\tilde\tau\partial_{\tilde\tau})^2 - \tilde G$ on $U \times [0,\delta)_{\tilde\tau}$, $\tilde G$ a Riemannian metric on U. In this case, if we define $\tau = \tilde\tau e^{\phi}$, ϕ a function on X, so $\frac{d\tau}{\tau} = \frac{d\tilde\tau}{\tilde\tau} + d\phi$, then

$$\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = \langle \frac{d\tilde\tau}{\tilde\tau}, \frac{d\tilde\tau}{\tilde\tau}\rangle_G - \langle d\phi, d\phi\rangle_G = 1 - \langle d\phi, d\phi\rangle_{\tilde G},$$

so $\frac{d\tau}{\tau}$ is time-like if $|d\phi|_{\tilde G} < 1$. Note that the effect of such a coordinate change on the Mellin transform of the normal operator of $\Box_g$ is conjugation by $e^{-\imath\sigma\phi}$ since $\tilde\tau^{-\imath\sigma} = \tau^{-\imath\sigma}e^{\imath\sigma\phi}$. Such a coordinate change is useful when G has a product structure on $U \times [0,\delta)_{\tilde\tau}$, but $\tilde\tau$ is only a local boundary defining function on $U \times [0,\delta)_{\tilde\tau}$ (the product structure might not extend smoothly beyond U), in which case it is useful to see if one can conserve the time-like nature of $\frac{d\tilde\tau}{\tilde\tau}$ for a global boundary defining function. This is directly relevant for the study of conformally compact spaces in Subsection 4.9.

3.3. **Wave equation localization.** In this section we recall energy estimates and their consequences when $P_\sigma \in \mathrm{Diff}^2(X)$ is the wave operator to leading order in a region $\mathcal{O} \subset X$, i.e. when for a Lorentzian metric h on $\mathcal{O}$, $P_\sigma - \Box_h \in \mathrm{Diff}^1(\mathcal{O})$, and indeed when an analogous statement holds in the 'large parameter' sense as well, with the latter naturally arising by Mellin transforming b-wave equations. These results are not needed for the Fredholm properties, but are very useful in describing the asymptotics of the solutions of the wave equation on Kerr-de Sitter space as they show that certain terms arising from cutoffs do not affect the solution in the region of interest. These are also useful for giving an alternative explanation why the choice of the extension of the modified Laplacian across the boundary is unimportant on an (asymptotically) hyperbolic space. *In this whole subsection we assume* $|\operatorname{Im}\sigma| < C_0$ *in our uniform estimates, i.e. we only work in strips in* $\mathbb{C}$.

So assume that one is also given a function $\mathfrak{t} : \mathcal{O} \to (\mathfrak{t}_0, \mathfrak{t}_1)$ with $d\mathfrak{t}$ time-like and with $\mathfrak{t}$ proper on $\mathcal{O}$. Then one has the standard energy estimate; see e.g. [51, Section 2.8] for a version of these estimates (in a slightly different setup):

Proposition 3.7. *Assume* $\mathfrak{t}_0 < T_0 < T_0' < T_1 < \mathfrak{t}_1$. *Then*

$$\|u\|_{H^1(\mathfrak{t}^{-1}([T_0',T_1]))} \leq C\left(\|u\|_{H^1(\mathfrak{t}^{-1}([T_0,T_0']))} + \|P_\sigma u\|_{L^2(\mathfrak{t}^{-1}([T_0,T_1]))}\right).$$

Before proceeding, we recall here Remark 2.2: the error terms in our estimates in Section 2 can be localized in the base space X when one use sufficiently localized pseudodifferential operators, i.e. one does not need the global space H^{-N} in the errors. This will be useful for us below in using the basic energy estimate to control such errors.

For us, even more important is a semiclassical version of Proposition 3.7. The setup is more conveniently formulated in the large parameter setting, where the large parameter is interpreted as the dual variable of an extra variable of which the operator is independent. So with $\mathcal{O}$ as above, consider the family $P_\sigma \in \mathrm{Diff}^2(\mathcal{O})$

with large parameter dependence, and assume that the large parameter principal symbol of P_σ, p_σ, is the dual metric function G on $T^*\mathcal{O}\times{}^{\mathrm{b}}T^*\overline{\mathbb{R}^+_\tau}$ of an $\mathbb{R}^+$-invariant (acting as dilations in the second factor on $\mathcal{O}\times\mathbb{R}^+$) Lorentzian b-metric g, where σ is the b-dual variable[42] on ${}^{\mathrm{b}}T^*\overline{\mathbb{R}^+}$. Suppose moreover that, as above, we are also given a function $\mathfrak{t}:\mathcal{O}\to(\mathfrak{t}_0,\mathfrak{t}_1)$ with $d\mathfrak{t}$ time-like and with $\mathfrak{t}$ proper on $\mathcal{O}$. Then

Proposition 3.8. *Assume* $\mathfrak{t}_0<T_0<T_0'<T_1<\mathfrak{t}_1$. *Then, with* $P_{h,z}=h^2P_{h^{-1}z}$,

$$\|u\|_{H^1_\hbar(\mathfrak{t}^{-1}([T_0',T_1]))}\leq C\left(\|u\|_{H^1_\hbar(\mathfrak{t}^{-1}([T_0,T_0']))}+h^{-1}\|P_{h,z}u\|_{L^2(\mathfrak{t}^{-1}([T_0,T_1]))}\right).$$

We remark that Proposition 3.7 implies the corresponding estimate for σ finite (with the standard proof giving uniform dependence on σ in compact sets), so we may assume $h<1$, i.e. $|\sigma|>1$, here without losing control in a bounded set of σ.

Proof. We start by remarking that the L^2 norm on the right hand side is just the $H^0_\hbar$ norm, so this is a non-microlocal real principal type estimate except that there is no error term $h\|u\|_{H^{-N}_\hbar(\mathfrak{t}^{-1}([T_0,T_1]))}$ norm inside the parentheses on the right hand side (cf. the displayed equation in Footnote 32), and except that one would usually expect that both $h^{-1}\|P_{h,z}u\|_{L^2(\mathfrak{t}^{-1}([T_0,T_1]))}$ and $h\|u\|_{H^{-N}_\hbar(\mathfrak{t}^{-1}([T_0,T_1]))}$ should in fact be taken on a larger set, such as $\mathfrak{t}^{-1}([T_0,T_1'])$, $T_1'>T_1$. The point is thus to gain these improvements; this is done by a version of the classical energy estimates. We note that of these observations the only truly important part is the absence of a term $h\|u\|_{H^{-N}_\hbar(\mathfrak{t}^{-1}([T_0,T_1']))}$, which is thus on a larger set – this would prevent the argument leading to Proposition 3.9 below.

The following is essentially the standard proof of energy estimates, see e.g. [51, Section 2.8], but in a different context. Here we phrase it as done in [56, Sections 3 and 4]. So consider $V_\sigma=-\imath Z_\sigma$, $Z_\sigma=\chi(\mathfrak{t})W_\sigma$, and let W_σ be given by $W=G(d\mathfrak{t},.)$ considered as a first order differential operator on $\mathcal{O}$. That is, on $\mathcal{O}\times\mathbb{R}^+$, $G(d\mathfrak{t},.)$ gives a vector field of the form $W=W'+a\,\tau\partial_\tau$, with W' a vector field on $\mathcal{O}$ and a a function on $\mathcal{O}$ (both independent of τ), and via the Mellin transform one can consider W as $W_\sigma=W'+\imath a\sigma$, a σ-dependent first order differential operator on $\mathcal{O}$, with standard large-parameter dependence. Let $\Box$ be the d'Alembertian of g, and let $\Box_\sigma$ be its Mellin conjugate, so $P_\sigma-\Box_\sigma$ is first order, even in the large parameter sense, on $\mathcal{O}$. As usual in energy estimates, we want to consider the 'commutator'

$$-\imath(V_\sigma^*\Box_\sigma-(\Box_\sigma)^*V_\sigma). \tag{3.20}$$

While this can easily be computed directly, in order to connect it to the wave equation, we first recall the computation of $-\imath(V^*\Box-\Box V)$ on $\mathcal{O}\times\mathbb{R}^+$ with adjoints taken using the Lorentzian density (so $\Box$ is formally self-adjoint), rephrase this in terms of $\mathrm{Diff}_{\mathrm{b}}(\mathcal{O}\times[0,\infty))$; note that $\Box,V\in\mathrm{Diff}_{\mathrm{b}}(\mathcal{O}\times[0,\infty))$ so

$$-\imath(V^*\Box-\Box V)\in\mathrm{Diff}^2_{\mathrm{b}}(\mathcal{O}\times[0,\infty)).$$

Since all operators here are $\mathbb{R}^+$-invariant, the b-expressions are mostly a matter of notation. We then Mellin transform to compute[43] (3.20).

[42]We could also work with $T^*\mathbb{R}$ and standard dual variables via a logarithmic change of variables, changing dilations to translations, but in view of the previous section, the b-setup is particularly convenient.

[43]Operating with $-\log\tau$ in place of τ one would have translation invariance, no changes required into the b-notation, and one would use the Fourier transform.

Then, the usual computation, see [56, Section 3] for it written down in this form, using the standard summation convention, yields

$$
\begin{aligned}
&-\imath(V^*\Box - \Box V) = d^* C d,\ C_i^j = g_{i\ell} B^{\ell j}\\
&B^{ij} = -J^{-1}\partial_k(JZ^k G^{ij}) + G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i),
\end{aligned}
\tag{3.21}
$$

where C_i^j are the matrix entries of C relative to the basis $\{dz_\ell\}$ of the fibers of the cotangent bundle (rather than the b-cotangent bundle), $z_\ell = y_\ell$ for $\ell \leq n-1$, the y_ℓ local coordinates on a chart in $\mathcal{O}$, $z_n = \tau$. Expanding B using $Z = \chi W$, and separating the terms with χ derivatives, gives

$$
\begin{aligned}
B^{ij} &= G^{ik}(\partial_k Z^j) + G^{jk}(\partial_k Z^i) - J^{-1}\partial_k(JZ^k G^{ij})\\
&= (\partial_k\chi)\big(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k\big)\\
&\qquad + \chi\big(G^{ik}(\partial_k W^j) + G^{jk}(\partial_k W^i) - J^{-1}\partial_k(JW^k G^{ij})\big).
\end{aligned}
\tag{3.22}
$$

Multiplying the first term on the right hand side by $\alpha_i\,\overline{\alpha_j}$ (and summing over i, j; here $\alpha \in \mathbb{C}^n \simeq \mathbb{C}T_q^*(\mathcal{O}\times\mathbb{R}^+)$, $q \in \mathcal{O}\times\mathbb{R}^+$) gives

$$
\begin{aligned}
E_{W,d\chi}(\alpha) &= (\partial_k\chi)(G^{ik}W^j + G^{jk}W^i - G^{ij}W^k)\alpha_i\,\overline{\alpha_j}\\
&= (\alpha, d\chi)_G\,\overline{\alpha(W)} + \alpha(W)\,(d\chi,\alpha)_G - d\chi(W)(\alpha,\alpha)_G = \chi'(\mathfrak{t})\,E_{W,d\mathfrak{t}},\\
E_{W,d\mathfrak{t}} &= (\alpha, d\mathfrak{t})_G\,\overline{\alpha(W)} + \alpha(W)\,(d\mathfrak{t},\alpha)_G - d\mathfrak{t}(W)(\alpha,\alpha)_G.
\end{aligned}
\tag{3.23}
$$

Now, $E_{W,d\mathfrak{t}}$ is twice the sesquilinear stress-energy tensor associated to α, W and $d\mathfrak{t}$. This is well-known to be positive definite in α, i.e. $E_{W,d\chi}(\alpha) \geq 0$, with vanishing if and only if $\alpha = 0$, when W and $d\mathfrak{t}$ are both forward time-like for smooth Lorentz metrics, see e.g. [51, Section 2.7] or [32, Lemma 24.1.2]; (7.10) below provides the computation when α is real.

We change to a local basis of the b-cotangent bundle and use the b-differential ${}^{\mathrm{b}}d = (d_X, \tau\partial_\tau)$ and the local basis $\{dy_1, \dots, dy_{n-1}, \frac{d\tau}{\tau}\}$ of the fibers of the b-cotangent bundle, $\hat{\partial}_j = \delta_j\partial_j$, $\delta_j = 1$ for $j \leq n-1$, $\delta_n = \tau$, for the local basis of the fibers of the b-tangent bundle, write $\hat{G}^{ij}$, $\hat{g}_{ij}$ for the corresponding metric entries, $\hat{Z}^i$ for the vector field components. This yields

$$
\begin{aligned}
&-\imath(V^*\Box - \Box V) = {}^{\mathrm{b}}d^*\,\hat{C}\,{}^{\mathrm{b}}d,\ \hat{C}_i^j = \hat{g}_{i\ell}\hat{B}^{\ell j}\\
&\hat{B}^{ij} = -J^{-1}\delta_k^{-1}\delta_i^{-1}\delta_j^{-1}\hat{\partial}_k(J\delta_k\hat{Z}^k\delta_i\delta_j\hat{G}^{ij}) + \hat{G}^{ik}(\delta_j^{-1}\hat{\partial}_k\delta_j\hat{Z}^j) + \hat{G}^{jk}(\delta_i^{-1}\hat{\partial}_k\delta_i\hat{Z}^i).
\end{aligned}
\tag{3.24}
$$

While the δ-factors may have non-vanishing derivatives in the above expression for $\hat{B}$, if these are differentiated, χ in $\hat{Z}^i = \chi\hat{W}^i$ is not, so we conclude that

$$
\begin{aligned}
\hat{B}^{ij} =& (\hat{\partial}_k\chi)\big(\hat{G}^{ik}\hat{W}^j + \hat{G}^{jk}\hat{W}^i - \hat{G}^{ij}\hat{W}^k\big)\\
&+ \chi\big(\hat{G}^{ik}\delta_j^{-1}(\hat{\partial}_k\delta_j W^j) + G^{jk}\delta_i^{-1}(\hat{\partial}_k\delta_i W^i)\\
&\qquad - J^{-1}\delta_k^{-1}\delta_i^{-1}\delta_j^{-1}\hat{\partial}_k(J\delta_k\delta_i\delta_j W^k G^{ij})\big),
\end{aligned}
$$

and so

$$
\hat{C} = \chi' A + \chi R,
$$

with A positive definite, i.e. fixing any positive definite $\mathbb{R}^+$-invariant inner product $\tilde{g}$ on ${}^{\mathrm{b}}T^*(\mathcal{O}\times[0,\infty))$, $\langle\hat{C}\alpha,\alpha\rangle_g \geq c\|\alpha\|_{\tilde{g}}^2$, $\alpha \in {}^{\mathbb{C}\mathrm{b}}T^*(\mathcal{O}\times[0,\infty))$. Notice that the inner product on the left hand side is with respect to g, and is thus not positive definite. It is thus useful to rewrite

$$ {}^{\mathrm{b}}d_g^*\,\hat{C}\,{}^{\mathrm{b}}d = {}^{\mathrm{b}}d_{\tilde{g}}^*\,\tilde{C}\,{}^{\mathrm{b}}d,\ \ \tilde{C} = \chi'\tilde{A} + \chi\tilde{R}, $$

where the subscript denotes the inner product with respect to which the adjoint is taken. Below it is convenient to take $\tilde{g}$ to be of product form, so $\tilde{g} = \frac{d\tau^2}{\tau^2} + \tilde{h}$, $\tilde{h}$ a Riemannian metric on $\mathcal{O}$. Further, the $\mathbb{R}^+$-invariant b-density $|dg|$ can be written as $\frac{|d\tau|}{\tau}\,\omega$, where ω is a smooth density on $\mathcal{O}$; we may choose $\tilde{h}$ so that ω is the volume density of $\tilde{h}$.

Then for any dilation invariant $L \in \mathrm{Diff}_{\mathrm{b}}^k(\mathcal{O} \times [0,\infty))$, we can write $L = \sum_{\alpha\leq k} L_\alpha(\tau D_\tau)^\alpha$, $L_\alpha \in \mathrm{Diff}^{k-\alpha}(\mathcal{O})$ lifted via the product structure and then $L_{dg}^* = \sum_\alpha (L_\alpha)_{\tilde{h}}^*(\tau D_\tau)^\alpha$, where the subscript under the adjoint sign denotes the density with respect to which it is taken. This gives under the Mellin transform $L_\sigma = \sum_\alpha L_\alpha\sigma^\alpha$, $(L_{dg}^*)_\sigma = \sum_\alpha (L_\alpha)_{\tilde{h}}^*\sigma^\alpha$, so

$$ (L_\sigma)_{\tilde{h}}^* = \sum_\alpha (L_\alpha)_{\tilde{h}}^*\overline{\sigma}^\alpha = (L_{dg}^*)_{\overline{\sigma}}. $$

In particular, if $L_{dg}^* = L$ then $(L_\sigma)_{\tilde{h}}^* = L_{\overline{\sigma}}$, so L_σ is formally self-adjoint with respect to $\tilde{h}$ when σ is real, and in general then $(L_\sigma)_{\tilde{h}}^* - L_\sigma = (\operatorname{Im}\sigma)\tilde{L}_\sigma$, where $\tilde{L}_\sigma$ is a large parameter differential operator of order $k-1$ (which is not holomorphic in σ). The analogous computations also work on cotangent bundle valued differential operators if the adjoint is taken with respect to $\tilde{g}$, so for instance it works for ${}^{\mathrm{b}}d_{\tilde{g}}^*\tilde{C}\,{}^{\mathrm{b}}d$ above.

The Mellin transformed version of (3.24) finally computes (3.20); it reads as

$$ -\imath((V^*)_\sigma\Box_\sigma - \Box_\sigma V_\sigma) = (d_{\tilde{g}}^*)_\sigma\tilde{C}d_\sigma,\ \ \tilde{C} = \chi'\tilde{A} + \chi\tilde{R} \tag{3.25} $$

where $d_\sigma = (d_X, \imath\sigma)$, with the last component being multiplication by $\imath\sigma$, while $(d_{\tilde{g}}^*)_\sigma = (d_{\overline{\sigma}})_{\tilde{g}}^*$ is the transpose of $(d_{X,\tilde{h}}^*, -\imath\sigma)$, and A is positive definite. In particular, $(\Box_\sigma)^* - \Box_\sigma = \Box_{\overline{\sigma}} - \Box_\sigma$, which in the strip $|\operatorname{Im}\sigma| < C_0$ is first order in the large parameter sense, i.e. is of the form $S + \sigma T' + \overline{\sigma}T''$, $S \in \mathrm{Diff}^1(\mathcal{O})$, $T', T'' \in \mathcal{C}^\infty(\mathcal{O})$ dependent on σ only via the bounded quantity $\operatorname{Im}\sigma$ (i.e. are uniformly controlled). Correspondingly, dropping the subscripts on adjoints,

$$ \begin{aligned} & -\imath((V^*)_\sigma P_\sigma - (P_\sigma)^*V_\sigma) \\ & = -\imath((V^*)_\sigma\Box_\sigma - \Box_\sigma V_\sigma) - \imath(V^*)_\sigma(P_\sigma - \Box_\sigma) + \imath(P_\sigma - \Box_\sigma)^*V_\sigma + \imath((\Box_\sigma)^* - \Box_\sigma)V_\sigma \\ & = (d_\sigma)^*\tilde{C}d_\sigma + \hat{E}\chi d_\sigma + (d_\sigma)^*\chi\hat{E}^*, \end{aligned} \tag{3.26} $$

where $\hat{E} = (\hat{E}_X, \hat{E}')$, $\hat{E}_X \in \mathcal{C}^\infty(\mathcal{O}; TX)$, $\hat{E}' \in \mathcal{C}^\infty(\mathcal{O})$, and

$$ \tilde{C} = \chi'\tilde{A} + \chi\tilde{R}^\sharp, $$

since the contribution of $P_\sigma - \Box_\sigma$ and $\Box_{\overline{\sigma}} - \Box_\sigma$ to second order terms in the large parameter sense is only via terms not differentiated in χ, and where we absorbed $(d_\sigma)^* - (d^*)_\sigma$ arising from the right hand side of (3.25) in the $\hat{E}$ term. A standard argument, given below, making χ' large relative to χ, completes the proof.

Indeed, let $\chi_0(s) = e^{-1/s}$ for $s > 0$, $\chi_0(s) = 0$ for $s \leq 0$, $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ identically 1 on $[1,\infty)$, vanishing on $(-\infty, 0]$, Thus, $s^2\chi_0'(s) = \chi_0(s)$ for $s \in \mathbb{R}$. Now let

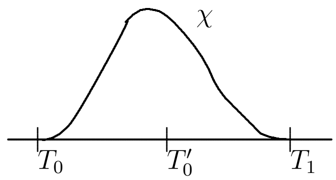

FIGURE 3. The graph of the function χ used below.

$T_1' \in (T_1, \mathfrak{t}_1)$, and consider (see Figure 3)

$$\chi(s) = \chi_0(-\digamma^{-1}(s - T_1'))\chi_1((s - T_0)/(T_0' - T_0)).$$

Then

- $\operatorname{supp}\chi \subset [T_0, T_1']$,
- $s \in [T_0', T_1'] \Rightarrow \chi' = -\digamma^{-1}\chi_0'(-\digamma^{-1}(s - T_1'))$,

so

$$s \in [T_0', T_1'] \Rightarrow \chi = -\digamma^{-1}(s - T_1')^2\chi',$$

so for $\digamma > 0$ sufficiently large, this is bounded by a small multiple of χ', namely

$$s \in [T_0', T_1'] \Rightarrow \chi \leq -\gamma\chi',\ \ \gamma = (T_1' - T_0')^2\digamma^{-1}. \tag{3.27}$$

In particular, for sufficiently large $\digamma$,

$$-(\chi'\tilde{A} + \chi\tilde{R}^\sharp) \geq -\chi_0'\chi_1\tilde{A}/2$$

on $[T_0', T_1']$. Thus,

$$\langle d_\sigma^*\tilde{C}d_\sigma u, u\rangle \geq -\frac{1}{2}\langle\chi_0'\chi_1\tilde{A}d_\sigma u, d_\sigma u\rangle - C'\|d_\sigma u\|^2_{L^2(\mathfrak{t}^{-1}([T_0,T_0']))}.$$

So for some $c_0 > 0$, by (3.26),

(3.28)

$$\begin{aligned} c_0\|(-\chi_0')^{1/2}\chi_1^{1/2}d_\sigma u\|^2 &\leq -\frac{1}{2}\langle\chi_0'\chi_1\tilde{A}d_\sigma u, d_\sigma u\rangle \\ &\leq C'\|d_\sigma u\|^2_{L^2(\mathfrak{t}^{-1}([T_0,T_0']))} + C'\|\chi^{1/2}P_\sigma u\|\|\chi^{1/2}d_\sigma u\| + C'\|\chi^{1/2}u\|\|\chi^{1/2}d_\sigma u\| \\ &\leq C'\|d_\sigma u\|^2_{L^2(\mathfrak{t}^{-1}([T_0,T_0']))} + C'\|\chi^{1/2}P_\sigma u\|^2 + 2C'\gamma\|(-\chi_0')^{1/2}\chi_1 d_\sigma u\|^2 \\ &\qquad + C'\gamma|\sigma|^{-1}\|(-\chi_0')^{1/2}\chi_1 d_\sigma u\|^2, \end{aligned}$$

where we used $\|\chi^{1/2}u\| \leq |\sigma|^{-1}\|\chi^{1/2}d_\sigma u\|$ (which holds in view of the last component of d_σ). Thus, choosing first $\digamma > 0$ sufficiently large (thus $\gamma > 0$ is sufficiently small), and then $|\sigma|$ sufficiently large, the last two terms on the right hand side can be absorbed into the left hand side. Rewriting in the semiclassical notation gives the desired result, except that $\|P_{h,z}u\|_{L^2_h(\mathfrak{t}^{-1}([T_0,T_1]))}$ is replaced by $\|P_{h,z}u\|_{L^2_h(\mathfrak{t}^{-1}([T_0,T_1']))}$ (or $\|\chi^{1/2}P_{h,z}u\|_{L^2_h(\mathfrak{t}^{-1}([T_0,T_1']))}$). This however is easily remedied by replacing χ by

$$\chi(s) = H(T_1 - s)\chi_0(-\digamma^{-1}(s - T_1'))\chi_1((s - T_0)/(T_0' - T_0)),$$

where H is the Heaviside step function (the characteristic function of $[0, \infty)$) so $\operatorname{supp}\chi \subset [T_0, T_1]$. Now χ is not smooth, but either approximating H by smooth bump functions and taking a limit, or indeed directly performing the calculation, integrating on the domain with boundary $\mathfrak{t} \leq T_1$, the contribution of the derivative of H to χ' is a delta distribution at $\mathfrak{t} = T_1$, corresponding to a boundary term on the domain, which has the same sign as the derivative of χ_0. Thus, with $\mathcal{S}_{T_1}$ the hypersurface $\{\mathfrak{t} = T_1\}$, (3.28) holds in the form

$$\begin{aligned} c_0\|d_\sigma u\|^2_{L^2(\mathfrak{t}^{-1}([T_0',T_1]))} + c_0\|d_\sigma u\|^2_{L^2(\mathcal{S}_{T_1})} \\ \leq C'\|d_\sigma u\|^2_{L^2(\mathfrak{t}^{-1}([T_0,T_0']))} + C'\|\chi^{1/2}P_\sigma u\|^2 + 2C'\gamma\|(-\chi_0')^{1/2}\chi_1 H d_\sigma u\|^2 \\ + C'\gamma^{-1}|\sigma|^{-1}\|(-\chi_0')^{1/2}\chi_1 H d_\sigma u\|^2. \end{aligned}$$

Now one can simply drop the second term from the left hand side and proceed as above; the semiclassical rewriting now proves the claimed result. □

Suppose now that $\mathfrak{t}$ is a globally defined function on X, with $\mathfrak{t}|_{\mathcal{O}}$ having the properties discussed above, and such that $p_{\hbar,z}$ is *semiclassically non-trapping, resp. mildly trapping*, in $\mathfrak{t}^{-1}((-\infty,T_1])$, in the sense that in Definitions 2.12, resp. Definition 2.16, $\mathrm{ell}(q_{\hbar,z})$ is replaced by $\overline{T}^*_{\mathfrak{t}=T_1}X$ (and X itself is replaced by $\mathfrak{t}^{-1}((-\infty,T_1])$). Proposition 3.8 can be used to show directly that when $\mathrm{Re}\,\sigma$ is large, σ is in a strip, if $\mathrm{supp}\,P_\sigma u \subset \mathfrak{t}^{-1}([T_1,+\infty))$, then $\mathrm{supp}\,u \subset \mathfrak{t}^{-1}([T_1,+\infty))$. Indeed, by the discussion preceding Theorem 2.14, if P_σ is semiclassically non-trapping, we have, with $T_0' < T_0'' < T_1$, and suitably large s (which we may take to satisfy $s \geq 1$),

$$\|u\|_{H^s_\hbar(\mathfrak{t}^{-1}(-\infty,T_0'])} \leq C\big(h^{-1}\|P_{h,z}u\|_{H^{s-1}_\hbar(\mathfrak{t}^{-1}(-\infty,T_0''])} + h\|u\|_{H^{-N}_\hbar(\mathfrak{t}^{-1}(-\infty,T_0''))}\big).$$

If instead $P_\sigma - \imath Q_\sigma$ is mildly trapping of order $\varkappa$ then

$$\|u\|_{H^s_\hbar(\mathfrak{t}^{-1}(-\infty,T_0'])} \leq C\big(h^{-\varkappa-1}\|P_{h,z}u\|_{H^{s-1}_\hbar(\mathfrak{t}^{-1}(-\infty,T_0''])} + h\|u\|_{H^{-N}_\hbar(\mathfrak{t}^{-1}(-\infty,T_0''))}\big).$$

Using this in combination with Proposition 3.8 yields

$$\|u\|_{H^1_\hbar(\mathfrak{t}^{-1}(-\infty,T_1])} \leq C\big(h^{-\varkappa-1}\|P_{h,z}u\|_{H^{s-1}_\hbar(\mathfrak{t}^{-1}(-\infty,T_1])} + h\|u\|_{H^{-N}_\hbar(\mathfrak{t}^{-1}(-\infty,T_0''])}\big).$$

Now, for sufficiently small h, the second term on the right hand side can be absorbed into the left hand side to yield

$$\|u\|_{H^1_\hbar(\mathfrak{t}^{-1}(-\infty,T_1])} \leq Ch^{-\varkappa-1}\|P_{h,z}u\|_{H^{s-1}_\hbar(\mathfrak{t}^{-1}(-\infty,T_1])}. \tag{3.29}$$

This shows that for h sufficiently small, i.e. $\mathrm{Re}\,\sigma$ sufficiently large, the vanishing of $P_\sigma u$ in $\{\mathfrak{t} < T_1\}$ gives that of u in the same region.

Turning to the operator $P_\sigma - \imath Q_\sigma$, assuming that Ω, the domain of definition of Q_σ, includes the set $C_- < \mathrm{Im}\,\sigma < C_+$ (with $C_+ > C_-$) for $|\mathrm{Re}\,\sigma|$ sufficiently large, as is the case for all the operators considered in Subsection 3.2, we thus conclude:

Proposition 3.9. *Suppose $\mathcal{O}$, $\mathfrak{t}$, P_σ are as discussed before the statement of Proposition 3.8, with $\mathfrak{t}$ globally defined on X, and P_σ, Q_σ as in Theorem 2.14 or as in Theorem 2.17, with domain satisfying the asymptotic strip condition as stated before the proposition. Suppose also that $p_{\hbar,z}$ is semiclassically mildly trapping in $\mathfrak{t}^{-1}((-\infty,T_1])$ in the sense discussed above. Finally, suppose that the Schwartz* *kernel of* Q_σ *is supported in* $\mathfrak{t}^{-1}((T_1,+\infty))\times\mathfrak{t}^{-1}((T_1,+\infty))$. *Then*[44]

$$\operatorname{supp} f\subset\mathfrak{t}^{-1}([T_1,+\infty))\Rightarrow\operatorname{supp}(P_\sigma-\imath Q_\sigma)^{-1}f\subset\mathfrak{t}^{-1}([T_1,+\infty)),$$
$$\operatorname{supp} g\subset\mathfrak{t}^{-1}((-\infty,T_1])\Rightarrow\operatorname{supp}(P^*_\sigma+\imath Q^*_\sigma)^{-1}g\subset\mathfrak{t}^{-1}((-\infty,T_1]).$$

Proof. Note that $u=(P_\sigma-\imath Q_\sigma)^{-1}f$ satisfies $(P_\sigma-\imath Q_\sigma)u=f$, so in view of the support condition on Q_σ, $\operatorname{supp} P_\sigma u\subset\mathfrak{t}^{-1}([T_1,+\infty))$. For $\sigma\in\Omega$ in the strip $C_-<\operatorname{Im}\sigma<C_+$ and with $|\operatorname{Re}\sigma|$ sufficiently large, the proposition then follows from (3.29). Thus, with ψ supported in $\mathfrak{t}^{-1}((-\infty,T_1))$, ϕ supported in $\mathfrak{t}^{-1}((T_1,+\infty))$, and σ in this region, $\psi(P_\sigma-\imath Q_\sigma)^{-1}\phi=0$. By the meromorphy of $\psi(P_\sigma-\imath Q_\sigma)^{-1}\phi$, $\psi(P_\sigma-\imath Q_\sigma)^{-1}\phi=0$ follows for all $\sigma\in\Omega$. This also gives $\phi(P^*_\sigma+\imath Q^*_\sigma)^{-1}\psi=0$. □

One reason this proposition is convenient is that it shows that for ψ supported in $\mathfrak{t}^{-1}((-\infty,T_1))$, $\psi(P_\sigma-\imath Q_\sigma)^{-1}$ is independent of the choice of Q_σ (satisfying the general conditions); analogous results hold for modifying P_σ in $\mathfrak{t}^{-1}((T_1,+\infty))$. Indeed, let Q'_σ be another operator satisfying conditions analogous to those on Q_σ for σ in some open set $\Omega'\subset\mathbb{C}$ (including the assumption of asymptotically, as $|\operatorname{Re}\sigma|\to\infty$, containing a strip $C_-<\operatorname{Im}\sigma<C_+$), and let $(\mathcal{X}')^s\subset H^s$ be the corresponding function space in place of $\mathcal{X}^s$ (note that $\mathcal{Y}^s=H^{s-1}$ is independent of Q_σ); thus $(P_\sigma-\imath Q_\sigma)^{-1}:\mathcal{Y}^s\to\mathcal{X}^s$ and $(P_\sigma-\imath Q'_\sigma)^{-1}\mathcal{Y}^s\to(\mathcal{X}')^s$ are meromorphic on $\Omega\cap\Omega'$. Then for $\sigma\in\Omega\cap\Omega'$ which is not a pole of either $(P_\sigma-\imath Q_\sigma)^{-1}$ or $(P_\sigma-\imath Q'_\sigma)^{-1}$, and for $f\in\mathcal{Y}^s$, let $u=(P_\sigma-\imath Q_\sigma)^{-1}f$, $u'=(P_\sigma-\imath Q'_\sigma)^{-1}f$. Then $(P_\sigma-\imath Q_\sigma)u'=f+\imath(Q_\sigma-Q'_\sigma)u'\in H^{s-2}$, and thus, provided $s-1>1/2-\beta\operatorname{Im}\sigma$ (rather than this inequality holding merely for s),

$$u'=(P_\sigma-\imath Q_\sigma)^{-1}f+(P_\sigma-\imath Q_\sigma)^{-1}\imath(Q_\sigma-Q'_\sigma)u',$$

so

$$\psi(P_\sigma-\imath Q'_\sigma)^{-1}f=\psi(P_\sigma-\imath Q_\sigma)^{-1}f$$

since $\psi(P_\sigma-\imath Q_\sigma)^{-1}\imath(Q_\sigma-Q'_\sigma)=0$ in view of the support properties of Q_σ and Q'_σ. In particular, one may drop the particular choice of Q_σ from the notation; note also that this also establishes the equality for $s>1/2-\beta\operatorname{Im}\sigma$ since $(P_\sigma-\imath Q_\sigma)^{-1}$ is independent of s in this range in the sense of Remark 2.9. This is particularly helpful if for σ in various open subsets Ω_j of $\mathbb{C}$ we construct different operators $Q^{(j)}_\sigma$; if for instance for each Ω_j, $(P_\sigma-\imath Q^{(j)}_\sigma)^{-1}$ is meromorphic, resp. holomorphic, the same follows for the single operator family (independent of j) $\psi(P_\sigma-\imath Q_\sigma)^{-1}$ where we now we write Q_σ for any of the valid choices (i.e. $Q_\sigma=Q^{(j)}_\sigma$ for any one of the j's such that $\sigma\in\Omega_j$). We then have the following extension of Lemma 3.1.

Corollary 3.10. *Suppose* $\mathcal{P}$ *is invariant under dilations in* τ *for functions supported near* $\tau=0$*, and the normal operator family* $\hat{N}(\mathcal{P})$ *is of the form* P_σ *satisfying*

[44]In particular, this shows that the support of the Schwartz kernel of the inverse, with the first (left) factor giving the 'outgoing' and the second (right) factor the 'incoming' (i.e. the one in which the integral is taken) variables, satisfies

$$\begin{aligned}\operatorname{supp} K_{(P_\sigma-\imath Q_\sigma)^{-1}}\subset&\big(X\times\mathfrak{t}^{-1}((-\infty,\mathfrak{t}_0])\big)\cup\big(\mathfrak{t}^{-1}([\mathfrak{t}_1,+\infty))\times X\big)\\&\cup(\mathfrak{t}\times\mathfrak{t})^{-1}\{(\mathfrak{t}',\mathfrak{t}'')\in(\mathfrak{t}_0,\mathfrak{t}_1)^2:\ \mathfrak{t}'\geq\mathfrak{t}''\};\end{aligned}$$

for $K_{(P^*_\sigma+\imath Q^*_\sigma)^{-1}}$ the two factors are reversed. This also gives that the Laurent coefficients have similar support properties at any pole. In summary, there is a 'block lower triangular' structure (with the first variable being on the vertical axis, increasing downwards, as in matrix notation) to $\operatorname{supp} K_{(P_\sigma-\imath Q_\sigma)^{-1}}$, with the middle piece, $(\mathfrak{t}_0,\mathfrak{t}_1)^2$, itself being lower triangular.

the conditions[45] *of Sections 2 and 7, and such that there are is an open cover of* $\mathbb{C}$ *by sets* Ω_j*, and for each* j *there is an operator* $Q_\sigma^{(j)}$ *satisfying the conditions of Section 2, including semiclassical non-trapping. Let* $\mathfrak{t}$ *be as in Proposition 3.9,* ψ *supported in* $\mathfrak{t}^{-1}((-\infty, T_1))$*, identically* 1 *on* $\mathfrak{t} < T_1'$*. Let* σ_j *be the poles of the meromorphic family*[46] $\psi(P_\sigma - \imath Q_\sigma)^{-1}$*. Then for* $\ell < \beta^{-1}(s-(k-1)/2)$*,* $\ell \neq -\operatorname{Im}\sigma_j$ *for any* j*,* $\mathcal{P}u = f$ *in* $\mathfrak{t} < T_1$*,* u *tempered, supported near* $\tau = 0$*,* $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$*, in* $\mathfrak{t} < T_1'$*,* u *has an asymptotic expansion*

$$u = \sum_j \sum_{\kappa \leq m_j} \tau^{\imath\sigma_j} (\log|\tau|)^\kappa a_{j\kappa} + u' \tag{3.30}$$

with $a_{j\kappa} \in \mathcal{C}^\infty(X)$ *and* $u' \in \tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$*.*

If instead the family $P_\sigma - \imath Q_\sigma$ *is semiclassically mildly trapping of order* $\varkappa$ *in a* C_0*-strip then for* $\ell < C_0$ *and* $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1+\varkappa}(\bar{M}_\infty)$ *one has, in* $\mathfrak{t} < T_1'$*,*

$$u = \sum_j \sum_{\kappa \leq m_j} \tau^{\imath\sigma_j} (\log|\tau|)^\kappa a_{j\kappa} + u' \tag{3.31}$$

with $a_{j\kappa} \in \mathcal{C}^\infty(X)$ *and* $u' \in \tau^\ell H_{\mathrm{b}}^s(\bar{M}_\infty)$*.*

Conversely, given f *in the indicated spaces, with* f *supported near* $\tau = 0$*, a solution* u *of* $\mathcal{P}u = f$ *of the form* (3.30)*, resp.* (3.31)*, supported near* $\tau = 0$ *exists in* $\mathfrak{t} < T_1'$*.*

In either case, the coefficients $a_{j\kappa}$ *are given by the Laurent coefficients of* $\psi(\mathcal{P} - \imath\mathcal{Q})^{-1}$ *at the poles* σ_j *applied to* f*, with simple poles corresponding to* $m_j = 0$*.*

If $f = \sum_j \sum_{\kappa \leq m_j'} \tau^{\alpha_j} (\log|\tau|)^\kappa b_{j\kappa} + f'$*, with* f' *in the spaces indicated above for* f*, and* $b_{j\kappa} \in H^{s-k+1}(X)$*, analogous results hold when the expansion of* f *is added to the form of* (3.30) *and* (3.31)*, in the sense of the extended union of index sets, see* [43, Section 5.18]*.*

Further, the result is stable under sufficiently small dilation-invariant perturbations in the b-sense, i.e. if $\mathcal{P}'$ *is sufficiently close to* $\mathcal{P}$ *in* $\Psi_{\mathrm{b}}^k(\bar{M}_\infty)$ *with real principal symbol, then there is a similar expansion for solutions of* $\mathcal{P}'u = f$ *in* $\mathfrak{t} < T_1'$*.*

For $\mathcal{P}^*$ *in place of* $\mathcal{P}$*, analogous results apply, but we need* $\ell < -\beta^{-1}(s-(k-1)/2)$*, and the* $a_{j\kappa}$ *are not smooth, but have wave front set in the Lagrangians* $\Lambda_\pm$*.*

Remark 3.11. Proposition 3.5 has an analogous extension, but we do not state it here explicitly.

Further, estimates analogous to Remark 3.4 are applicable, with the norms of restrictions to $\mathfrak{t} < T_1'$ bounded in terms of the norms of restrictions to $\mathfrak{t} < T_1$.

Proof. We follow the proof of Lemma 3.1 closely. Again, we first consider the expansion, and let $\alpha, r \in \mathbb{R}$ be such that $u \in \tau^\alpha H_{\mathrm{b}}^r(\bar{M}_\infty)$ and $\psi(P_\sigma - \imath Q_\sigma)^{-1}$ has[47] no poles on $\operatorname{Im}\sigma = -\alpha$. These α, r exist due to the vanishing of u for $\tau > 1$ and the absence of poles of $\psi(P_\sigma - \imath Q_\sigma^{(j)})^{-1}$ for $\operatorname{Re}\sigma$ large, σ in a strip; then also $u \in \tau^\alpha (1+\tau)^{-N} H_{\mathrm{b}}^r(\bar{M}_\infty)$ for all N. The Mellin transform of the PDE, a priori on $\operatorname{Im}\sigma = -\alpha$, is $P_\sigma \mathcal{M}u = \mathcal{M}f$, and thus $(P_\sigma - \imath Q_\sigma)\mathcal{M}u = f - \imath Q_\sigma \mathcal{M}u$. Thus,

$$\mathcal{M}u = (P_\sigma - \imath Q_\sigma)^{-1}\mathcal{M}f - (P_\sigma - \imath Q_\sigma)^{-1}\imath Q_\sigma \mathcal{M}u \tag{3.32}$$

[45] Again, as discussed in Remark 7.4, the large $\operatorname{Im}\sigma$ assumptions only affect the existence part below, and do so relatively mildly.

[46] As remarked above, these are independent of the choice of j for $\sigma \in \mathbb{C}$ as long as $\sigma \in \Omega_j$.

[47] Recall that this operator, when considered as a product, refers to $\psi(P_\sigma - \imath Q_\sigma^{(j)})^{-1}$, with j appropriately chosen.

there. Restricting to $\mathfrak{t} < T_1$, the last term vanishes by Proposition 3.9, so

$$\mathcal{M}u|_{\mathfrak{t}<T_1'} = (P_\sigma - \imath Q_\sigma)^{-1}\mathcal{M}f|_{\mathfrak{t}<T_1'}$$

If $f \in \tau^\ell H_{\mathrm{b}}^{s-k+1}(\bar{M}_\infty)$, then shifting the contour of integration to $\operatorname{Im}\sigma = -\ell$, we obtain contributions from the poles of $(P_\sigma - \imath Q_\sigma)^{-1}$, giving the expansion in (3.30) and (3.31) by Cauchy's theorem. The error term u' is what one obtains by integrating along the new contour in view of the high energy bounds on $(P_\sigma - \imath Q_\sigma)^{-1}$ (which differ as one changes one's assumption from non-trapping to mild trapping), and the assumptions on f.

Conversely, to obtain existence, let $\alpha < \min(\ell, -\sup \operatorname{Im}\sigma_j)$ and define $u \in \tau^\alpha H_{\mathrm{b}}^s(\bar{M}_\infty)$ by

$$\mathcal{M}u = (P_\sigma - \imath Q_\sigma)^{-1}\mathcal{M}f,$$

using the inverse Mellin transform with $\operatorname{Im}\sigma = -\alpha$. Then

$$P_\sigma \mathcal{M}u = \mathcal{M}f + \imath Q_\sigma \mathcal{M}u,$$

and so $P_\sigma\mathcal{M}u|_{\mathfrak{t}<T_1} = \mathcal{M}f|_{\mathfrak{t}<T_1}$. Thus, the expansion follows by the first part of the argument. The support property of u follows from Paley-Wiener, taking into account holomorphy in $\operatorname{Im}\sigma > -\alpha$, and the estimates on $\mathcal{M}f$ and $\psi(P_\sigma - \imath Q_\sigma)^{-1}$ there.

Finally, stability under perturbations follows for the same reasons as those stated in Lemma 3.1 once one remarks that the $Q_\sigma^{(j)}$ used for $\mathcal{P}$ can actually be used for $\mathcal{P}'$ as well provided these two are sufficiently close, since the relationship between P_σ and $Q_\sigma^{(j)}$ is via ellipticity considerations, and these are preserved under small perturbations of P_σ. □

4. De Sitter space and conformally compact spaces

In this section we show how de Sitter space and conformally compact spaces fit into the general framework we have developed. We start by describing de Sitter space and hyperbolic space from this perspective, then in Subsection 4.8 we discuss more general operators, and then in Subsection 4.9 we show that even asymptotically hyperbolic spaces indeed fit into this framework, and we state our results in this setting.

4.1. **De Sitter space as a symmetric space.** Rather than starting with the static picture of de Sitter space, we consider it as a Lorentzian symmetric space. We follow the treatment of [53] and [40]. De Sitter space M is given by the hyperboloid

$$z_1^2 + \ldots + z_n^2 = z_{n+1}^2 + 1 \text{ in } \mathbb{R}^{n+1}$$

equipped with the pull-back g of the Minkowski metric

$$dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2.$$

Introducing polar coordinates (R, θ) in $(z_1, \ldots, z_n)$, so

$$R = \sqrt{z_1^2 + \ldots + z_n^2} = \sqrt{1 + z_{n+1}^2},\ \theta = R^{-1}(z_1, \ldots, z_n) \in \mathbb{S}^{n-1},\ \tilde{\tau} = z_{n+1},$$

the hyperboloid can be identified with $\mathbb{R}_{\tilde{\tau}} \times \mathbb{S}_\theta^{n-1}$ with the Lorentzian metric

$$g = \frac{d\tilde{\tau}^2}{\tilde{\tau}^2 + 1} - (\tilde{\tau}^2 + 1)\, d\theta^2,$$

where $d\theta^2$ is the standard Riemannian metric on the sphere. For $\tilde\tau > 1$, set $x = \tilde\tau^{-1}$, so the metric becomes

$$g = \frac{(1+x^2)^{-1}\,dx^2 - (1+x^2)\,d\theta^2}{x^2}.$$

An analogous formula holds for $\tilde\tau < -1$, so compactifying the real line to an interval $[0,1]_T$, with $T = x = \tilde\tau^{-1}$ for $x < \frac{1}{4}$ (i.e. $\tilde\tau > 4$), say, and $T = 1 - |\tilde\tau|^{-1}$, $\tilde\tau < -4$, gives a compactification, $\hat M$, of de Sitter space on which the metric is conformal to a non-degenerate Lorentz metric. There is natural generalization, to *asymptotically de Sitter-like spaces* $\hat M$, which are diffeomorphic to compactifications $[0,1]_T \times Y$ of $\mathbb{R}_{\tilde\tau} \times Y$, where Y is a compact manifold without boundary, and $\hat M$ is equipped with a Lorentz metric on its interior which is conformal to a Lorentz metric smooth up to the boundary. These space-times are Lorentzian analogues of the much-studied conformally compact (Riemannian) spaces. On this class of space-times the solutions of the Klein-Gordon equation were analyzed by Vasy in [53], and were shown to have simple asymptotics analogous to those for generalized eigenfunctions on conformally compact manifolds.

Theorem. *(*[53, Theorem 1.1.]*) Set* $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}$. *If* $s_+(\lambda) - s_-(\lambda) \notin \mathbb{N}$, *any solution* u *of the Cauchy problem for* $\Box - \lambda$, *with* $\mathcal{C}^\infty$ *initial data imposed at* $\tilde\tau = 0$, *is of the form*[48]

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-,\ \ v_\pm \in \mathcal{C}^\infty(\hat M).$$

If $s_+(\lambda) - s_-(\lambda)$ *is an integer, the same conclusion holds if* $v_- \in \mathcal{C}^\infty(\hat M)$ *is replaced by* $v_- = \mathcal{C}^\infty(\hat M) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(\hat M)$.

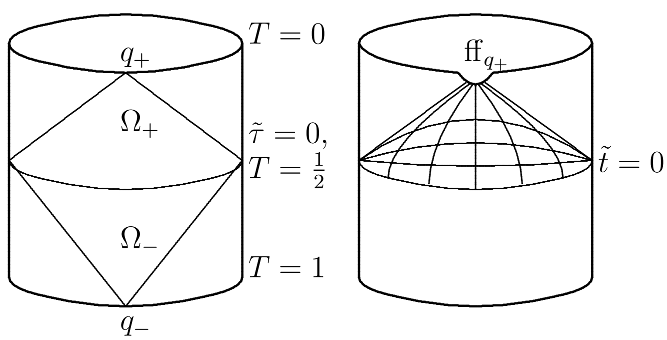

FIGURE 4. On the left, the compactification of de Sitter space with the backward light cone from $q_+ = (1,0,0,0)$ and forward light cone from $q_- = (-1,0,0,0)$ are shown. Ω_+, resp. Ω_-, denotes the intersection of these light cones with $\tilde\tau > 0$, resp. $\tilde\tau < 0$. On the right, the blow up of de Sitter space at q_+ is shown. The interior of the light cone inside the front face ff_{q_+} can be identified with the spatial part of the static model of de Sitter space. The spatial and temporal coordinate lines for the static model are also shown.

[48]Here the asymptotic behavior as $x \to 0$ is the interesting statement.

One important feature of asymptotically de Sitter spaces is the following: a conformal factor, such as x^{-2} above, does not change the image of null-geodesics, only reparameterizes them. More precisely, recall that null-geodesics are merely projections to M of null-bicharacteristics of the metric function in T^*M. Since $p \mapsto \mathsf{H}_p$ is a derivation, $ap \mapsto a\mathsf{H}_p + p\mathsf{H}_a$, which is $a\mathsf{H}_p$ on the characteristic set of p. Thus, the null-geodesics of de Sitter space are the same (up to reparameterization) as those of the metric

$$(1+x^2)^{-1}\,dx^2 - (1+x^2)\,d\theta^2$$

which is smooth on the compact space $\hat{M}$.

4.2. **The static model of a part of de Sitter space.** The simple structure of de Sitter metric (and to some extent of the asymptotically de Sitter-like metrics) can be hidden by blowing up certain submanifolds of the boundary of $\hat{M}$. In particular, the *static model* of de Sitter space arises by singling out a point on $\mathbb{S}^{n-1}_\theta$, e.g. $q_0 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Note that $(\theta_2, \dots, \theta_n) \in \mathbb{R}^{n-1}$ are local coordinates on $\mathbb{S}^{n-1}$ near q_0. Now consider the intersection of the backward light cone from q_0 considered as a point q_+ at future infinity, i.e. where $T = 0$, and the forward light cone from q_0 considered as a point q_- at past infinity, i.e. where $T = 1$. These intersect the equator $T = 1/2$ (here $\tilde{\tau} = 0$) in the same set, and together form a 'diamond', $\hat{\Omega}$, with a conic singularity at q_+ and q_-. Explicitly $\hat{\Omega}$ is given by the two inequalities $z_2^2 + \dots + z_n^2 \le 1$, $z_1 \ge 0$, inside the hyperboloid. If q_+, q_- are blown up, as well as the corner $\partial\hat{\Omega} \cap \{\tilde{\tau} = 0\}$, i.e. where the light cones intersect $\tilde{\tau} = 0$ in $\hat{\Omega}$, we obtain a manifold $\bar{M}$, which can be blown down to (i.e. is a blow up of) the space-time product $[0,1] \times \mathbb{B}^{n-1}$, with $\mathbb{B}^{n-1} = \{Z \in \mathbb{R}^{n-1} :\ |Z| \le 1\}$ on which the Lorentz metric has a time-translation invariant warped product form. Namely, first considering the interior Ω of $\hat{\Omega}$ we introduce the global (in Ω) standard static coordinates $(\tilde{t}, Z)$, given by (with the expressions involving x valid near $T = 0$)

$$\begin{aligned}
&(\mathbb{B}^{n-1})^\circ \ni Z = (z_2, \dots, z_n) = x^{-1}\sqrt{1+x^2}(\theta_2, \dots, \theta_n),\\
&\qquad\qquad \sinh \tilde{t} = \frac{z_{n+1}}{\sqrt{z_1^2 - z_{n+1}^2}} = (x^2 - (1+x^2)(\theta_2^2 + \dots + \theta_n^2))^{-1/2}.
\end{aligned}$$

It is convenient to rewrite these as well in terms of polar coordinates in Z (valid away from $Z = 0$):

$$\begin{aligned}
r &= \sqrt{z_2^2 + \dots + z_n^2} = \sqrt{1 + z_{n+1}^2 - z_1^2} = x^{-1}\sqrt{1+x^2}\sqrt{\theta_2^2 + \dots + \theta_n^2},\\
\sinh \tilde{t} &= \frac{z_{n+1}}{\sqrt{z_1^2 - z_{n+1}^2}} = (x^2 - (1+x^2)(\theta_2^2 + \dots + \theta_n^2))^{-1/2} = x^{-1}(1-r^2)^{-1/2},\\
\omega &= r^{-1}(z_2, \dots, z_n) = (\theta_2^2 + \dots + \theta_n^2)^{-1/2}(\theta_2, \dots, \theta_n) \in \mathbb{S}^{n-2}.
\end{aligned}$$

In these coordinates the metric becomes

$$(1-r^2)\,d\tilde{t}^2 - (1-r^2)^{-1}dr^2 - r^2\,d\omega^2, \tag{4.1}$$

which is a special case of the de Sitter-Schwarzschild metrics with vanishing mass, $M = 0$, and cosmological constant $\Lambda = 3$, see Section 6. Correspondingly, the dual metric is

$$(1-r^2)^{-1}\partial_{\tilde{t}}^2 - (1-r^2)\partial_r^2 - r^{-2}\partial_\omega^2. \tag{4.2}$$

We also rewrite this in terms of coordinates valid at the origin, namely $Y = r\omega$:

$$(1-|Y|^2)^{-1}\partial_{\tilde t}^2 + (\sum_{j=1}^{n-1} Y_j\partial_{Y_j})^2 - \sum_{j=1}^{n-1}\partial_{Y_j}^2. \tag{4.3}$$

4.3. **Blow-up of the static model.** We have already seen that de Sitter space has a smooth conformal compactification; the singularities in the metric of the form (4.1) at $r = 1$ must thus be artificial. On the other hand, the metric is already well-behaved for $r < 1$ bounded away from 1, so we want the coordinate change to be smooth there — this means smoothness in valid coordinates (Y above) at the origin as well. This singularity can be removed by a blow-up on an appropriate compactification. We phrase this at first in a way that is closely related to our treatment of Kerr-de Sitter space, and the Kerr-star-type coordinates used there, see (6.4)-(6.5). So let

$$t = \tilde t + h(r),\ h(r) = \frac{1}{2}\log\mu,\ \mu = 1 - r^2.$$

Note that h is smooth at the origin. A key feature of this change of coordinates is

$$h'(r) = -\frac{r}{\mu} = -\frac{1}{\mu} + \frac{1}{1+r},$$

which is $-\mu^{-1}$ near $r = 1$ modulo terms smooth at $r = 1$. Other coordinate changes with this property would also work. Let

$$\tau = e^{-t} = \frac{e^{-\tilde t}}{\mu^{1/2}}.$$

Thus, if we compactify static space-time as $\mathbb{B}^{n-1}_{r\omega} \times [0,1]_{\tilde T}$, with $\tilde T = e^{-\tilde t}$ for say $\tilde t > 4$, then this procedure amounts to blowing up the corner $\tilde T = 0$, $\mu = 0$ parabolically[49]. Then the dual metric becomes

$$-\mu\partial_r^2 - 2r\partial_r\tau\partial_\tau + \tau^2\partial_\tau^2 - r^{-2}\partial_\omega^2,$$

or

$$-4r^2\mu\partial_\mu^2 + 4r^2\tau\partial_\tau\partial_\mu + \tau^2\partial_\tau^2 - r^{-2}\partial_\omega^2, \tag{4.4}$$

which is a non-degenerate Lorentzian b-metric[50] on $\mathbb{R}^{n-1}_{r\omega} \times [0,1)_\tau$, i.e. it extends smoothly and non-degenerately across the 'event horizon', $r = 1$. Note that in coordinates valid near $r = 0$ this becomes

$$(\sum_j Y_j\partial_{Y_j})^2 - 2(\sum_j Y_j\partial_{Y_j})\tau\partial_\tau + \tau^2\partial_\tau^2 - \sum_j \partial_{Y_j}^2 = (\tau\partial_\tau - \sum_j Y_j\partial_{Y_j})^2 - \sum_j\partial_{Y_j}^2.$$

In slightly different notation, this agrees with the symbol of [53, Equation (7.3)].

We could have used other equivalent local coordinates; for instance replaced $e^{-\tilde t}$ by $(\sinh\tilde t)^{-1}$, in which case the coordinates (r, τ, ω) we obtained are replaced by

$$r,\ \rho = (\sinh\tilde t)^{-1}/(1-r^2)^{1/2} = x,\ \omega. \tag{4.5}$$

As expected, in these coordinates the metric would still be a smooth and non-degenerate b-metric. These coordinates also show that Kerr-star-type coordinates

[49] If we used $\tau = e^{-2t}$ and $\tilde T = e^{-2\tilde t}$, everything would go through, except there would be many additional factors of 2; then the blow-up would be homogeneous, i.e. spherical. See Footnote 14 for a description of spherical blow-ups. See [43] for parabolic blow-ups.

[50] See Section 3 for a quick introduction to b-geometry and further references.

are smooth in the interior of the front face on the blow-up of our conformal compactification of de Sitter space at q_+.[51] In summary we have reproved (modulo a few details):

Lemma 4.1. *(See* [40, Lemma 2.1] *for a complete version.) The lift of $\hat{\Omega}$ to the blow up $[\hat{M}; q_+, q_-]$ is a $\mathcal{C}^\infty$ manifold with corners, $\bar{\Omega}$. Moreover, near the front faces $\mathrm{ff}_{q_\pm}$, i.e. away from $\tilde{\tau} = 0$, $\bar{\Omega}$ is naturally diffeomorphic to a neighborhood of the temporal faces $\mathrm{tf}_\pm$ in the $\mathcal{C}^\infty$ manifold with corners obtained from $[0,1]_T \times \mathbb{B}^{n-1}$ by blowing up the corners $\{0\} \times \partial\mathbb{B}^{n-1}$ and $\{1\} \times \partial\mathbb{B}^{n-1}$ in the parabolic manner indicated in* (4.5)*; here $\mathrm{tf}_\pm$ are the lifts of $\{0\} \times \partial\mathbb{B}^{n-1}$ and $\{1\} \times \partial\mathbb{B}^{n-1}$.*

It is worthwhile comparing the de Sitter space wave asymptotics of [53],

$$u = x^{n-1} v_+ + v_-, \ v_+ \in \mathcal{C}^\infty(\hat{M}), \ v_- \in \mathcal{C}^\infty(\hat{M}) + x^{n-1}(\log x)\mathcal{C}^\infty(\hat{M}), \tag{4.6}$$

with our main result, Theorem 1.4. The fact that the coefficients in the de Sitter expansion are $\mathcal{C}^\infty$ on $\hat{M}$ means that on $\bar{M}$, the leading terms are constant. Thus, (4.6) implies (and is much stronger than) the statement that u decays to a constant on $\bar{M}$ at an exponential rate.

4.4. **D'Alembertian and its Mellin transform.** Consider the d'Alembertian, $\Box_g$, whose principal symbol, including subprincipal terms, is given by the metric function. Thus, writing b-covectors on $[\hat{M}; q_+, q_-]$ near the interior of the front face (away from $r = 0$, to be precise), where μ, ω, τ are coordinates with $\tau = 0$ being the boundary, i.e. τ the defining function of the front face, as

$$\xi\, d\mu + \sigma\, \frac{d\tau}{\tau} + \eta\, d\omega,$$

we have, by (4.4),

$$G = \sigma_{\mathrm{b},2}(\Box) = -4r^2\mu\xi^2 + 4r^2\sigma\xi + \sigma^2 - r^{-2}|\eta|^2, \tag{4.7}$$

with $|\eta|^2_\omega$ denoting the dual metric function on the sphere. Note that there is a polar coordinate singularity at $r = 0$; this is resolved by using actually valid coordinates $Y = r\omega$ on $\mathbb{R}^{n-1}$ near the origin; writing b-covectors as

$$\sigma\, \frac{d\tau}{\tau} + \zeta\, dY,$$

we have

$$\begin{gathered} G = \sigma_{\mathrm{b},2}(\Box) = (Y\cdot\zeta)^2 - 2(Y\cdot\zeta)\sigma + \sigma^2 - |\zeta|^2 = (Y\cdot\zeta - \sigma)^2 - |\zeta|^2, \\ Y\cdot\zeta = \sum_j Y_j \cdot \zeta_j, \ |\zeta|^2 = \sum_j \zeta_j^2. \end{gathered} \tag{4.8}$$

Since there are no interesting phenomena at the origin, we may ignore this point below.

We now describe the normal operator of the d'Alembertian at $\tau = 0$. Via conjugation by the (inverse) Mellin transform, see Subsection 3.1, we obtain the normal operator family P_σ depending on σ, the b-dual variable of τ, on $\mathbb{R}^{n-1}_{r\omega}$ with both principal and high energy ($|\sigma| \to \infty$) symbol given by (4.7). Thus, the

[51] If we had worked with e^{-2t} instead of e^{-t} above, we would obtain x^2 as the defining function of the temporal face, rather than x.

principal symbol of $P_\sigma \in \mathrm{Diff}^2(\mathbb{R}^{n-1})$, including in the high energy sense ($\sigma \to \infty$), is

$$\begin{aligned} p_{\mathrm{full}} &= -4r^2\mu\xi^2 + 4r^2\sigma\xi + \sigma^2 - r^{-2}|\eta|^2_\omega \\ &= (Y\cdot\zeta)^2 - 2(Y\cdot\zeta)\sigma + \sigma^2 - |\zeta|^2 = (Y\cdot\zeta - \sigma)^2 - |\zeta|^2. \end{aligned} \tag{4.9}$$

The Hamilton vector field is

$$\begin{aligned} \mathsf{H}_{p_{\mathrm{full}}} &= 4r^2(-2\mu\xi + \sigma)\partial_\mu - r^{-2}\mathsf{H}_{|\eta|^2_\omega} - (4(1-2r^2)\xi^2 - 4\sigma\xi - r^{-4}|\eta|^2_\omega)\partial_\xi \\ &= 2(Y\cdot\zeta - \sigma)(Y\cdot\partial_Y - \zeta\cdot\partial_\zeta) - 2\zeta\cdot\partial_Y, \end{aligned} \tag{4.10}$$

with $\zeta\cdot\partial_Y = \sum \zeta_j\partial_{Y_j}$, etc. Thus, in the standard 'classical' sense, which effectively means letting $\sigma = 0$, the principal symbol is

$$\begin{aligned} p = \sigma_2(P_\sigma) &= -4r^2\mu\xi^2 - r^{-2}|\eta|^2_\omega \\ &= (Y\cdot\zeta)^2 - |\zeta|^2, \end{aligned} \tag{4.11}$$

while the Hamilton vector field is

$$\begin{aligned} \mathsf{H}_p &= -8r^2\mu\xi\partial_\mu - r^{-2}\mathsf{H}_{|\eta|^2_\omega} - (4(1-2r^2)\xi^2 - r^{-4}|\eta|^2_\omega)\partial_\xi \\ &= 2(Y\cdot\zeta)(Y\cdot\partial_Y - \zeta\cdot\partial_\zeta) - 2\zeta\cdot\partial_Y, \end{aligned} \tag{4.12}$$

Moreover, the imaginary part of the subprincipal symbol, given by the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$, is

$$\sigma_1(\frac{1}{2\imath}(P_\sigma - P_\sigma^*)) = 4r^2(\operatorname{Im}\sigma)\xi = -2(Y\cdot\zeta)\operatorname{Im}\sigma.$$

When comparing these with [53, Section 7], it is important to keep in mind that what is denoted by σ there (which we refer to as $\tilde\sigma$ here to avoid confusion) is $\imath\sigma$ here corresponding to the Mellin transform, which is a decomposition in terms of $\tau^{\imath\sigma} \sim x^{\imath\sigma}$, being replaced by weights $x^{\tilde\sigma}$ in [53, Equation (7.4)].

One important feature of this operator is that

$$N^*\{\mu = 0\} = \{(\mu,\omega,\xi,\eta):\ \mu = 0,\ \eta = 0\}$$

is invariant under the classical flow (i.e. effectively letting $\sigma = 0$); moreover, the Hamilton vector field is radial there. Let

$$N^*S\setminus o = \Lambda_+\cup\Lambda_-, \qquad \Lambda_\pm = N^*S\cap\{\pm\xi > 0\}, \qquad S = \{\mu = 0\}.$$

Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*\mathbb{R}^{n-1}$. Next we analyze the flow at $\Lambda_\pm$. First,

$$\mathsf{H}_p|\eta|^2_\omega = 0 \tag{4.13}$$

and

$$\mathsf{H}_p\mu = -8r^2\mu\xi = -8\xi\mu + a\mu^2\xi \tag{4.14}$$

with a being $\mathcal{C}^\infty$ in T^*X, and homogeneous of degree 0. While, in the spirit of linearizations, we used an expression in (4.14) that is linear in the coordinates whose vanishing defines N^*S, one should note that μ is an elliptic multiple of p in the sense of linearizations (i.e. the differentials at N^*S are elliptic multiples of each other), so one can simply use $\hat p = p/|\xi|^2$ (which is homogeneous of degree 0, like μ) in its place.

It is convenient to rehomogenize (4.13) in terms of $\hat\eta = \eta/|\xi|$. To phrase this more invariantly, consider the fiber-compactification $\overline{T}^*\mathbb{R}^{n-1}$ of $T^*\mathbb{R}^{n-1}$, see Subsection 2.2. On this space, the classical principal symbol, p, is (essentially) a function on $\partial\overline{T}^*\mathbb{R}^{n-1} = S^*\mathbb{R}^{n-1}$. Then at fiber infinity near N^*S, we can take $(|\xi|^{-1},\hat\eta)$

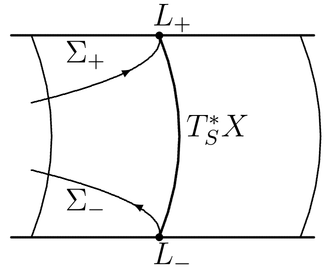

FIGURE 5. The cotangent bundle near the event horizon $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view. The boundary of the fiber compactificaton is the cosphere bundle $S^*\mathbb{R}^{n-1}$; it is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu \leq 0$, only meeting $S^*_S\mathbb{R}^{n-1}$ at $L_\pm$. Semiclassically, i.e. in the interior of $\overline{T}^*\mathbb{R}^{n-1}$, for $z = h^{-1}\sigma > 0$, only the component of the semiclassical characteristic set containing L_+ can enter $\mu > 0$. This is reversed for $z < 0$.

as coordinates on the fibers of the cotangent bundle, with $\tilde{\rho} = |\xi|^{-1}$ defining S^*X in $\overline{T}^*X$. Then $|\xi|^{-1}\mathsf{H}_p$ is a $\mathcal{C}^\infty$ vector field in this region and

$$|\xi|^{-1}\mathsf{H}_p|\hat{\eta}|^2 = |\hat{\eta}|^2\mathsf{H}_p|\xi|^{-1} = -4(\operatorname{sgn}\xi)|\hat{\eta}|^2 + \tilde{a}, \tag{4.15}$$

where $\tilde{a}$ vanishes cubically at N^*S, i.e. (2.3) holds. In similar notation we have

$$\begin{aligned} \mathsf{H}_p|\xi|^{-1} &= -4\operatorname{sgn}(\xi) + \tilde{a}', \\ |\xi|^{-1}\mathsf{H}_p\mu &= -8(\operatorname{sgn}\xi)\mu. \end{aligned} \tag{4.16}$$

with $\tilde{a}'$ smooth (indeed, homogeneous degree zero without the compactification) vanishing at N^*S. As the vanishing of $\hat{\eta}, |\xi|^{-1}$ and μ defines ∂N^*S, we conclude that $L_- = \partial\Lambda_-$ is a source, while $L_+ = \partial\Lambda_+$ is a sink, in the sense that all nearby bicharacteristics (in fact, including semiclassical (null)bicharacteristics, since $\mathsf{H}_p|\xi|^{-1}$ contains the additional information needed) converge to $L_\pm$ as the parameter along the bicharacteristic goes to $\pm\infty$. In particular, the quadratic defining function of $L_\pm$ given by

$$\rho_0 = \widehat{\widehat{p}} + \hat{p}^2, \text{ where } \hat{p} = |\xi|^{-2}p,\ \widehat{\widehat{p}} = |\hat{\eta}|^2,$$

satisfies (2.4).

The imaginary part of the subprincipal symbol at $L_\pm$ is given by

$$(4\operatorname{sgn}(\xi))\operatorname{Im}\sigma|\xi|;$$

here $(4\operatorname{sgn}(\xi))$ is pulled out due to (4.16), namely its size relative to $\mathsf{H}_p|\xi|^{-1}$ matters, with a change of sign, see Subsection 2.2, thus (2.5)-(2.6) hold. This corresponds to the fact[52] that $(\mu \pm \imath 0)^{\imath\sigma}$, which are Lagrangian distributions associated to $\Lambda_\pm$, solve the PDE modulo an error that is two orders lower than what one might a priori expect, i.e. $P_\sigma(\mu \pm \imath 0)^{\imath\sigma} \in (\mu \pm i0)^{\imath\sigma}\mathcal{C}^\infty(\mathbb{R}^{n-1})$. Note that P_σ is second order, so one should lose two orders a priori; the characteristic nature of $\Lambda_\pm$ reduces the

[52]This needs the analogous statement for full subprincipal symbol, not only its imaginary part.

loss to 1, and the particular choice of exponent eliminates the loss. This has much in common with $e^{\imath\lambda/x}x^{(n-1)/2}$ being an approximate solution in asymptotically Euclidean scattering. The precise situation for Kerr-de Sitter space is more delicate as the Hamilton vector field does not vanish at $L_\pm$, but this[53] is irrelevant for our estimates: only a quantitative version of the source/sink statements and the imaginary part of the subprincipal symbol are relevant.

While $(\mu\pm\imath 0)^{\imath\sigma}$ is singular regardless of σ apart from integer coincidences (when this should be corrected anyway), it is interesting to note that for $\operatorname{Im}\sigma > 0$ this is not bounded at $\mu = 0$, while for $\operatorname{Im}\sigma < 0$ it vanishes there. This is interesting because if one reformulates the problem as one in $\mu \geq 0$, as was done for instance by Sá Barreto and Zworski [47], and later by Melrose, Sá Barreto and Vasy [40] for de Sitter-Schwarzschild space then one obtains an operator that is essentially (up to a conjugation and a weight, see below) the Laplacian on an asymptotically hyperbolic space at energy $\sigma^2 + \frac{(n-2)^2}{4}$ — more precisely its normal operator (which encodes its behavior near $\mu = 0$) is a multiple of that of the hyperbolic Laplacian. Then the growth/decay behavior corresponds to the usual scattering theory phenomena, but in our approach smooth extendability across $\mu = 0$ is the distinguishing feature of the solutions we want, not growth/decay. See Remark 4.5 for more details.

4.5. **Global behavior of the characteristic set.** First remark that $\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = 1 > 0$, so $\frac{d\tau}{\tau}$ is time-like. Correspondingly all the results of Subsection 3.2 apply. In particular, (3.16) gives that the characteristic set is divided into two components with $\Lambda_\pm$ in different components. It is easy to make this explicit: points with $\xi = 0$, or equivalently $Y \cdot \zeta = 0$, cannot lie in the characteristic set. Thus,

$$\Sigma_\pm = \Sigma \cap \{\pm\xi > 0\} = \Sigma \cap \{\mp(Y \cdot \zeta) > 0\}.$$

While it is not important here since the characteristic set in $\mu \geq 0$ is localized at N^*S, hence one has a similar localization for nearby μ, for global purposes (which we do not need here), we point out that $\mathsf{H}_p\mu = -8r^2\mu\xi$. Since $\xi \neq 0$ on Σ, and in Σ, $r = 1$ can only happen at N^*S, i.e. only at the radial set, the $\mathcal{C}^\infty$ function μ provides a negative global escape function which is increasing on Σ_+, decreasing on Σ_-. Correspondingly, bicharacteristics in Σ_+ travel from infinity to L_+, while in Σ_- they travel from L_- to infinity.

4.6. **High energy, or semiclassical, asymptotics.** We are also interested in the high energy behavior, as $|\sigma| \to \infty$. For the associated semiclassical problem one obtains a family of operators

$$P_{h,z} = h^2 P_{h^{-1}z},$$

with $h = |\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. Then the semiclassical principal symbol $p_{\hbar,z}$ of $P_{h,z}$ is a function on $T^*\mathbb{R}^{n-1}$. As in Section 2, we are mostly interested in $|\operatorname{Im} z| \leq Ch$, which corresponds to $|\operatorname{Im}\sigma| \leq C'$; in the limit $h \to 0$ this means z is real. It is sometimes convenient to think of $p_{\hbar,z}$, and its rescaled Hamilton vector field, as objects on $\overline{T}^*\mathbb{R}^{n-1}$. Thus,

$$\begin{aligned} p_{\hbar,z} = \sigma_{2,h}(P_{h,z}) &= -4r^2\mu\xi^2 + 4r^2 z\xi + z^2 - r^{-2}|\eta|^2_\omega \\ &= (Y\cdot\zeta)^2 - 2(Y\cdot\zeta)z + z^2 - |\zeta|^2 = (Y\cdot\zeta - z)^2 - |\zeta|^2. \end{aligned} \tag{4.17}$$

[53]This would be relevant for a full Lagrangian analysis, as done e.g. in [39], or in a somewhat different, and more complicated, context by Hassell, Melrose and Vasy in [30, 31].

We make the general discussion of Subsection 3.2 and Section 7 explicit; *for z non-real one can jump to the next paragraph.* First,

$$\operatorname{Im} p_{\hbar,z} = 2\operatorname{Im} z(2r^2\xi + \operatorname{Re} z) = -2\operatorname{Im} z(Y\cdot\zeta - \operatorname{Re} z). \tag{4.18}$$

In particular, for z non-real, $\operatorname{Im} p_{\hbar,z} = 0$ implies $2r^2\xi + \operatorname{Re} z = 0$, i.e. $Y\cdot\zeta - \operatorname{Re} z = 0$, which means that $\operatorname{Re} p_{\hbar,z}$ is

$$-r^{-2}(\operatorname{Re} z)^2 - (\operatorname{Im} z)^2 - r^{-2}|\eta|^2_\omega = -(\operatorname{Im} z)^2 - |\zeta|^2 < 0, \tag{4.19}$$

i.e. $p_{\hbar,z}$ is semiclassically elliptic on $T^*\mathbb{R}^{n-1}$, but *not* at fiber infinity, i.e. at $S^*\mathbb{R}^{n-1}$ (standard ellipticity is lost only in $r \geq 1$, of course). Explicitly, if we introduce for instance

$$(\mu, \omega, \nu, \hat\eta), \qquad \nu = |\xi|^{-1},\ \hat\eta = \eta/|\xi|,$$

as valid projective coordinates in a (large!) neighborhood of $L_\pm$ in $\overline{T}^*\mathbb{R}^{n-1}$, then

$$\nu^2 p_{\hbar,z} = -4r^2\mu + 4r^2(\operatorname{sgn}\xi)z\nu + z^2\nu^2 - r^{-2}|\hat\eta|^2_\omega$$

so

$$\nu^2 \operatorname{Im} p_{\hbar,z} = 4r^2(\operatorname{sgn}\xi)\nu\operatorname{Im} z + 2\nu^2\operatorname{Re} z\operatorname{Im} z$$

which automatically vanishes at $\nu = 0$, i.e. at $S^*\mathbb{R}^{n-1}$. Thus, for σ large and pure imaginary, the semiclassical problem adds no complexity to the 'classical' quantum problem, but of course it does not simplify it. In fact, we need somewhat more information at the characteristic set, which is thus at $\nu = 0$ when $\operatorname{Im} z$ is bounded away from 0:

$$\begin{aligned}&\nu \text{ small},\ \operatorname{Im} z \geq 0 \Rightarrow (\operatorname{sgn}\xi)\operatorname{Im} p_{\hbar,z} \geq 0 \Rightarrow \pm\operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm},\\ &\nu \text{ small},\ \operatorname{Im} z \leq 0 \Rightarrow (\operatorname{sgn}\xi)\operatorname{Im} p_{\hbar,z} \leq 0 \Rightarrow \pm\operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm},\end{aligned}$$

which, as we have seen, means that for $P_{h,z}$ with $\operatorname{Im} z > 0$ one can propagate estimates forwards along the bicharacteristics where $\xi < 0$ (in particular, away from L_-, as the latter is a source) and backwards where $\xi > 0$ (in particular, away from L_+, as the latter is a sink), while for $P^*_{\hbar,z}$ the directions are reversed. The directions are also reversed if $\operatorname{Im} z$ switches sign. This is important because it gives invertibility for $z = \imath$ (corresponding to $\operatorname{Im}\sigma$ large positive, i.e. the physical halfplane), but does not give invertibility for $z = -\imath$ negative.

We now return to the claim that even semiclassically, for z real real the characteristic set can be divided into two components $\Sigma_{\hbar,\pm}$, with $L_\pm$ in different components. As explained in Subsection 3.2 the vanishing of the factor following $\operatorname{Im} z$ in (4.18) gives a hypersurface that separates $\Sigma_\hbar$ into two parts; this can be easily checked also by a direct computation. Concretely, this is the hypersurface given by

$$0 = 2r^2\xi + z = -(Y\cdot\zeta - z), \tag{4.20}$$

and so

$$\Sigma_{\hbar,\pm} = \Sigma_\hbar \cap \{\mp(Y\cdot\zeta - z) > 0\}.$$

We finally need more information about the global semiclassical dynamics. Here all null-bicharacteristics go to either L_+ in the forward direction or to L_- in the backward direction, and escape to infinity in the other direction. Rather than proving this at once, which depends on the global non-trapping structure on $\mathbb{R}^{n-1}$, we first give an argument that is local near the event horizon, and suffices for the extension discussed below for asymptotically hyperbolic spaces.

As stated above, first, we are only concerned about semiclassical dynamics in $\mu > \mu_0$, where $\mu_0 < 0$ might be close to 0. To analyze this (with z real as usual), we observe that the semiclassical Hamilton vector field is

$$\begin{aligned}\mathsf{H}_{p_{\hbar,z}} &= 4r^2(-2\mu\xi + z)\partial_\mu - r^{-2}\mathsf{H}_{|\eta|^2_\omega} - (4(1-2r^2)\xi^2 - 4z\xi - r^{-4}|\eta|^2_\omega)\partial_\xi \\ &= 2(Y\cdot\zeta - z)(Y\cdot\partial_Y - \zeta\cdot\partial_\zeta) - 2\zeta\cdot\partial_Y.\end{aligned} \tag{4.21}$$

Thus,

$$\mathsf{H}_{p_{\hbar,z}}(Y\cdot\zeta) = -2|\zeta|^2,$$

and $\zeta = 0$ implies $p_{\hbar,z} = z^2$, so $\mathsf{H}_{p_{\hbar,z}}(Y\cdot\zeta)$ has a negative upper bound on the characteristic set in compact subsets of $T^*\{r < 1\}$; note that the characteristic set is compact in $T^*\{r \le r_0\}$ if $r_0 < 1$ by standard ellipticity. Thus, bicharacteristics have to leave $\{r \le r_0\}$ for $r_0 < 1$ in both the forward and backward direction (as $Y\cdot\zeta$ is bounded over this region on the characteristic set). We already know the dynamics near $L_\pm$, which is the only place where the characteristic set intersects $S^*_S\mathbb{R}^{n-1}$, namely L_+ is a sink and L_- is a source. Now, at $\mu = 0$, $\mathsf{H}_{p_{\hbar,z}}\mu = z$, which is positive when $z > 0$, so bicharacteristics can only cross $\mu = 0$ in the inward direction. In view of our preceding observations, thus, once a bicharacteristic crossed $\mu = 0$, it has to tend to L_+. As bicharacteristics in a neighborhood of L_+ (even in $\mu < 0$) tend to L_+ since L_+ is a sink, it follows that in $\Sigma_{\hbar,+}$ the same is true in $\mu > \mu_0$ for some $\mu_0 < 0$. On the other hand, in a neighborhood of L_- all bicharacteristics emanate from L_- (but cannot cross into $\mu > 0$ by our observations), so leave $\mu > \mu_0$ in the forward direction. These are all the relevant features of the bicharacteristic flow for our purposes as we shall place a complex absorbing potential near $\mu = \mu_0$ in the next subsection.

However, it is easy to see the global claim by noting that $\mathsf{H}_{p_{\hbar,z}}\mu = 4r^2(-2\mu\xi + z)$, and this cannot vanish on $\Sigma_\hbar$ in $\mu < 0$, since where it vanishes, a simple calculation gives $p_{\hbar,z} = 4\mu\xi^2 - r^{-2}|\eta|^2$. Thus, $\mathsf{H}_{p_{\hbar,z}}\mu$ has a constant sign on $\Sigma_{\hbar,\pm}$ in $\mu < 0$, so combined with the observation above that all bicharacteristics escape to $\mu = \mu_0$ in the appropriate direction, it shows that all bicharacteristics in fact escape to infinity in that direction[54].

In fact, for applications, it is also useful to remark that for $\alpha \in T^*X$,

$$0 < \mu(\alpha) < 1,\ p_{\hbar,z}(\alpha) = 0 \text{ and } (\mathsf{H}_{p_{\hbar,z}}\mu)(\alpha) = 0 \Rightarrow (\mathsf{H}^2_{p_{\hbar,z}}\mu)(\alpha) < 0. \tag{4.23}$$

Indeed, as $\mathsf{H}_{p_{\hbar,z}}\mu = 4r^2(-2\mu\xi + z)$, the hypotheses imply $z = 2\mu\xi$ and $\mathsf{H}^2_{p_{\hbar,z}}\mu = -8r^2\mu\mathsf{H}_{p_{\hbar,z}}\xi$, so we only need to show that $\mathsf{H}_{p_{\hbar,z}}\xi > 0$ at these points. Since

$$\mathsf{H}_{p_{\hbar,z}}\xi = -4(1-2r^2)\xi^2 + 4z\xi + r^{-4}|\eta|^2_\omega = 4\xi^2 + r^{-4}|\eta|^2_\omega = 4r^{-2}\xi^2,$$

[54]There is in fact a not too complicated global escape function, e.g.

$$f = \frac{2Y\cdot\zeta - z}{2\sqrt{1+|Y|^2}(Y\cdot\zeta - z)} = \frac{2Y\cdot\hat\zeta - z|\zeta|^{-1}}{2\sqrt{1+|Y|^2}(Y\cdot\hat\zeta - z|\zeta|^{-1})},$$

which is a smooth function on the characteristic set in $T^*\mathbb{R}^{n-1}$ as $Y\cdot\zeta \ne z$ there; further, it extends smoothly to the characteristic set in $\overline{T}^*\mathbb{R}^{n-1}$ away from $L_\pm$ since $\sqrt{1+|Y|^2}(Y\cdot\hat\zeta - z|\zeta|^{-1})$ vanishes only there near $S^*\mathbb{R}^{n-1}$ (where these are valid coordinates), at which it has conic points. This function arises in a straightforward manner when one reduces Minkowski space, $\mathbb{R}^n = \mathbb{R}^{n-1}_{z'}\times\mathbb{R}_t$ with metric g_0, to the boundary of its radial compactification, as described in Section 5, and uses the natural escape function

$$\tilde f = \frac{tt^* - z'(z')^*}{t^*\sqrt{t^2 + |z'|^2}} \tag{4.22}$$

there; here t^* is the dual variable of t and $(z')^*$ of z', outside the origin.

where the second equality uses $\mathsf{H}_{p_{\hbar,z}}\mu = 0$ and the third uses that in addition $p_{\hbar,z} = 0$, this follows from $2\mu\xi = z \neq 0$, so $\xi \neq 0$. Thus, μ can be used for gluing constructions as in [15].

4.7. **Complex absorption.** The final step of fitting P_σ into our general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. We thus consider a compact manifold without boundary X for which $X_{\mu_0} = \{\mu > \mu_0\}$, $\mu_0 < 0$, say, is identified as an open subset with smooth boundary; it is convenient to take X to be the double[55] of X_{μ_0}.

It is convenient to separate the 'classical' (i.e. quantum!) and 'semiclassical' problems, for in the former setting trapping does not matter, while in the latter it does.

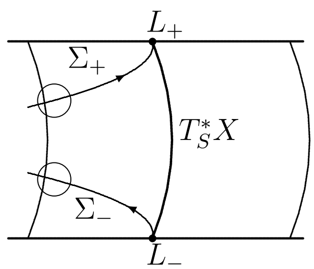

FIGURE 6. The cotangent bundle near the event horizon $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view, as in Figure 5. The circles on the left show the support of q; it has opposite signs on the two disks corresponding to the opposite directions of propagation relative to the Hamilton vector field.

Ultimately, we want to extend P_σ to X (as currently it is only defined near X_{μ_0}), and introduce a complex absorbing operator $Q_\sigma \in \Psi^2_{\mathrm{cl}}(X)$ with principal symbol q, such that $h^2 Q_{h^{-1}z} \in \Psi^2_{\hbar,\mathrm{cl}}(X)$ with semiclassical principal symbol $q_{\hbar,z}$, and such that $p \pm \imath q$ is elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, Q_σ is supported there (say, in $\mu < \mu_0/2$) and which satisfies that the $\mp q \geq 0$ on $\Sigma_\pm$. In fact, it is convenient to also arrange that $p \pm \imath q$ are elliptic near $X \setminus X_{\mu_0}$; the region we added is thus irrelevant in the sense that it does not complicate the analysis. In view of Subsection 3.3, the solution in, say, $\mu > \mu_0/2$ is *unaffected* by thus modifying P_σ, i.e. working with P_σ and $P_\sigma - \imath Q_\sigma$ is equivalent for this purpose, so the region we added is irrelevant in this sense as well.

First, for the 'classical' problem (i.e. completely ignoring the issue of uniform bounds as $\sigma \to \infty$), we can make Q_σ actually independent of σ. Indeed, it is straightforward to write down q with the required properties (so q is independent of σ), as we now do; quantizing it in a standard σ-independent manner gives a desired Q_σ; now Q_σ depends holomorphically on σ (since there is no σ-dependence

[55]In fact, in the de Sitter context, this essentially means moving to the boundary of n-dimensional Minkowski space, where our $(n-1)$-dimensional model is the 'upper hemisphere', see Section 5. Thus, doubling over means working with the whole boundary, but putting an absorbing operator near the equator, corresponding to the usual Cauchy hypersurface in Minkowski space, and solving from the radial points at both the future and past light cones towards the equator — this would be impossible without the complex absorption.

at all). Concretely, as discussed in Subsection 3.2, this can be achieved by defining $\tilde{p} = \chi_1 p - \chi_2 \hat{p}$, where $\hat{p}$ is a Riemannian metric function (or is simply a homogeneous degree 2, positive function), $\chi_1 + \chi_2 = 1$, $\chi_1 = 1$ on X_{μ_0}, is supported nearby, $\chi_j \geq 0$, letting $q = -2r^2\xi\tilde{p}^{1/2}\chi(\mu) = (Y \cdot \zeta)\tilde{p}^{1/2}\chi(\mu)$, $\chi \geq 0$ supported near μ_0, identically equal to a positive constant where neither χ_1 nor χ_2 vanishes. As $p < 0$ when $\xi = 0$, on the set where $\chi = 1$, $\tilde{p} \pm \imath q$ does not vanish (for the vanishing of q there implies that $p < 0$, and $-\hat{p}$ is always negative), while if $\chi_2 = 1$, $\tilde{p} < 0$ so $\tilde{p} \pm \imath q$ does not vanish, and if $\chi_1 = 1$, $\tilde{p} \pm \imath q = p \pm \imath q$ behave the required way as $\mp q \geq 0$ on $\Sigma_\pm$. Thus, renaming $\tilde{p}$ as p (since we consider it an extension of p) $p \pm \imath q$ satisfy our requirements.

An alternative to this extension would be simply adding a boundary at $\mu = \mu_0$; this is easy to do since this is a space-like hypersurface, but this is slightly unpleasant from the point of view of microlocal analysis as one has to work on a manifold with boundary (though as mentioned this is easily done, see Remark 2.6).

For the semiclassical problem we need to increase the requirements on Q_σ. For this we need in addition, in the semiclassical notation, semiclassical ellipticity near $\mu = \mu_0$, i.e. that $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that the $\mp q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. While this is extremely easy to arrange if we ignore holomorphy in z, a bit of care is required to ensure the latter. Following (3.17), and taking into account (4.18), we take

$$q_{\hbar,z} = -(2r^2\xi + z)f_z\chi(\mu) = (Y \cdot \zeta - z)f_z\chi(\mu),$$

with

$$f_z = (|\zeta|^{2j} + z^{2j})^{1/2j},$$

$\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi \geq 0$ is supported near μ_0, and is identically 1 in a smaller neighborhood of μ_0, $j \geq 1$ integer and the branch of the $2j$th root function is chosen so that it is defined on $\mathbb{C} \setminus (-\infty, 0)$ and is non-negative when the argument is non-negative, thus the real part of this root is ≥ 0 on $\mathbb{C} \setminus (-\infty, 0)$ with vanishing only at the origin. In fact, we can make Q_σ a (standard) quantization of

$$(Y \cdot \zeta - \sigma)(|\zeta|^{2j} + \sigma^{2j} + C^{2j})^{1/2j}\chi(\mu),$$

where $C > 0$ is chosen suitably large; then Q_σ is holomorphic away from the inverse images of the branch cuts, and in particular when $|\operatorname{Im}\sigma| < C \operatorname{Im} e^{\imath\pi/j}$. Here we can take even $j = 1$ and then choose C greater than the width of the strip we want to study. However, by allowing j to vary, we obtain an open cover of $\mathbb{C}$ by domains Ω_j of holomorphy for $Q_\sigma^{(j)}$ as discussed in Subsection 3.3, and as $\operatorname{Re} f \geq 0$, Subsection 3.2 shows that $Q_\sigma^{(j)}$ in fact satisfies the required semiclassical properties in Ω_j. Again, we extend P_σ to X (Q_σ can already be considered as being defined on X in view of $\operatorname{supp}\chi$), for instance in a manner analogous to the 'classical' one discussed above, i.e. replacing $p_{\hbar,z}$ by $\chi_1 p_{\hbar,z} - \chi_2 \hat{p}_{\hbar,z}$, with $\hat{p} = (\|\zeta\|^{2j} + z^{2j})^{1/j}$, with $\|.\|^2$ denoting a Riemannian metric function on X. Then $p \pm \imath q$ and $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near $X \setminus X_{\mu_0}$, as desired, as discussed in Subsection 3.2.

4.8. **More general metrics.** If the operator is replaced by one on a neighborhood of $Y_y \times (-\delta, \delta)_\mu$ with full principal symbol (including high energy terms)

$$-4(1 + a_1)\mu\xi^2 + 4(1 + a_2)\sigma\xi + (1 + a_3)\sigma^2 - |\eta|_h^2, \tag{4.24}$$

and h a family of Riemannian metrics on Y depending smoothly on μ, a_j vanishing at $\mu = 0$, then the local behavior of this operator P_σ near the 'event horizon' $Y \times \{0\}$

is exactly as in the de Sitter setting. If we start with a compact manifold X_0 with boundary Y and a neighborhood of the boundary identified with $Y \times [0,\delta)_\mu$ with the operator of the form above, and which is elliptic in X_0 (we only need to assume this away from $Y \times [0,\delta/2)$, say), including in the non-real high energy sense (i.e. for z away from $\mathbb{R}$ when $\sigma = h^{-1}z$) then we can extend the operator smoothly to one on X_{μ_0}, $\mu_0 = -\delta$, which enjoys all the properties above, except semiclassical non-trapping. If we assume that X_0° is non-trapping in the usual sense, the semiclassical non-trapping property also follows. In addition, for $\mu > 0$ sufficiently small, (4.23) also holds since η is small when $\mathsf{H}_{p_{\hbar,z}}\mu = 0$ and $p_{\hbar,z} = 0$, for the former gives $z = 2(1+a_2)^{-1}(1+a_1)\mu\xi$, and then the latter gives

$$4(1+a_1)\left(1+\frac{(1+a_1)(1+a_3)}{(1+a_2)^2}\mu\right)\mu\xi^2 = |\eta|_h^2,$$

so the contribution of $|\eta|_h^2$ to $\mathsf{H}_{p_{\hbar,z}}\xi$, which can be large elsewhere even at $\mu = 0$, is actually small.

We show below in the proof of Theorem 4.3 that (4.24) holds on general even asymptotically hyperbolic spaces; as mentioned above, the non-trapping property of the asymptotically hyperbolic space then also implies that of the extended operator.

4.9. **Results for asymptotically hyperbolic and de Sitter metrics.** The preceding subsections show that for the Mellin transform of $\Box_g$ on n-dimensional de Sitter space, all the hypotheses needed in Section 2 are satisfied, thus analogues of the results stated for Kerr-de Sitter space in the introduction, Theorems 1.1-1.4, hold. It is important to keep in mind, however, that there is no trapping to remove, so Theorem 1.1 applies with Q_σ supported outside the event horizon, and one does not need gluing or the result of Wunsch and Zworski [61]. In particular, Theorem 1.3 holds with arbitrary C', without the logarithmic or polynomial loss. As already mentioned when discussing [53, Theorem 1.1] at the beginning of this section, this is weaker than the result of [53, Theorem 1.1], since there one has smooth asymptotics without a blow-up of a boundary point[56].

We now reinterpret our results on the Mellin transform side in terms of $(n-1)$-dimensional hyperbolic space. Let $\mathbb{B}^{n-1}_{1/2}$ be $\mathbb{B}^{n-1} = \{r \le 1\}$ with $\nu = \sqrt{\mu}$ added to the smooth structure. For the purposes of the discussion below, we identify the interior $\{r < 1\}$ of $\mathbb{B}^{n-1}_{1/2}$ with a Poincaré ball model[57] of hyperbolic $(n-1)$-space $(\mathbb{H}^{n-1}, g_{\mathbb{H}^{n-1}})$. Using polar coordinates around the origin, let $\cosh\rho = \nu^{-1}$, ρ is the distance from the origin. The Laplacian on $\mathbb{H}^{n-1}$ in these coordinates is

$$\Delta_{\mathbb{H}^{n-1}} = D_\rho^2 - \imath(n-2)\coth\rho\, D_\rho + (\sinh\rho)^{-2}\Delta_\omega.$$

[56] Note that our methods work equally well for asymptotically de Sitter spaces in the sense of [53]; after the blow up, the boundary metric is 'frozen' at the point that is blown up, hence the induced problem at the front face is the same as for the de Sitter metric with asymptotics given by this 'frozen' metric.

[57] The standard Poincaré ball model is the metric $4\frac{dw^2}{(1-|w|^2)^2}$ on $\mathbb{B}^{n-1}$. Introducing polar coordinates, $w = \tilde r\omega$, the present form arises by letting $\nu = \frac{1-\tilde r^2}{1+\tilde r^2}$, i.e. $\tilde r^2 = \frac{1-\nu}{1+\nu}$ with $\nu = \sqrt{1-r^2}$; recall that $\mu = 1-r^2$. Thus, ν and $1-\tilde r^2$ are equivalent boundary defining functions.

It is shown in [53, Lemma 7.10] that in $r < 1$, and with s be such that $2s = \imath\sigma - \frac{n}{2}$,

$$(4.25)\quad \begin{aligned} &(1-r^2)^{-s} P_\sigma (1-r^2)^s = \nu^{\frac{n}{2}-\imath\sigma} P_\sigma \nu^{\imath\sigma - \frac{n}{2}} \\ &= -\nu^{-1}\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - \nu^2 \frac{n(n-2)}{4}\right)\nu^{-1} \\ &= -\cosh\rho\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - (\cosh\rho)^{-2}\frac{n(n-2)}{4}\right)\cosh\rho. \end{aligned}$$

We thus deduce:

Proposition 4.2. *The inverse* $\mathcal{R}(\sigma)$ *of*

$$\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - (\cosh\rho)^{-2}\frac{n(n-2)}{4}$$

has a meromorphic continuation from $\operatorname{Im}\sigma > 0$ *to* $\mathbb{C}$ *with poles with finite rank residues as a map* $\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1}) \to \mathcal{C}^{-\infty}(\mathbb{B}^{n-1})$*, and with non-trapping estimates in every strip* $-C < \operatorname{Im}\sigma < C_+$*,* $|\operatorname{Re}\sigma| \gg 0$*:* $s > \frac{1}{2} + C$*,*

$$(4.26)\quad \|(\cosh\rho)^{(n-2)/2-\imath\sigma}\mathcal{R}(\sigma)f\|_{H^s_{|\sigma|^{-1}}(\mathbb{B}^{n-1})} \leq C|\sigma|^{-1}\|(\cosh\rho)^{(n+2)/2-\imath\sigma} f\|_{H^{s-1}_{|\sigma|^{-1}}(\mathbb{B}^{n-1})},$$

where the Sobolev spaces are those on $\mathbb{B}^{n-1}$ *(rather than* $\mathbb{B}^{n-1}_{1/2}$*). If* $\operatorname{supp} f \subset (\mathbb{B}^{n-1})^\circ$*, the* $s-1$ *norm on* f *can be replaced by the* $s-2$ *norm.*

The same conclusion holds for small even $\mathcal{C}^\infty$ *perturbations, vanishing at* $\partial\mathbb{B}^{n-1}_{1/2}$*, of* $g_{\mathbb{H}^{n-1}}$ *in the class of conformally compact metrics, or the addition of (not necessarily small)* $V \in \mu\mathcal{C}^\infty(\mathbb{B}^{n-1})$*.*

Proof. By self-adjointness and positivity of $\Delta_{\mathbb{H}^{n-1}}$,

$$\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - \nu^2\frac{n(n-2)}{4}\right) u = f \in \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$$

has a unique solution $u = \mathcal{R}(\sigma)f \in L^2(\mathbb{B}^{n-1}_{1/2}, |dg_{\mathbb{H}^{n-1}}|)$ when when $\operatorname{Im}\sigma \gg 0$. On the other hand, let $\tilde{f}_0 = \nu^{\imath\sigma - n/2}\nu^{-1} f$ in $r \leq 1$, and $\tilde{f}_0$ still vanishes to infinite order at $r = 1$. Let $\tilde{f}$ be an arbitrary smooth extension of $\tilde{f}_0$ to the compact manifold X on which $P_\sigma - \imath Q_\sigma$ is defined. Let $\tilde{u} = (P_\sigma - \imath Q_\sigma)^{-1}\tilde{f}$, with $(P_\sigma - \imath Q_\sigma)^{-1}$ given by our results in Section 2; this satisfies $(P_\sigma - \imath Q_\sigma)\tilde{u} = \tilde{f}$ and $\tilde{u} \in \mathcal{C}^\infty(X)$. Thus, $u' = -\nu^{-\imath\sigma + n/2}\nu^{-1}\tilde{u}|_{r<1}$ satisfies $u' \in \nu^{(n-2)/2 - \imath\sigma}\mathcal{C}^\infty(\mathbb{B}^{n-1})$, and

$$\left(\Delta_{\mathbb{H}^{n-1}} - \sigma^2 - \left(\frac{n-2}{2}\right)^2 - \nu^2\frac{n(n-2)}{4}\right) u' = f$$

by (4.25) (as Q_σ is supported in $r > 1$). Since $u' \in L^2(\mathbb{B}^{n-1}, |dg_{\mathbb{H}^{n-1}}|)$ for $\operatorname{Im}\sigma > 0$, by the aforementioned uniqueness, $u = u'$.

To make the extension from $\mathbb{B}^{n-1}$ to X more systematic, let $E_s : H^s(\mathbb{B}^{n-1}) \to H^s(X)$ be a continuous extension operator, $R_s : H^s(X) \to H^s(\mathbb{B}^{n-1})$ the restriction map. Then, as we have just seen, for $f \in \dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$,

$$(4.27)\quad \mathcal{R}(\sigma)f = -\nu^{-\imath\sigma+n/2}\nu^{-1}R_s(P_\sigma - \imath Q_\sigma)^{-1}E_{s-1}\nu^{\imath\sigma - n/2}\nu^{-1}f.$$

Thus, the first half of the proposition (including the non-trapping estimate) follows immediately from the results of Section 2. Note also that this proves that every pole of $\mathcal{R}(\sigma)$ is a pole of $(P_\sigma - \imath Q_\sigma)^{-1}$ (for otherwise (4.27) would show $\mathcal{R}(\sigma)$ does not have a pole either), but $(P_\sigma - \imath Q_\sigma)^{-1}$ may have poles which are not poles of $\mathcal{R}(\sigma)$. However, in the latter case, the Laurent coefficients of $(P_\sigma - \imath Q_\sigma)^{-1}$ would be annihilated by multiplication by R_s from the left when applied to elements of $\dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$, regarded as a subset of $\mathcal{C}^\infty(X)$. If $(\sigma-\sigma_0)^{-j}F_j$ is the most singular Laurent term at σ_0, and $F_j = \sum_i \phi_i\langle\psi_i, .\rangle$, then $(P_{\sigma_0} - \imath Q_{\sigma_0})\phi_i = 0$, $(P^*_{\sigma_0} + \imath Q^*_{\sigma_0})\psi_i = 0$; we refer to ϕ_i as resonant states and ψ_i as dual states. This also holds in $\mu > 0$ if $(\sigma-\sigma_0)^{-j}F_j$ is just the most singular term with support intersecting $(0,\infty)\times(0,\infty)$ in μ (i.e. $\mu^{-1}((0,\infty))\times\mu^{-1}((0,\infty))$) non-trivially. Thus, if σ is not a pole of $\mathcal{R}(\sigma)$, the resonant states of $P_\sigma - \imath Q_\sigma$ (which are $\mathcal{C}^\infty$) are supported in $\mu \leq 0$, in particular vanish to infinite order at $\mu = 0$, unless the corresponding dual state vanishes when paired with $\dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$, i.e. is supported in $\mu \leq 0$.[58]

We now turn to the perturbation. After the conjugation, division by $\mu^{1/2}$ from both sides, elements of $V \in \mu\mathcal{C}^\infty(\mathbb{B}^{n-1})$ can be extended to become elements of $\mathcal{C}^\infty(\mathbb{R}^{n-1})$, and they do not affect any of the structures discussed in Section 2, so the results automatically go through. Operators of the form x^2L, $L \in \mathrm{Diff}_{\mathrm{b,even}}(\mathbb{B}^{n-1}_{1/2})$, i.e. with even coefficients with respect to the local product structure, become elements of $\mathrm{Diff}_{\mathrm{b}}(\mathbb{B}^{n-1})$ after conjugation and division by $\mu^{1/2}$ from both sides. Hence, they can be smoothly extended across $\partial\mathbb{B}^{n-1}$, and they do not affect either the principal or the subprincipal symbol at $L_\pm$ in the classical sense. They do, however, affect the classical symbol elsewhere and the semiclassical symbol everywhere, thus the semiclassical Hamilton flow, but under the smallness assumption the required properties are preserved, since the dynamics is non-degenerate (the rescaled Hamilton vector field on $\overline{T}^*\mathbb{R}^{n-1}$ does not vanish) away from the radial points. □

Without the non-trapping estimate, this is a special case of a result of Mazzeo and Melrose [37], with improvements by Guillarmou [28]. The point is that first, we do not need the machinery of the zero calculus here, and second, the analogous result holds true on arbitrary asymptotically hyperbolic spaces, with the non-trapping estimates holding under dynamical assumptions (namely, no trapping). The poles were actually computed in [53, Section 7] using special algebraic properties, within the Mazzeo-Melrose framework; however, given the Fredholm properties our methods here give, the rest of the algebraic computation in [53] go through. Indeed,

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[58] In fact, a stronger statement can be made: a calculation completely analogous to what we just performed, see Remark 4.6, shows that in $\mu < 0$, P_σ is a conjugate (times a power of μ) of a Klein-Gordon-type operator on $(n-1)$-dimensional de Sitter space with $\mu = 0$ being the boundary (i.e. where time goes to infinity). Thus, if σ is not a pole of $\mathcal{R}(\sigma)$ and $(P_\sigma - \imath Q_\sigma)\tilde{u} = 0$ then one would have a solution u of this Klein-Gordon-type equation near $\mu = 0$, i.e. infinity, that rapidly vanishes at infinity. It is shown in [53, Proposition 5.3] by a Carleman-type estimate that this cannot happen; although there $\sigma^2 \in \mathbb{R}$ is assumed, the argument given there goes through almost verbatim in general. Thus, if Q_σ is supported in $\mu < c$, $c < 0$, i.e. the Schwartz kernel is supported in $(-\infty, c)\times(-\infty, c)$ in terms of μ, then $\tilde{u}$ is also supported in $\mu < c$. This argument can be applied to the highest order Laurent term which has support intersecting $(c,\infty)\times(c,\infty)$ non-trivially (which need not be the overall highest order term), so if σ is a pole of $(P_\sigma - \imath Q_\sigma)^{-1}$ with a Laurent coefficient with support intersecting $(c,\infty)\times(c,\infty)$ non-trivially, but σ is not a pole of $\mathcal{R}(\sigma)$, then the corresponding resonant state is supported in $\mu < c$, unless the dual state is supported in $\mu \leq 0$. Applying the argument to the highest order terms with support intersecting $(c,\infty)\times(0,\infty)$ non-trivially (with the first factor corresponding to the resonant state, the second to the dual state), we see that all poles of $(P_\sigma - \imath Q_\sigma)^{-1}$ with Laurent coefficients with support intersecting $(c,\infty)\times(0,\infty)$ non-trivially are given by poles of $\mathcal{R}(\sigma)$.

the results are stable under perturbations[59], provided they fit into the framework after conjugation and the weights. In the context of the perturbations (so that the asymptotically hyperbolic structure is preserved) though with evenness conditions relaxed, the non-trapping estimate is almost the same as in [41], where it is shown by a parametrix construction; here the estimates are slightly stronger.

In fact, by the discussion of Subsection 4.8, we deduce a more general result, which in particular, for even metrics, generalizes the results of Mazzeo and Melrose [37], Guillarmou [28], and adds high-energy non-trapping estimates under non-degeneracy assumptions. It also adds the semiclassically outgoing property which is useful for resolvent gluing, including for proving non-trapping bounds microlocally away from trapping, provided the latter is mild, as shown by Datchev and Vasy [15, 16].

Theorem 4.3. *Suppose that (X_0, g_0) is an $(n-1)$-dimensional manifold with boundary with an even conformally compact metric and boundary defining function x. Let $X_{0,\mathrm{even}}$ denote the even version of X_0, i.e. with the boundary defining function replaced by its square with respect to a decomposition in which g_0 is even. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-2}{2}\right)^2 - \sigma^2,$$

written as $\mathcal{R}(\sigma) : L^2 \to L^2$, has a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$,

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. Further, if (X_0, g_0) is non-trapping, then non-trapping estimates hold in every strip $-C < \operatorname{Im}\sigma < C_+$, $|\operatorname{Re}\sigma| \gg 0$: for $s > \frac{1}{2} + C$,
(4.28)
$$\|x^{-(n-2)/2+\imath\sigma}\mathcal{R}(\sigma)f\|_{H^s_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+2)/2+\imath\sigma}f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}.$$

If f is supported in X_0°, the $s-1$ norm on f can be replaced by the $s-2$ norm.

If instead $\Delta_{g_0} - \sigma^2$ satisfies mild trapping assumptions with order $\varkappa$ estimates in a C_0-strip, see Definition 2.16, then the mild trapping estimates hold, with $|\sigma|^{\varkappa-1}$ replacing $|\sigma|^{-1}$ on the right hand side of (4.28), as long as $C \leq C_0$.

Furthermore, for $\operatorname{Re} z > 0$, $\operatorname{Im} z = \mathcal{O}(h)$, the resolvent $\mathcal{R}(h^{-1}z)$ is semiclassically outgoing *with a loss of h^{-1} in the sense that if f has compact support in X_0°, $\alpha \in T^*X$ is in the semiclassical characteristic set and if $\mathrm{WF}_h^{s-1}(f)$ is disjoint from the backward bicharacteristic from α, then $\alpha \notin \mathrm{WF}_h^s(h^{-1}\mathcal{R}(h^{-1}z)f)$.*

We remark that although in order to go through without changes, our methods require the evenness property, it is not hard to deduce more restricted results without this. Essentially one would have operators with coefficients that have a conormal singularity at the event horizon; as long as this is sufficiently mild relative to what is required for the analysis, it does not affect the results. The problems arise for the analytic continuation, when one needs strong function spaces (H^s with s large); these are not preserved when one multiplies by the singular coefficients.

[59]Though of course the resonances vary with the perturbation, in the same manner as they would vary when perturbing any other Fredholm problem.

Proof. Suppose that g_0 is an even asymptotically hyperbolic metric. Then we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 + h}{x^2} \tag{4.29}$$

there, where h is an even family of metrics; it is convenient to take x to be a globally defined boundary defining function. Then

$$\Delta_{g_0} = (xD_x)^2 + \imath(n-2+x^2\gamma)(xD_x) + x^2\Delta_h, \tag{4.30}$$

with γ even. Changing to coordinates (μ, y), $\mu = x^2$, we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\imath(n-2+\mu\gamma)(\mu D_\mu) + \mu\Delta_h, \tag{4.31}$$

Now we conjugate by $\mu^{-\imath\sigma/2+n/4}$ to obtain

$$\begin{aligned}
&\mu^{\imath\sigma/2-n/4}(\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+n/4}\\
&= 4(\mu D_\mu - \sigma/2 - \imath n/4)^2 + 2\imath(n-2+\mu\gamma)(\mu D_\mu - \sigma/2 - \imath n/4)\\
&\qquad\qquad + \mu\Delta_h - \frac{(n-2)^2}{4} - \sigma^2\\
&= 4(\mu D_\mu)^2 - 4\sigma(\mu D_\mu) + \mu\Delta_h - 4\imath(\mu D_\mu) + 2\imath\sigma - 1 + 2\imath\mu\gamma(\mu D_\mu - \sigma/2 - \imath n/4).
\end{aligned}$$

Next we multiply by $\mu^{-1/2}$ from both sides to obtain

$$\begin{aligned}
&\mu^{-1/2}\mu^{\imath\sigma/2-n/4}(\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+n/4}\mu^{-1/2}\\
&= 4\mu D_\mu^2 - \mu^{-1} - 4\sigma D_\mu - 2\imath\sigma\mu^{-1} + \Delta_h - 4\imath D_\mu + 2\mu^{-1} + 2\imath\sigma\mu^{-1} - \mu^{-1}\\
&\qquad + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-2)/4)\\
&= 4\mu D_\mu^2 - 4\sigma D_\mu + \Delta_h - 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-2)/4).
\end{aligned} \tag{4.32}$$

This is certainly in $\mathrm{Diff}^2(X)$, and for σ with bounded imaginary part, is equivalent to the form we want via conjugation by a smooth function, with exponent depending on σ. The latter would make no difference even semiclassically in the real regime as it is conjugation by an elliptic semiclassical FIO. However, in the non-real regime (where we would like ellipticity) it does; the present operator is not semiclassically elliptic at the zero section. So finally we conjugate by $(1+\mu)^{\imath\sigma/4}$ to obtain

$$4\mu D_\mu^2 - 4\sigma D_\mu - \sigma^2 + \Delta_h - 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-2)/4) \tag{4.33}$$

modulo terms that can be absorbed into the error terms in the *negative* of operators in the class (4.24).

We still need to check that μ can be appropriately chosen in the interior away from the region of validity of the product decomposition (4.29) (where we had no requirements so far on μ). This only matters for semiclassical purposes, and (being smooth and non-zero in the interior) the factor $\mu^{-1/2}$ multiplying from both sides does not affect any of the relevant properties (semiclassical ellipticity and possible non-trapping properties), so can be ignored — the same is true for σ independent powers of μ.

To do so, it is useful to think of $(\tilde\tau\partial_{\tilde\tau})^2 - G_0$, G_0 the dual metric of g_0, as a Lorentzian b-metric on $X_0^\circ \times [0,\infty)_{\tilde\tau}$. From this perspective, we want to introduce a new boundary defining function $\tau = \tilde\tau e^{\phi}$, with our σ the b-dual variable of τ and ϕ a function on X_0, i.e. with our τ already given, at least near $\mu = 0$, i.e. ϕ already fixed there, namely $e^\phi = \mu^{1/2}(1+\mu)^{-1/4}$. Recall from the end of Subsection 3.2 that such a change of variables amounts to a conjugation on the Mellin transform side by $e^{-\imath\sigma\phi}$. Further, properties of the Mellin transform are preserved provided $\frac{d\tau}{\tau}$ is globally time-like, which, as noted at the end of Subsection 3.2, is satisfied if $|d\phi|_{G_0} < 1$. But, reading off the dual metric from the principal symbol of (4.31),

$$\frac{1}{4}\left|d(\log\mu - \frac{1}{2}\log(1+\mu))\right|^2_{G_0} = \left(1 - \frac{\mu}{2(1+\mu)}\right)^2 < 1$$

for $\mu > 0$, with a strict bound as long as μ is bounded away from 0. Correspondingly, $\mu^{1/2}(1+\mu)^{-1/4}$ can be extended to a function e^ϕ on all of X_0 so that $\frac{d\tau}{\tau}$ is time-like, and we may even require that ϕ is constant on a fixed (but arbitrarily large) compact subset of X_0°. Then, after conjugation by $e^{-\imath\sigma\phi}$ all of the semiclassical requirements of Section 2 are satisfied. Naturally, the semiclassical properties could be easily checked directly for the conjugate of $\Delta_{g_0} - \sigma^2$ by the so-extended μ.

Thus, all of the results of Section 2 apply. The only part that needs some explanation is the direction of propagation for the semiclassically outgoing condition. For $z > 0$, as in the de Sitter case, null-bicharacteristics in X_0° must go to L_+, hence lie in $\Sigma_{\hbar,+}$. Theorem 2.15 states *backward* propagation of regularity for the operator considered there. However, the operator we just constructed is the negative of the class considered in (4.24), and under changing the sign of the operator, the Hamilton vector field also changes direction, so semiclassical estimates (or WF_h) indeed propagate in the forward direction. □

Remark 4.4. We note that if the dual metric G_1 on X_0 is of the form $\kappa^2 G_0$, G_0 the dual of g_0 as in (4.29), then

$$\Delta_{G_1} - \kappa^2\frac{(n-2)^2}{4} - \sigma^2 = \kappa^2(\Delta_{G_0} - \frac{(n-2)^2}{4} - (\sigma/\kappa)^2).$$

Thus, with μ as above, and with $\tilde{P}_\sigma$ the conjugate of $\Delta_{G_0} - \frac{(n-2)^2}{4} - (\sigma/\kappa)^2$, of the form (4.33) (modulo error terms as described there) then with $e^\phi = \mu^{1/(2\kappa)}(1+\mu)^{-1/(4\kappa)}$ extended into the interior of X_0 as above, we have

$$\mu^{-1/2}\mu^{n/4}e^{\imath\sigma\phi}(\Delta_{g_1} - \kappa^2\frac{(n-2)^2}{4} - \sigma^2)e^{-\imath\sigma\phi}\mu^{n/4}\mu^{-1/2} = \kappa^2\tilde{P}_{\sigma/\kappa}.$$

Now, $P_\sigma = \kappa^2\tilde{P}_{\sigma/\kappa}$ still satisfies all the assumptions of Section 2, thus directly conjugation by $e^{-\imath\sigma\phi}$ and multiplication from both sides by $\mu^{-1/2}$ gives an operator to which the results of Section 2 apply. This is relevant because if we have an asymptotically hyperbolic manifold with ends of different sectional curvature, the manifold fits into the general framework directly, including the semiclassical estimates[60]. A particular example is de Sitter-Schwarzschild space, on which resonances and wave propagation were analyzed from an asymptotically hyperbolic perspective in [47, 5, 40]; this is a special case of the Kerr-de Sitter family discussed in Section 6. The stability of estimates for operators such as P_σ under small smooth, in the b-sense, perturbations of the coefficients of the associated d'Alembertian means that all the properties of de Sitter-Schwarzschild obtained by this method are also valid for Kerr-de Sitter with sufficiently small angular momentum. However, working directly with Kerr-de Sitter space, and showing that it satisfies the assumptions of Section 2 on its own, gives a better result; we accomplish this in Section 6.

[60]For 'classical' results, the interior is automatically irrelevant.

Remark 4.5. We now return to our previous remarks regarding the fact that our solution disallows the conormal singularities $(\mu \pm i0)^{\imath\sigma}$ from the perspective of conformally compact spaces of dimension $n-1$. The two indicial roots on these spaces[61] correspond to the asymptotics $\mu^{\pm\imath\sigma/2+(n-2)/4}$ in $\mu > 0$. Thus for the operator

$$\mu^{-1/2}\mu^{\imath\sigma/2-n/4}(\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+n/4}\mu^{-1/2},$$

or indeed P_σ, they correspond to

$$\left(\mu^{-\imath\sigma/2+n/4}\mu^{-1/2}\right)^{-1}\mu^{\pm\imath\sigma/2+(n-2)/4} = \mu^{\imath\sigma/2\pm\imath\sigma/2}.$$

Here the indicial root $\mu^0 = 1$ corresponds to the smooth solutions we construct for P_σ, while $\mu^{\imath\sigma}$ corresponds to the conormal behavior we rule out. Back to the original Laplacian, thus, $\mu^{-\imath\sigma/2+(n-2)/4}$ is the allowed asymptotics and $\mu^{\imath\sigma/2+(n-2)/4}$ is the disallowed one. Notice that $\operatorname{Re}\imath\sigma = -\operatorname{Im}\sigma$, so the disallowed solution is growing at $\mu = 0$ relative to the allowed one, as expected in the physical half plane, and the behavior reverses when $\operatorname{Im}\sigma < 0$. Thus, in the original asymptotically hyperbolic picture one has to distinguish two different rates of growths, whose relative size changes. On the other hand, in our approach, we rule out the singular solution and allow the non-singular (smooth one), so there is no change in behavior at all for the analytic continuation.

Remark 4.6. For *even* asymptotically de Sitter metrics on an $(n-1)$-dimensional manifold X_0' with boundary, the methods for asymptotically hyperbolic spaces work. For the *past* de Sitter problem, solving locally the Klein-Gordon equation propagating *away from* the boundary, $P_\sigma - \imath Q_\sigma$ plays the same role as for asymptotically hyperbolic spaces; for the *future* de Sitter problem, propagating *towards* the boundary, $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma^*$ switch roles, which does not affect Fredholm properties, see Remark 2.8. Again, evenness means that we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 - h}{x^2} \tag{4.34}$$

there, where h is an even family of Riemannian metrics; as above, we take x to be a globally defined boundary defining function. For the *past problem*, with $\tilde\mu = x^2$, so $\tilde\mu > 0$ is the Lorentzian region, the above calculations for $\Box_{g_0} - \frac{(n-2)^2}{4} - \sigma^2$ in place of $\Delta_{g_0} - \frac{(n-2)^2}{4} - \sigma^2$ leading to (4.32) all go through with μ replaced by $\tilde\mu$, and Δ_h replaced by $-\Delta_h$. Letting $\mu = -\tilde\mu$, and conjugating by $(1+\mu)^{\imath\sigma/4}$ as above, yields

$$-4\mu D_\mu^2 + 4\sigma D_\mu + \sigma^2 - \Delta_h + 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-2)/4), \tag{4.35}$$

modulo terms that can be absorbed into the error terms in operators in the class (4.24); this is the negative of the operator (4.33) apart from the γ term, which is not important for our framework. For the *future problem* the calculations are analogous except we work with $\overline{\sigma}$ in place of σ since our aim is to get to $P_\sigma^* + \imath Q_\sigma^*$; the above calculations for $\Box_{g_0} - \frac{(n-2)^2}{4} - \overline{\sigma}^2$ yield

$$-4\mu D_\mu^2 + 4\overline{\sigma} D_\mu + \overline{\sigma}^2 - \Delta_h + 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \overline{\sigma}/2 - \imath(n-2)/4), \tag{4.36}$$

[61] Note that $\mu = x^2$.

again modulo terms that can be absorbed into the error terms in operators in the class (4.24), i.e. this is indeed of the form $P_\sigma^* + \imath Q_\sigma^*$ in the framework of Subsection 4.8, at least near $\tilde\mu = 0$. If now X_0' is extended to a manifold without boundary so that in $\tilde\mu < 0$, i.e. $\mu > 0$, one has a classically elliptic, semiclassically either non-trapping or mildly trapping problem, then all the results of Section 2 are applicable.

For the past problem one concretely obtains a formula analogous to (4.27), namely when f is supported in $(c,0)_\mu$, and Q_σ is supported in $\mu < c$, the past solution of $(\Box_{g_0} - \frac{(n-2)^2}{4} - \sigma^2)u = f$, i.e. the solution vanishing for μ sufficiently close to 0, which exists and is unique for all $\sigma \in \mathbb{C}$, is

$$\tilde\mu^{-\imath\sigma+n/2}\tilde\mu^{-1}\tilde R_s(P_\sigma - \imath Q_\sigma)^{-1}\tilde E_{s-1}\tilde\mu^{\imath\sigma-n/2}\tilde\mu^{-1}f, \tag{4.37}$$

where $\tilde R_s$ is restriction to $(c,0)_\mu$, and $\tilde E_{s-1}$ is an H^{s-1} extension to X, in a manner that vanishes in $\mu \geq 0$. In particular, for all poles σ of $(P_\sigma - \imath Q_\sigma)^{-1}$ either the resonant state or the corresponding dual state vanishes in $(c,0)_\mu$. If the resonant state does (which is in $\mathcal{C}^\infty(X)$), then, as in Footnote 58, a unique continuation statement on the asymptotically hyperbolic side shows that it also vanishes for $\mu > 0$, so in particular such a term does not contribute to the Laurent coefficients of the asympotically hyperbolic resolvent, $\mathcal{R}(\sigma)$, either. If the resonant state does not vanish identically in $(c,0)_\mu$ (and thus the dual state does vanish there identically), then it does not vanish in $\mu > 0$ either identically (again, by unique continuation, now on the de Sitter side). There are two possibilities then: either the support of the dual state intersects $\mu > 0$ non-trivially, and then σ is a pole of $\mathcal{R}(\sigma)$ and the dual state is supported[62] in $\mu^{-1}([0,\infty))$, or its support intersects $\mu > c$ precisely at $\mu = 0$, in which case it is a differentiated delta distribution. Since it lies in H^{1-s}, this can only happen for $s > 3/2$; since the poles of $(P_\sigma - \imath Q_\sigma)^{-1}$ in $\operatorname{Im}\sigma > -C$ are independent of s as long as $s > \frac{1}{2} + C$, in $\operatorname{Im}\sigma > -1$ the poles of $\psi(P_\sigma - \imath Q_\sigma)^{-1}\phi$ when ϕ, ψ are supported in $\mu > c$ are exactly those of $\mathcal{R}(\sigma)$. Indeed, using the form of $P_\sigma^* + \imath Q_\sigma^*$ we see that differentiated delta distributions can lie in its nullspace only if $\sigma \in -\imath\mathbb{N}^+$ (with $\mathbb{N}^+$ standing for positive integers), so these are the only possible poles of $\psi(P_\sigma - \imath Q_\sigma)^{-1}\phi$ in addition to those of $\mathcal{R}(\sigma)$; for all of these poles the dual states are necessarily supported in[63] $\mu^{-1}([0,\infty))$.

5. Minkowski space

Perhaps our simplest example is Minkowski space $M = \mathbb{R}^n$ with the metric

$$g_0 = dz_n^2 - dz_1^2 - \ldots - dz_{n-1}^2.$$

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[62] There can be no support in $\mu^{-1}((-\infty, c])$ in view of Footnote 44.

[63] In view of the block lower triangular structure of the Schwartz kernel of $(P_\sigma - \imath Q_\sigma)^{-1}$, as explained in Footnote 44, at least for σ near a fixed σ_0, one can change Q_σ by a holomorphic finite rank operator family, keeping its support in $\mu < c$ in both factors, so that for the new Q_σ the poles of $(P_\sigma - \imath Q_\sigma)^{-1}$ near σ_0 are exactly those of $\psi(P_\sigma - \imath Q_\sigma)^{-1}\phi$, with multiplicities. This in particular implies that in the perturbation framework of Subsection 2.7, for perturbations $P_\sigma(w) - \imath Q_\sigma(w)$, w close to w_0, the poles of $\psi(P_\sigma(w) - \imath Q_\sigma(w))^{-1}\phi$ are necessarily close to the poles of $\psi(P_\sigma(w_0) - \imath Q_\sigma(w_0))^{-1}\phi$, with multiplicities. In the Kerr-de Sitter setting for small angular momentum, a, as in Theorem 1.4, this justifies the simplicity and one dimensionality of the zero resonance: while for $a = 0$, $(P_\sigma(a) - \imath Q_\sigma(a))^{-1}$ may have other resonant states at $\sigma = 0$, only 1 contributes to $\psi(P_\sigma(a) - \imath Q_\sigma(a))^{-1}\phi$ with $a = 0$, with a resulting simple resonance, hence for small a the same holds.

Also, let $\hat{M} = \overline{\mathbb{R}^n}$ be the radial (or geodesic) compactification of space-time, see [39, Section 1]; thus $\hat{M}$ is the n-ball, with boundary $X = \mathbb{S}^{n-1}$. Writing $z' = (z_1, \ldots, z_{n-1}) = r\omega$ in terms of Euclidean product coordinates, and $t = z_n$, local coordinates on $\hat{M}$ in $|z'| > \epsilon|z_n|$, $\epsilon > 0$, are given by

$$s = \frac{t}{r},\ \rho = r^{-1},\ \omega, \tag{5.1}$$

while in $|z_n| > \epsilon|z'|$, by

$$\tilde{\rho} = |t|^{-1},\ Z = \frac{z'}{|t|}. \tag{5.2}$$

Note that in the overlap, the curves given by Z constant are the same as those given by s, ω constant, but the actual defining function of the boundary we used, namely $\tilde{\rho}$ vs. ρ, differs, and does so by a factor which is constant on each fiber. For some purposes it is useful to fix a global boundary defining function, such as $\hat{\rho} = (r^2+t^2)^{-1/2}$. We remark that if one takes a Mellin transform of functions supported near infinity along these curves, and uses conjugation by the Mellin transform to obtain families of operators on $X = \partial\hat{M}$, the effect of changing the boundary defining function in this manner is conjugation by a non-vanishing factor which does not affect most relevant properties of the induced operator on the boundary, so one can use local defining functions when convenient.

The metric g_0 is a Lorentzian scattering metric in the sense of Melrose [39] (where, however, only the Riemannian case was discussed) in that it is a symmetric non-degenerate bilinear form on the scattering tangent bundle of $\hat{M}$ of Lorentzian signature. This would be the appropriate locus of analysis of the Klein-Gordon operator, $\Box_{g_0} - \lambda$ for $\lambda > 0$, but for $\lambda = 0$ the scattering problem becomes degenerate at the zero section of the scattering cotangent bundle at infinity. However, one can convert $\Box_{g_0}$ to a non-degenerate b-operator on $\hat{M}$: it is of the form $\hat{\rho}^2\tilde{P}$, $\tilde{P} \in \mathrm{Diff}^2_{\mathrm{b}}(X)$, where $\hat{\rho}$ is a defining function of the boundary. In fact, following Wang [60], we consider (taking into account the different notation for dimension)

$$\begin{aligned} &\rho^{-2}\rho^{-(n-2)/2}\Box_{g_0}\rho^{(n-2)/2} = \Box_{\tilde{g}_0} + \frac{(n-2)(n-4)}{4};\\ &\tilde{G}_0 = (1-s^2)\partial_s^2 - 2(s\partial_s)(\rho\partial_\rho) - (\rho\partial_\rho)^2 - \partial_\omega^2, \end{aligned} \tag{5.3}$$

with $\tilde{G}_0$ being the dual metric of $\tilde{g}_0$; here we write $\Box_{\tilde{g}_0}$ of the d'Alembertian of $\tilde{g}_0$. Again, this ρ is not a globally valid defining function, but changing to another one does not change the properties we need[64] where this is a valid defining function. It is then a straightforward calculation that the induced operator on the boundary is

$$P'_\sigma = D_s(1-s^2)D_s - \sigma(sD_s + D_s s) - \sigma^2 - \Delta_\omega + \frac{(n-2)(n-4)}{4},$$

In the other coordinate region, where $\tilde{\rho}$ is a valid defining function, and $t > 0$, it is even easier to compute

$$\Box_{g_0} = \tilde{\rho}^2\left((\tilde{\rho}D_{\tilde{\rho}})^2 + 2(\tilde{\rho}D_{\tilde{\rho}})ZD_Z + (ZD_Z)^2 - \Delta_Z - \imath(\tilde{\rho}D_{\tilde{\rho}}) - \imath ZD_Z\right), \tag{5.4}$$

so after Mellin transforming $\tilde{\rho}^{-2}\Box$, we obtain

$$\tilde{L}_\sigma = (\sigma - \imath/2)^2 + \frac{1}{4} + 2(\sigma - \imath/2)ZD_Z + (ZD_Z)^2 - \Delta_Z.$$

[64] Only when $\operatorname{Im}\sigma \to \infty$ can such a change matter.

Conjugation by $\tilde\rho^{(n-2)/2}$ simply replaces σ by $\sigma-\imath\frac{n-2}{2}$, yielding that the Mellin transform L_σ of $\tilde\rho^{-(n-2)/2}\tilde\rho^{-2}\Box_{g_0}\tilde\rho^{(n-2)/2}$ is

$$\begin{aligned}L_\sigma&=\Big(\sigma-\imath\frac{n-1}{2}\Big)^2+\frac14+2\Big(\sigma-\imath\frac{n-1}{2}\Big)ZD_Z+(ZD_Z)^2-\Delta_Z\\&=\Big(ZD_Z+\sigma-\imath\frac{n-1}{2}\Big)^2+\frac14-\Delta_Z.\end{aligned}\tag{5.5}$$

Note that L_σ and L'_σ are not the same operator in different coordinates; they are related by a σ-dependent conjugation. The operator L_σ in (5.5) is *almost* exactly the operator arising from de Sitter space on the front face, see the displayed equation after [53, Equation 7.4] (the σ in [53, Equation 7.4] is $\imath\sigma$ in our notation as already remarked in Section 4), with the only change that our σ would need to be replaced by $-\sigma$, and we need to add $\frac{(n-1)^2}{4}-\frac14$ to our operator[65]. However, due to the way we need to propagate estimates, as explained below, we need to think of this as the *adjoint* of an operator of the type we considered in Section 4 up to Remark 4.6, or after [53, Equation 7.4]. Thus, we think of L_σ as the adjoint (with respect to $|dZ|$) of

$$\begin{aligned}P_{\overline\sigma}=L_\sigma^*&=(ZD_Z+\overline\sigma-\imath(n-1)/2)^2+\frac14-\Delta_Z\\&=(ZD_Z+\overline\sigma-\imath(n-1))(ZD_Z+\overline\sigma)-\Delta_Z+\frac14-\frac{(n-1)^2}{4},\end{aligned}$$

which is the de Sitter operator after [53, Equation 7.4], except, denoting σ of that paper by $\check\sigma$, $\imath\check\sigma=\overline\sigma$, and we need to take $\lambda=\frac{(n-1)^2}{4}-\frac14$ in [53, Equation 7.4]. This is also of the form in Section 4, but with σ of that section, denoted temporarily by $\hat\sigma$, being given by $-\overline\sigma$. Thus, all of the analysis of Section 4 applies.

In particular, note that $P_{\overline\sigma}$ is elliptic inside the light cone, where $s>1$, and hyperbolic outside the light cone, where $s<1$. It follows from Subsection 4.9 that $P_{\overline\sigma}$ is a conjugate of the hyperbolic Laplacian inside the light cones[66], and of the Klein-Gordon operator on de Sitter space outside the light cones: with[67] $\nu=(1-|Z|^2)^{1/2}=(\cosh\rho_{\mathbb{H}^{n-1}})^{-1}$,

$$\nu^{\frac n2+\imath\overline\sigma}P_{-\overline\sigma}\nu^{-\imath\overline\sigma-\frac n2}=\nu^{\frac n2-\imath\overline\sigma}P_{\overline\sigma}\nu^{\imath\overline\sigma-\frac n2}=-\nu^{-1}\left(\Delta_{\mathbb{H}^{n-1}}-\overline\sigma^2-\frac{(n-2)^2}{4}\right)\nu^{-1}.$$

We remark that in terms of dynamics on ${}^{\mathrm b}S^*\hat M$, as discussed in Subsection 3.1, there is a sign difference in the normal to the boundary component of the Hamilton vector field (normal in the b-sense, only), so in terms of the full b-dynamics (rather than normal family dynamics) the radial points here are sources/sinks, unlike the saddle points in the de Sitter case. This is closely related to the appearance of adjoints in the Minkowski problem (as compared to the de Sitter one).

This immediately assures that not only the wave equation on Minkowski space fits into our framework, wave propagation on it is stable under small smooth perturbation in $\mathrm{Diff}^2_{\mathrm b}(X)$ of $\hat\rho^2\Box_{g_0}$ which have real principal symbol.

[65]Since replacing $t>0$ by $t<0$ in the region we consider reverses the sign when relating D_ρ and D_t, the signs would agree with those from the discussion after [53, Equation 7.4] at the backward light cone.

[66]As pointed out to the author by Gunther Uhlmann, this means that the Klein model of hyperbolic space is the one induced by the Minkowski boundary reduction.

[67]Recall Footnote 57 for the connection to the standard Poincaré model.

Further, it is shown in [53, Corollary 7.18] that the problem for $P_{-\overline{\sigma}}$ is invertible in the interior of hyperbolic space as an operator on weighted zero-Sobolev spaces, with the inverse mapping $\dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$ to $\nu^{2\imath\overline{\sigma}}\mathcal{C}^\infty(\mathbb{B}^{n-1})$, unless

$$\imath\overline{\sigma} \in -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda} - \mathbb{N} = -\frac{n-1}{2} \pm \frac{1}{2} - \mathbb{N} = -\frac{n-2}{2} - \mathbb{N},$$

i.e.

$$\sigma \in -\imath(\frac{n-2}{2} + \mathbb{N}); \tag{5.6}$$

this amounts to the invertibility of $P_{\overline{\sigma}}$, with the inverse mapping $\dot{\mathcal{C}}^\infty(\mathbb{B}^{n-1})$ to $\mathcal{C}^\infty(\mathbb{B}^{n-1})$. Recall also from Remark 4.6 that, with ϕ, ψ supported in $s-1 > c$, $c < 0$, as there (with the role of μ there played by $s-1$ here), $\phi(P_{\overline{\sigma}} - \imath Q_{\overline{\sigma}})^{-1}\psi$, and thus also its adjoint, may have additional poles as compared to the resolvent of the asymptotically hyperbolic model, but for these $-\overline{\sigma} = \hat{\sigma} \in -\imath\mathbb{N}^+$, i.e. $\sigma \in -\imath\mathbb{N}^+$. Thus, if f is supported in $s-1 > c$ then $((P^*_{\overline{\sigma}} + \imath Q^*_{\overline{\sigma}})^{-1}f)|_{s-1>c}$ only has the poles given by the asymptotically hyperbolic model in $s-1 > c$ except possibly $-\imath\mathbb{N}^+$. Further for these non-asymptotically hyperbolic poles, the resonant states of $(P^*_{\overline{\sigma}} + \imath Q^*_{\overline{\sigma}})^{-1}$ are the dual states of $(P_\sigma - \imath Q_\sigma)^{-1}$ and vice versa, so either the dual states of $(P^*_{\overline{\sigma}} + \imath Q^*_{\overline{\sigma}})^{-1}$ are supported in $s-1 < c$, in which case they are not important to us since they do not affect the solution of the standard forward problem when the forcing is supported in $s-1 > c$, or the states are supported in $s-1 \leq 0$, in which case they still do not affect the solution in the elliptic region $s-1 > 0$; see Footnote 58.

To recapitulate, L_σ is of the form described in Section 2, at least if we restrict away from the backward light cone[68]. To be more precise, for the forward problem for the wave equation, the *adjoint* $P_{\overline{\sigma}}$ of the operator L_σ we need to study satisfies the properties in Section 2, i.e. singularities are propagated towards the radial points at the forward light cone, which means that our solution lies in the 'bad' dual spaces – of course, these are just the singularities corresponding to the radiation field of Friedlander [25], see also [46], which is singular on the radial compactification of Minkowski space. However, by elliptic regularity or microlocal propagation of singularities, we of course automatically have estimates in better spaces away from the boundary of the light cone. We also need complex absorption supported, say, near $s = -1/2$ in the coordinates (5.1), as in Subsection 4.7. If we wanted to, we could instead add a boundary at $s = -1/2$, or indeed at $s = 0$ (which would give the standard Cauchy problem), see Remark 2.6. By Subsection 3.3, this does not affect the solution in $s > 0$, say, when the forcing f vanishes in $s < 0$ and we want the solution u to vanish there as well.

We thus deduce from Lemma 3.1 and the analysis of Section 2:

Theorem 5.1. *Let K be a compact subset of the interior of the light cone at infinity on $\hat{M}$. Suppose that g is a Lorentzian scattering metric and $\hat{\rho}^2\Box_g$ is sufficiently close to $\hat{\rho}^2\Box_{g_0}$ in $\mathrm{Diff}^2_{\mathrm{b}}(\hat{M})$, with n the dimension of $\hat{M}$. Then solutions of the*

[68]The latter is only done to avoid combining for the same operator the estimates we state below for an operator L_σ and its adjoint; as follows from the remark above regarding the sign of σ, for the operator here, the microlocal picture near the backward light cone is like that for the L_σ considered in Section 2, and near the forward light cone like that for L^*_σ. It is thus fine to include both the backward and the forward light cones; we just end up with a combination of the problem we study here and its adjoint, and with function spaces much like in [39, 57].

wave equation $\Box_g u = f$ vanishing in $t < 0$ and $f \in \dot{\mathcal{C}}^\infty(\hat{M}) = \mathcal{S}(\mathbb{R}^n)$ have a polyhomogeneous asymptotic expansion in the sense of [43] *in K of the form $\sim \sum_j \sum_{k \leq m_j} a_{jk} \hat{\rho}^{\delta_j} (\log|\hat{\rho}|)^k$, with a_{jk} in $\mathcal{C}^\infty$, and with*

$$\delta_j = \imath\sigma_j + \frac{n-2}{2},$$

with σ_j being a point of non-invertibility of L_σ on the appropriate function spaces. On Minkowski space, the exponents are given by

$$\delta_j = \imath(-\imath\frac{n-2}{2} - \imath j) + \frac{n-2}{2} = n-2+j,\ j \in \mathbb{N},$$

and they depend continuously on the perturbation if one perturbs the metric. A distributional version holds globally.

For polyhomogeneous f the analogous conclusion holds, except that one has to add to the set of exponents (index set) the index set of f, increased by 2 (corresponding factoring out $\hat{\rho}^2$ in (5.4)), in the sense of extended unions [43, Section 5.18].

Remark 5.2. Here a compact K is required since we allow drastic perturbations that may change where the light cone hits infinity. If one imposes more structure, so that the light cone at infinity is preserved, one can get more precise results.

As usual, the smallness of the perturbation is only relevant to the extent that rough properties of the global dynamics and the local dynamics at the radial points are preserved (so the analysis is only impacted via dynamics). There are no size restrictions on perturbations if one keeps the relevant features of the dynamics.

In a different class of spaces, namely asymptotically conic Riemannian spaces, analogous and more precise results exist for the induced product wave equation, see especially the work of Guillarmou, Hassell and Sikora [27]; the decay rate in their work is the same in *odd* dimensional space-time (i.e. even dimensional space). In terms of space-time, these spaces look like a blow-up of the 'north and south poles' $Z = 0$ of Minkowski space, with product type structure in terms of space time, but general smooth dependence on ω (with the sphere in ω replaceable by another compact manifold). In that paper a parametrix is constructed for Δ_g at all energies by combining a series of preceding papers. Their conclusion in even dimensional space-time is one order better; this is presumably the result of a cancellation. It is a very interesting question whether our analysis can be extended to non-product versions of their setting.

Note that for the Mellin transform of $\Box_{g_0}$ one can perform a more detailed analysis, giving Lagrangian regularity at the light cone, with high energy control. This would be preserved for other metrics that preserve the light cone at infinity to sufficiently high order. The result is an expansion on the $\hat{M}$ blown up at the boundary of the light cone, with the singularities corresponding to the Friedlander radiation field. However, in this relatively basic paper we do not pursue this further.

6. The Kerr-de Sitter metric

6.1. **The basic geometry.** We now give a brief description of the Kerr-de Sitter metric on

$$M_\delta = X_\delta \times [0,\infty)_\tau,\ X_\delta = (r_- - \delta, r_+ + \delta)_r \times \mathbb{S}^2,$$
$$X_+ = (r_-, r_+)_r \times \mathbb{S}^2,\ X_- = \big((r_- - \delta, r_+ + \delta)_r \setminus [r_-, r_+]_r\big) \times \mathbb{S}^2,$$

where $r_\pm$ are specified later. We refer the reader to the excellent treatments of the geometry by Dafermos and Rodnianski [13, 14] and Tataru and Tohaneanu [50, 49] for details, and Dyatlov's paper [20] for the set-up and most of the notation we adopt.

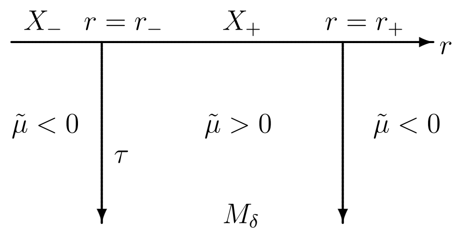

FIGURE 7. The basic diagram of Kerr-de Sitter space without the factor $\mathbb{S}^2$. The function $\tilde{\mu} = \tilde{\mu}(r)$ is defined below in (6.1); the event horizons are $\tilde{\mu} = 0$ corresponding to $r = r_+$ (the de Sitter end) and to $r = r_-$ (the Kerr end). The Mellin transform below is taken in τ, corresponding to the 'infinity' $\tau = 0$.

Away from the north and south poles $q_\pm$ we use spherical coordinates (θ, ϕ) on $\mathbb{S}^2$:

$$\mathbb{S}^2 \setminus \{q_+, q_-\} = (0, \pi)_\theta \times \mathbb{S}^1_\phi.$$

Thus, away from $(r_- - \delta, r_+ + \delta)_r \times [0, \infty)_\tau \times \{q_+, q_-\}$, the Kerr-de Sitter space-time is

$$(r_- - \delta, r_+ + \delta)_r \times [0, \infty)_\tau \times (0, \pi)_\theta \times \mathbb{S}^1_\phi$$

with the metric we specify momentarily.

The Kerr-de Sitter metric has a very similar microlocal structure at the event horizon to de Sitter space. We first start with a coordinate system in which the metric is usually expressed but in which the metric is singular at r_- and r_+, which are roots of the function $\tilde{\mu}$ defined below. The metric g is

$$\begin{aligned} g = -\rho^2\Big(\frac{dr^2}{\tilde{\mu}} + \frac{d\theta^2}{\kappa}\Big) - \frac{\kappa \sin^2\theta}{(1+\gamma)^2\rho^2}(a\,d\tilde{t} - (r^2+a^2)\,d\tilde{\phi})^2 \\ + \frac{\tilde{\mu}}{(1+\gamma)^2\rho^2}(d\tilde{t} - a\sin^2\theta\,d\tilde{\phi})^2, \end{aligned}$$

while the dual metric is

$$\begin{aligned} G = -\rho^{-2}\Big(\tilde{\mu}\partial_r^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(a\sin^2\theta\,\partial_{\tilde{t}} + \partial_{\tilde{\phi}})^2 + \kappa\partial_\theta^2 \\ - \frac{(1+\gamma)^2}{\tilde{\mu}}((r^2+a^2)\partial_{\tilde{t}} + a\,\partial_{\tilde{\phi}})^2\Big) \end{aligned} \tag{6.1}$$

with r_s, Λ, a constants, $r_s, \Lambda \geq 0$,

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta, \\ \tilde{\mu} &= (r^2+a^2)(1 - \frac{\Lambda r^2}{3}) - r_s r, \\ \kappa &= 1 + \gamma\cos^2\theta, \\ \gamma &= \frac{\Lambda a^2}{3}. \end{aligned}$$

While G is defined for all values of the parameters r_s, Λ, a, with $r_s, \Lambda \geq 0$, we make further restrictions. Note that under the rescaling

$$r' = \sqrt{\Lambda}r,\ \tilde{t}' = \sqrt{\Lambda}\tilde{t},\ r_s' = \sqrt{\Lambda}r_s,\ a' = \sqrt{\Lambda}a,\ \Lambda' = 1,$$

$\Lambda^{-1}G$ would have the same form, but with all the unprimed variables replaced by the primed ones. Thus, effectively, the general case $\Lambda > 0$ is reduced to $\Lambda = 1$.

Our first assumption is that $\tilde{\mu}(r) = 0$ has two positive roots $r = r_\pm$, $r_+ > r_-$, with

$$F_\pm = \mp \frac{\partial \tilde{\mu}}{\partial r}|_{r=r_\pm} > 0; \tag{6.2}$$

r_+ is the de Sitter end, r_- is the Kerr end. Since $\tilde{\mu}$ is a quartic polynomial, is > 0 at $r = 0$ if $|a| > 0$, and goes to $-\infty$ at $\pm\infty$, it can have at most 3 positive roots; the derivative requirements imply that these three positive roots exist, and $r_\pm$ are the larger two of these. If $a = 0$, (6.2) is satisfied if and only if $0 < \frac{9}{4}r_s^2\Lambda < 1$. Indeed, if (6.2) is satisfied, $\frac{\partial}{\partial r}(r^{-4}\tilde{\mu}) = -2r^{-3}\left(1 - \frac{3r_s}{2r}\right)$ must have a zero between r_- and r_+, where $\tilde{\mu}$ must be positive; $\frac{\partial}{\partial r}(r^{-4}\tilde{\mu}) = 0$ gives $r = \frac{3}{2}r_s$, and then $\tilde{\mu}(r) > 0$ gives $1 > \frac{9}{4}r_s^2\Lambda$. Conversely, if $0 < \frac{9}{4}r_s^2\Lambda < 1$, then the cubic polynomial $r^{-1}\tilde{\mu} = r - \frac{\Lambda}{3}r^3 - r_s$ is negative at 0 and at $+\infty$, and thus will have exactly two positive roots if it is positive at one point, which is the case at $r = \frac{3}{2}r_s$. Indeed, note that $r^{-4}\tilde{\mu} = r^{-2} - \frac{\Lambda}{3} - r_s r^{-3}$ is a cubic polynomial in r^{-1}, and $\partial_r(r^{-4}\tilde{\mu}) = -2r^{-3}\left(1 - \frac{3r_s}{2r}\right)$, so $r^{-4}\tilde{\mu}$ has a non-degenerate critical point at $r = \frac{3}{2}r_s$, and if $0 \leq \frac{9}{4}r_s^2\Lambda < 1$, then the value of $\tilde{\mu}$ at this critical point is positive. Thus, for small a (depending on $\frac{9}{4}r_s^2\Lambda$, but with uniform estimates in compact subintervals of $(0,1)$), $r_\pm$ satisfying (6.2) still exist.

We next note that for a not necessarily zero, if (6.2) is satisfied then $\frac{d^2\tilde{\mu}}{dr^2} = 2 - \frac{2}{3}\Lambda a^2 - 4\Lambda r^2$ must have a positive zero, so we need

$$0 \leq \gamma = \frac{\Lambda a^2}{3} < 1, \tag{6.3}$$

i.e. (6.2) implies (6.3).

Physically, Λ is the cosmological constant, $r_s = 2M$ the Schwarzschild radius, with M being the mass of the black hole, a the angular momentum. Thus, de Sitter-Schwarzschild space is the particular case with $a = 0$, while further de Sitter space is the case when $r_s = 0$ in which limit r_- goes to the origin and simply 'disappears', and Schwarzschild space is the case when $\Lambda = 0$, in which case r_+ goes to infinity, and 'disappears', creating an asymptotically Euclidean end. On the other hand, Kerr is the special case $\Lambda = 0$, with again $r_+ \to \infty$, so the structure near the event horizon is unaffected, but the de Sitter end is replaced by a different, asymptotically Minkowski, end. One should note, however, that of the limits $\Lambda \to 0$, $a \to 0$ and $r_s \to 0$, the only non-degenerate one is $a \to 0$; in both other cases the geometry changes drastically corresponding to the disappearance of the de Sitter, resp. the black hole, ends. Thus, arguably, from a purely mathematical point of view, de Sitter-Schwarzschild space-time is the most natural limiting case. Perhaps the best way to follow this section then is to keep de Sitter-Schwarzschild space in mind. Since our methods are stable, this automatically gives the case of small a; of course working directly with a gives better results.

In fact, from the point of view of our setup, all the relevant features are symbolic (in the sense of principal symbols), including dependence on the Hamiltonian dynamics. Thus, the only not completely straightforward part in showing that our abstract hypotheses are satisfied is the semi-global study of dynamics. The dynamics of the rescaled Hamilton flow depends smoothly on a, so it is automatically well-behaved for finite times for small a if it is such for $a = 0$; here rescaling is understood on the fiber-radially compactified cotangent bundle $\overline{T}^* X_\delta$ (so that one has a smooth dynamical system whose only non-compactness comes from that of the base variables). The only place where dynamics matters for unbounded times are critical points or trapped orbits of the Hamilton vector field. In $S^* X_\delta = \partial\overline{T}^* X_\delta$, one can analyze the structure easily for all a, and show that for a specific range of a, given below implicitly by (6.13), the only critical points/trapping is at fiber-infinity SN^*Y of the conormal bundle of the event horizon Y. We also analyze the semiclassical dynamics (away from $S^* X_\delta = \partial\overline{T}^* X_\delta$) directly for a satisfying (6.27), which allows a to be comparable to r_s. We show that in this range of a (subject to (6.2) and (6.13)), the only trapping is hyperbolic trapping, which was analyzed by Wunsch and Zworski [61]; further, we also show that the trapping is normally hyperbolic for small a, and is thus structurally stable then.

In summary, apart from the full analysis of semiclassical dynamics, we work with arbitrary a for which (6.2) and (6.13) holds, which are both natural constraints, since it is straightforward to check the requirements of Section 2 in this generality. Even in the semiclassical setting, we work under the relatively large a bound, (6.27), to show hyperbolicity of the trapping, and it is only for normal hyperbolicity that we deal with (unspecified) small a.

We now put the metric (6.1) into a form needed for the analysis. Since the metric is not smooth b-type in terms of $r, \theta, \tilde\phi, e^{-\tilde t}$, in order to eliminate the $\tilde\mu^{-1}$ terms we let

$$t = \tilde t + h(r),\ \ \phi = \tilde\phi + P(r) \tag{6.4}$$

with

$$h'(r) = \mp\frac{1+\gamma}{\tilde\mu}(r^2+a^2) \mp c_\pm, \qquad P'(r) = \mp\frac{1+\gamma}{\tilde\mu}a \tag{6.5}$$

near $r_\pm$. Here $c = c_\pm = c_\pm(r)$ is a smooth function of r (unlike $\tilde\mu^{-1}$!), that is to be specified. One also needs to specify the behavior in $\tilde\mu > 0$ bounded away from 0, much like we did so in the asymptotically hyperbolic setting. This only affects semiclassical properties when σ away from the reals, however; so the choice is not relevant for most purposes.

We at first focus on the 'classical' problem. Then the dual metric becomes

$$\begin{aligned} G = -\rho^{-2}\Big(&\tilde\mu\big(\partial_r \mp c\partial_t\big)^2 \mp 2(1+\gamma)(r^2+a^2)\big(\partial_r \mp c\partial_t\big)\partial_t \\ &\mp 2(1+\gamma)a\big(\partial_r \mp c\partial_t\big)\partial_\phi + \kappa\partial_\theta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(a\sin^2\theta\partial_t + \partial_\phi)^2\Big). \end{aligned}$$

We write $\tau = e^{-t}$, so $-\tau\partial_\tau = \partial_t$, and b-covectors as

$$\xi\,dr + \sigma\,\frac{d\tau}{\tau} + \eta\,d\theta + \zeta\,d\phi,$$

so

$$\begin{aligned} \rho^2 G = -\tilde\mu\big(\xi \pm c\sigma\big)^2 &\mp 2(1+\gamma)(r^2+a^2)\big(\xi \pm c\sigma\big)\sigma \\ &\pm 2(1+\gamma)a\big(\xi \pm c\sigma\big)\zeta - \kappa\eta^2 - \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta\sigma + \zeta)^2. \end{aligned}$$

Note that the sign of ξ here is the *opposite* of the sign in our de Sitter discussion in Section 4 where it was the dual variable (thus the symbol of D_μ) of μ, which is $r^{-2}\tilde\mu$ in the present notation, since $\frac{d\tilde\mu}{dr}<0$ at the de Sitter end, $r=r_+$.

A straightforward calculation shows $\det g=(\det G)^{-1}=-(1+\gamma)^{-4}\rho^4\sin^2\theta$, so apart from the usual polar coordinate singularity at $\theta=0,\pi$, which is an artifact of the spherical coordinates and is discussed below, we see at once that g is a smooth Lorentzian b-metric. In particular, it is non-degenerate, so the d'Alembertian $\Box_g=d^*d$ is a well-defined b-operator, and

$$\sigma_{\mathrm{b},2}(\rho^2\Box_g)=\rho^2 G.$$

Factoring out ρ^2 does not affect any of the statements below but simplifies some formulae, see Footnote 12 and Footnote 19 for general statements; one could also work with G directly.

6.2. **The 'spatial' problem: the Mellin transform.** The Mellin transform, P_σ, of $\rho^2\Box_g$ has the same principal symbol, including in the high energy sense,

$$\begin{aligned}p_{\mathrm{full}}=\sigma_{\mathrm{full}}(P_\sigma)=&-\tilde\mu\big(\xi\pm c\sigma\big)^2\mp 2(1+\gamma)(r^2+a^2)\big(\xi\pm c\sigma\big)\sigma\\&\pm 2(1+\gamma)a\big(\xi\pm c\sigma\big)\zeta-\tilde p_{\mathrm{full}}\end{aligned}\tag{6.6}$$

with

$$\tilde p_{\mathrm{full}}=\kappa\eta^2+\frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta\sigma+\zeta)^2,$$

so $\tilde p_{\mathrm{full}}\geq 0$ for real σ. Thus,

$$\begin{aligned}\mathsf{H}_{p_{\mathrm{full}}}=&\Big(-2\tilde\mu\big(\xi\pm c\sigma\big)\mp 2(1+\gamma)(r^2+a^2)\sigma\pm 2(1+\gamma)a\zeta\Big)\partial_r\\&-\Big(-\frac{\partial\tilde\mu}{\partial r}\big(\xi\pm c\sigma\big)^2\mp 4r(1+\gamma)\sigma\big(\xi\pm c\sigma\big)\pm\frac{\partial c}{\partial r}\tilde c\sigma\Big)\partial_\xi\\&\pm 2(1+\gamma)a\big(\xi\pm c\sigma\big)\partial_\phi-\mathsf{H}_{\tilde p_{\mathrm{full}}},\\\tilde c=&-2\tilde\mu\big(\xi\pm c\sigma\big)\mp 2(1+\gamma)(r^2+a^2)\sigma\pm 2(1+\gamma)a\zeta.\end{aligned}$$

To deal with q_+ given by $\theta=0$ (q_- being similar), let

$$y=\sin\theta\sin\phi,\ z=\sin\theta\cos\phi,\ \text{so}\ \cos^2\theta=1-(y^2+z^2).$$

We can then perform a similar calculation yielding that if λ is the dual variable to y and ν is the dual variable to z then

$$\zeta=z\lambda-y\nu$$

and

$$\begin{aligned}\tilde p_{\mathrm{full}}&=(1+\gamma\cos^2\theta)^{-1}\Big((1+\gamma)^2(\lambda^2+\nu^2)+\tilde p''\Big)+\tilde p^\sharp_{\mathrm{full}},\\\tilde p^\sharp_{\mathrm{full}}&=(1+\gamma\cos^2\theta)^{-1}(1+\gamma)^2(a\sin^2\theta\sigma-2\zeta)a\sigma,\end{aligned}$$

with $\tilde p''$ smooth, independent of σ and vanishing quadratically at the origin. Correspondingly, by (6.6), P_σ is indeed smooth at $q_\pm$. Thus, one can perform all symbol calculations away from $q_\pm$, since the results will extend smoothly to $q_\pm$, and correspondingly from now on we do not emphasize these two poles.

In the sense of 'classical' microlocal analysis, we thus have:

$$\begin{aligned} p &= \sigma_2(P_\sigma) = -\tilde{\mu}\xi^2 \pm 2(1+\gamma)a\xi\zeta - \tilde{p}, \qquad \tilde{p} = \kappa\eta^2 + \frac{(1+\gamma)^2}{\kappa \sin^2\theta}\zeta^2 \geq 0, \\ \mathsf{H}_p &= \Big(-2\tilde{\mu}\xi \pm 2(1+\gamma)a\zeta\Big)\partial_r \pm 2(1+\gamma)a\xi\partial_\phi + \frac{\partial\tilde{\mu}}{\partial r}\xi^2\partial_\xi - \mathsf{H}_{\tilde{p}}. \end{aligned} \tag{6.7}$$

6.3. **Microlocal geometry of Kerr-de Sitter space-time.** As already stated in Section 2, it is often convenient to consider the fiber-radial compactification $\overline{T}^* X_\delta$ of the cotangent bundle T^*X_δ, with S^*X_δ considered as the boundary at fiber-infinity of $\overline{T}^* X_\delta$.

We let

$$\Lambda_+ = N^*\{\tilde{\mu} = 0\} \cap \{\mp\xi > 0\},\ \Lambda_- = N^*\{\tilde{\mu} = 0\} \cap \{\pm\xi > 0\},$$

with the sign inside the braces corresponding to that of $r_\pm$. This is consistent with our definition of $\Lambda_\pm$ in the de Sitter case. We let $L_\pm = \partial\Lambda_\pm \subset S^*X_\delta$. Since $\Lambda_+ \cup \Lambda_-$ is given by $\eta = \zeta = 0$, $\tilde{\mu} = 0$, $\Lambda_\pm$ are preserved by the *classical* dynamics (i.e. with $\sigma = 0$), but they are not radial (everywhere) if $a \neq 0$. Note that the special structure of $\tilde{p}$ is irrelevant for the purposes of this observation; only the quadratic vanishing at $L_\pm$ matters. Even for other local aspects of analysis, considered below, the only relevant part[69], is that $\mathsf{H}_p\tilde{p}$ vanishes cubically at $L_\pm$, which in some sense reflects the behavior of the linearization of $\tilde{p}$.

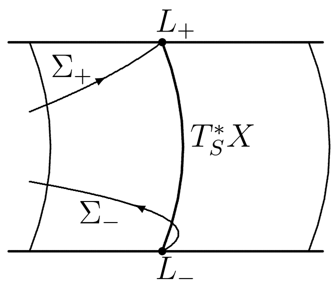

FIGURE 8. The cotangent bundle near the event horizon $S = \{\tilde{\mu} = 0\}$. It is drawn in a fiber-radially compactified view. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. The characteristic set crosses the event horizon on both components; here the part near L_+ is hidden from view. The projection of this region to the base space is the ergoregion. Semiclassically, i.e. the interior of $\overline{T}^* X_\delta$, for $z = h^{-1}\sigma > 0$, only $\Sigma_{\hbar,+}$ can enter $\tilde{\mu} > a^2$, see the paragraph after (6.15).

To analyze the dynamics near $L_\pm$ on the characteristic set, starting with the classical dynamics, we note that

$$\mathsf{H}_{\tilde{p}} r = 0,\ \mathsf{H}_{\tilde{p}}\xi = 0,\ \mathsf{H}_p\zeta = 0,\ \mathsf{H}_{\tilde{p}}\tilde{p} = 0,\ \mathsf{H}_p\tilde{p} = 0;$$

[69]This could be relaxed: quadratic behavior with small leading term would be fine as well; quadratic behavior follows from H_p being tangent to $\Lambda_\pm$; smallness is needed so that $\mathsf{H}_p|\xi|^{-1}$ can be used to dominate this in terms of homogeneous dynamics, so that the dynamical character of $L_\pm$ (sink/source) is as desired.

note that $\mathsf{H}_p\tilde{p} = 0$ and $\mathsf{H}_p\zeta = 0$ correspond to the integrability of the Hamiltonian dynamical system; these were observed by Carter [8] in the Kerr setting. Furthermore, with $|\xi|^{-1} = \tilde{\rho}$ in the notation of Subsection 2.2,

$$\mathsf{H}_p|\xi|^{-1}|_{S^*X_\delta} = -(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r}, \tag{6.8}$$

so at $\partial N^*\{\tilde{\mu} = 0\}$ it is given by $\pm(\operatorname{sgn}\xi)\digamma_\pm$, so $\beta_0 = \digamma_\pm$ which is bounded away from 0. We note that, with $\rho_0 = |\xi|^{-2}\tilde{p}$ in the notation of Subsection 2.2,

$$|\xi|^{-1}\mathsf{H}_p(|\xi|^{-2}\tilde{p})|_{S^*X_\delta} = -2(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r}\tilde{p}|\xi|^{-2}. \tag{6.9}$$

Since $\tilde{p} = 0$ and $\tilde{\mu} \neq 0$ implies $\xi = 0$ on the classical characteristic set (i.e. when we take $\sigma = 0$), which cannot happen on S^*X (we are away from the zero section!), this shows that the Hamilton vector field is non-radial except possibly at $\Lambda_\pm$. Moreover,

$$\mathsf{H}_p\left(\tilde{\mu} \mp 2(1+\gamma)a\frac{\zeta}{\xi}\right)|_{S^*X_\delta} = -2|\xi|\left(\tilde{\mu} \mp 2(1+\gamma)a\frac{\zeta}{\xi}\right)(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r};$$

as usual, this corresponds to $\hat{p} = |\xi|^{-2}p$ at $L_\pm$. Finally, the imaginary part of the subprincipal symbol at $L_\pm$ is

$$\begin{aligned}&((\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r})(\beta_\pm \operatorname{Im}\sigma)|\xi|, \text{ where}\\ &\beta_\pm = \mp 2\left(\frac{d\tilde{\mu}}{dr}\right)^{-1}(1+\gamma)(r^2+a^2)|_{r=r_\pm} = 2\digamma_\pm^{-1}(1+\gamma)(r_\pm^2+a^2) > 0;\end{aligned} \tag{6.10}$$

here $(\operatorname{sgn}\xi)\frac{\partial\tilde{\mu}}{\partial r}$ was factored out in view of (6.8), (2.5) and (2.3), so $\tilde{\beta}$ at $r_\pm$ is $\beta_\pm$.

Thus, L_+ is a sink, L_- a source. Furthermore, in the classical sense, $\xi = 0$ is disjoint from the characteristic set in the region of validity of the form (6.6) of the operator, as well as at the poles of the sphere (i.e. the only issue is when r is farther from $r_\pm$), so the characteristic set has two components there with $L_\pm$ lying in different components. We note that that as $\gamma < 1$, $\kappa\sin^2\theta = \sin^2\theta(1+\gamma) - \gamma\sin^4\theta$ has its maximum in $[0,\pi]_\theta$ at $\theta = \pi/2$, where it is 1. Since on the characteristic set

$$a^2\xi^2 + (1+\gamma)^2\zeta^2 \geq \pm 2(1+\gamma)a\xi\zeta = \tilde{p} + \tilde{\mu}\xi^2 \geq \eta^2 + (1+\gamma)^2\zeta^2 + \tilde{\mu}\xi^2 \tag{6.11}$$

and $\xi \neq 0$, we conclude that

$$\tilde{\mu} \leq a^2 \tag{6.12}$$

there, so this form of the operator remains valid, and the characteristic set can indeed be divided into two components, separating $L_\pm$.

Next, we note that if a is so large that at $r = r_0$ with $\frac{d\tilde{\mu}}{dr}(r_0) = 0$, one has $\tilde{\mu}(r_0) = a^2$, then letting $\eta_0 = 0$, $\theta_0 = \frac{\pi}{2}$, $\xi_0 \neq 0$, $\zeta_0 = \pm\frac{a}{1+\gamma}\xi_0$, the bicharacteristics through $(r_0, \theta_0, \phi_0, \xi_0, \eta_0, \zeta_0)$ are stationary for any ϕ_0, so the operator is classically trapping in the strong sense that not only is the Hamilton vector field radial, but it vanishes. Since such vanishing means that weights cannot give positivity in positive commutator estimates, see Section 2, it is natural to impose the restriction on a that

$$r_0 \in (r_+, r_-),\ \frac{d\tilde{\mu}}{dr}(r_0) = 0 \Rightarrow a^2 < \tilde{\mu}(r_0). \tag{6.13}$$

Under this assumption, by (6.12), the ergoregions from the two ends do not intersect.

Finally, we show that bicharacteristics leave the region $\tilde\mu > \tilde\mu_0$, where $\tilde\mu_0 < 0$ is such that $\frac{d\tilde\mu}{dr}$ is bounded away from 0 on $[\tilde\mu_0,(1+\epsilon)a^2]_{\tilde\mu}$ for some $\epsilon > 0$, which completes checking the hypotheses in the classical sense. Note that by (6.2) and (6.13) such $\tilde\mu_0$ and ϵ exists. To see this, we use $\tilde p$ to measure the size of the characteristic set over points in the base. Using $2ab \le (1+\epsilon)a^2 + b^2/(1+\epsilon)$ and $\kappa\sin^2\theta \le 1$, we note that on the characteristic set

$$(1+\epsilon)a^2\xi^2 + \frac{(1+\gamma)^2}{1+\epsilon}\zeta^2 \ge \tilde p + \tilde\mu\xi^2 \ge \frac{\epsilon}{1+\epsilon}\tilde p + \frac{(1+\gamma)^2}{1+\epsilon}\zeta^2 + \tilde\mu\xi^2,$$

so

$$((1+\epsilon)a^2 - \tilde\mu) \ge \frac{\epsilon}{1+\epsilon}|\xi|^{-2}\tilde p,$$

where now both sides are homogeneous of degree zero, or equivalently functions on S^*X_δ. Note that $\tilde p = 0$ implies that $\xi \ne 0$ on S^*X_δ, so our formulae make sense. By (6.9), using that $\frac{\partial\tilde\mu}{\partial r}$ is bounded away from 0, $|\xi|^{-2}\tilde p$ is growing exponentially in the forward/backward direction along the flow as long as the flow remains in a region $\tilde\mu \ge \tilde\mu_0$, where the form of the operator is valid (which is automatic in this region, as farther on 'our side' of the event horizon, X_+, where the form of the operator is not valid, it is elliptic), which shows that the bicharacteristics have to leave this region. As noted already, this proves that the operator fits into our framework in the classical sense.

6.4. **Semiclassical behavior.** The semiclassical principal symbol is

(6.14)

$$p_{\hbar,z} = -\tilde\mu\big(\xi \pm cz\big)^2 \mp 2(1+\gamma)(r^2+a^2)\big(\xi \pm cz\big)z \pm 2(1+\gamma)a\big(\xi \pm cz\big)\zeta - \tilde p_{\hbar,z}$$

with

$$\tilde p_{\hbar,z} = \kappa\eta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta z + \zeta)^2.$$

Recall now that $M_\delta = X_\delta \times [0,\infty)_\tau$, and that, due to Section 7, when we want to consider $\operatorname{Im}\sigma$ bounded away from 0, we need to choose c in our definition of τ so that $\frac{d\tau}{\tau}$ is time-like with respect to G. But

$$\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = -\tilde\mu c^2 - 2c(1+\gamma)(r^2+a^2) - \frac{a^2(1+\gamma)^2\sin^2\theta}{\kappa},$$

and as this must be positive for all θ, we need to arrange that

$$\tilde\mu c^2 + 2c(1+\gamma)(r^2+a^2) + a^2(1+\gamma)^2 < 0, \tag{6.15}$$

and this in turn suffices. Note that $c = -\tilde\mu^{-1}(1+\gamma)(r^2+a^2)$ automatically satisfies this in $\tilde\mu > 0$; this would correspond to undoing our change of coordinates in (6.5) (which is harmless away from $\tilde\mu = 0$, but of course c needs to be smooth at $\tilde\mu = 0$). At $\tilde\mu = 0$, (6.15) gives a (negative) upper bound for c; for $\tilde\mu > 0$ we have an interval of possible values of c; for $\tilde\mu < 0$ large negative values of c always work. Thus, we may choose a smooth function c such that (6.15) is satisfied everywhere, and we may further arrange that $c = -\tilde\mu^{-1}(1+\gamma)(r^2+a^2)$ for $\tilde\mu > \tilde\mu_1$ where $\tilde\mu_1$ is an arbitrary positive constant; in this case, as discussed in Section 7, $p_{\hbar,z}$ is semiclassically elliptic when $\operatorname{Im} z \ne 0$.

Note also that, as discussed in Subsection 3.2, there is only one component of the characteristic set in $\tilde\mu > a^2$ by (6.12), namely $\Sigma_{\hbar,\operatorname{sgn} z}$.

It remains to discuss trapping. Note that the dynamics depends continuously on a, with $a = 0$ being the de Sitter-Schwarzschild case, when there is no trapping near the event horizon, so the same holds for Kerr-de Sitter with slow rotation. Below we first describe the dynamics in de Sitter-Schwarzschild space explicitly, and then, in (6.27), give an explicit range of a in which the non-trapping dynamical assumption of Section 2, apart from hyperbolic trapping, is satisfied.

First, on de Sitter-Schwarzschild space, recalling that $c = c_\pm$ is irrelevant for the dynamics for real z, we may take $c_\pm = 0$, at least away from $r_\mp$ (i.e. otherwise we would simply change this calculation by the effect of a symplectomorphism, corresponding to a conjugation, which we note does not affect the 'base' variables on the cotangent bundle). Further, if we take a well-defined function on T^*X_δ (independently of c), such as r or ζ, and we consider objects such as $\mathsf{H}_{p_{\hbar,z}} r$ and $\mathsf{H}^2_{p_{\hbar,z}} r$, and whether the vanishing of the former on the characteristic set implies the positivity or negativity of the latter, any choice of $c_\pm$ can be used. Then

$$p_{\hbar,z} = -\tilde{\mu}\xi^2 \mp 2r^2\xi z - \tilde{p}_{\hbar,z}, \qquad \tilde{p}_{\hbar,z} = \eta^2 + \frac{\zeta^2}{\sin^2\theta},$$

so

$$\mathsf{H}_{p_{\hbar,z}} = -2(\tilde{\mu}\xi \pm r^2 z)\partial_r + \left(\frac{\partial\tilde{\mu}}{\partial r}\xi^2 \pm 4rz\xi\right)\partial_\xi - \mathsf{H}_{\tilde{p}_{\hbar,z}},$$

hence $\mathsf{H}_{p_{\hbar,z}} r = -2(\tilde{\mu}\xi \pm r^2 z)$, and so $\mathsf{H}_{p_{\hbar,z}} r = 0$ implies $\mp z = r^{-2}\tilde{\mu}\xi$. We first note that $\mathsf{H}_{p_{\hbar,z}} r$ cannot vanish in T^*X_δ in $\tilde{\mu} \le 0$ (though it can vanish at fiber infinity at $L_\pm$) since (for $z \neq 0$)

$$\tilde{\mu} \le 0 \text{ and } \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \tilde{\mu}\xi \neq 0 \text{ and } p_{\hbar,z} = \tilde{\mu}\xi^2 - \tilde{p}_{\hbar,z} < 0. \tag{6.16}$$

It remains to consider $\mathsf{H}_{p_{\hbar,z}} r = 0$ in $\tilde{\mu} > 0$. At such a point

$$\mathsf{H}^2_{p_{\hbar,z}} r = -2\tilde{\mu}\mathsf{H}_{p_{\hbar,z}}\xi = -2\tilde{\mu}\xi^2\left(\frac{\partial\tilde{\mu}}{\partial r} - 4r^{-1}\tilde{\mu}\right) = -2\tilde{\mu}\xi^2 r^4 \frac{\partial(r^{-4}\tilde{\mu})}{\partial r},$$

so as $\mp z = r^{-2}\tilde{\mu}\xi$, so $\xi \neq 0$, by the discussion after (6.2),

$$\tilde{\mu} > 0,\ \pm(r - \frac{3}{2}r_s) > 0,\ \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \pm\mathsf{H}^2_{p_{\hbar,z}} r > 0,$$

and thus the gluing hypotheses of [15] are satisfied arbitrarily close to[70] $r = \frac{3}{2}r_s$. Furthermore, as $p_{\hbar,z} = -\tilde{\mu}^{-1}(\tilde{\mu}\xi \pm r^2 z)^2 + \tilde{\mu}^{-1}r^4z^2 - \tilde{p}_{\hbar,z}$, if $r = \frac{3}{2}r_s$, $\mathsf{H}_{p_{\hbar,z}} r = 0$ and $p_{\hbar,z} = 0$ then $\tilde{p}_{\hbar,z} = \tilde{\mu}^{-1}r^4z^2$, so with

$$\Gamma_z = \{r = \frac{3}{2}r_s,\ \tilde{\mu}\xi \pm r^2 z = 0,\ \tilde{p}_{\hbar,z} = \tilde{\mu}^{-1}r^4z^2\},$$

we have

$$p_{\hbar,z}(\varpi) = 0,\ \tilde{\mu}(\varpi) > 0,\ \varpi \notin \Gamma_z,\ (\mathsf{H}_{p_{\hbar,z}} r)(\varpi) = 0 \Rightarrow (\pm\mathsf{H}^2_{p_{\hbar,z}} r)(\varpi) > 0, \tag{6.17}$$

with $\pm$ corresponding to whether $r > \frac{3}{2}r_s$ or $r < \frac{3}{2}r_s$. In particular, taking into account (6.16), r gives rise to an escape function in $T^*X_\delta \setminus \Gamma_z$ as discussed in Footnote 34, and Γ_z is the only possible trapping. (In this statement $L_\pm$ does not count as trapping.) To make this concrete, note that

(6.18)

$$\tilde{F} = (r - \frac{3}{2}r_s)^2 \Rightarrow \mathsf{H}_{p_{\hbar,z}}\tilde{F} = 2(r - \frac{3}{2}r_s)\mathsf{H}_{p_{\hbar,z}} r,\ \mathsf{H}^2_{p_{\hbar,z}}\tilde{F} = 2(\mathsf{H}_{p_{\hbar,z}} r)^2 + 2(r - \frac{3}{2}r_s)\mathsf{H}^2_{p_{\hbar,z}} r.$$

In particular, if $\mathsf{H}_{p_{\hbar,z}}\tilde{F} = 0$ then either $r = \frac{3}{2}r_s$, in which case $\mathsf{H}^2_{p_{\hbar,z}}\tilde{F} = 2(\mathsf{H}_{p_{\hbar,z}} r)^2$, which is positive unless $\mathsf{H}_{p_{\hbar,z}} r = 0$, or $\mathsf{H}_{p_{\hbar,z}} r = 0$ in which case $\mathsf{H}^2_{p_{\hbar,z}}\tilde{F} = 2(r -$

[70] Or far from, in $\tilde{\mu} > 0$.

$\frac{3}{2}r_s)\mathsf{H}^2_{p_{\hbar,z}}r$, which is positive on $\Sigma_\hbar$ unless $r = \frac{3}{2}r_s$. Thus, $p_{\hbar,z} = 0$ and $\mathsf{H}_{p_{\hbar,z}}\tilde{F} = 0$ imply that either the point in question is in Γ_z, or alternatively $\mathsf{H}^2_{p_{\hbar,z}}\tilde{F} > 0$. In particular, for any (null-)bicharacteristic γ outside Γ_z, any critical point of $\tilde{F}\circ\gamma$ is necessarily a strict local minimum, and in both the future or the past directions any bicharacteristic either escapes $r = r_\pm$ in finite time, tends to $L_\pm$[71], or tends to Γ_z. Moreover, it cannot tend to Γ_z in both directions without reaching $r_\pm$ since then $\tilde{F}\circ\gamma$ would have a local maximum. This shows that Γ_z is the only trapping in $[r_-, r_+]$, and indeed nearby as already discussed. As shown in Footnote 35, one can construct an escape function near Γ_z once it is known that Γ_z only exhibits normally hyperbolic trapping, which we prove below; further, with O and K as in the footnote, i.e. K the compact closure of a small neighborhood of Γ_z, O a small neighborhood of this, the dynamical parts of the semiclassical mild trapping hypotheses (namely (i) of the local definition plus the global flow assumptions stated after this) are satisfied in view of what we just showed regarding bicharacteristics outside Γ_z, once $q_{\hbar,z}$ is appropriately arranged in the next subsection.[72]

Since in [61, Section 2] Wunsch and Zworski only check normal hyperbolicity in Kerr space-times with sufficiently small angular momentum, in order to use their general results for normally hyperbolic trapped sets, we need to check that Kerr-de Sitter space-times are still normally hyperbolic. For this, with small a, we follow [61, Section 2], and note that for $a = 0$ the linearization of the flow at Γ_z in the normal variables $r - \frac{3}{2}r_s$ and $\tilde{\mu}\xi \pm r^2 z$ is

$$\begin{bmatrix} r-\frac{3}{2}r_s \\ \tilde{\mu}\xi \pm r^2 z\end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ -2(\frac{3}{2}r_s)^4 z^2 (\tilde{\mu}|^{-1}_{r=\frac{3}{2}r_s}) & 0\end{bmatrix}\begin{bmatrix} r-\frac{3}{2}r_s \\ \tilde{\mu}\xi \pm r^2 z\end{bmatrix} + \mathcal{O}((r-\frac{3}{2}r_s)^2 + (\tilde{\mu}\xi \pm r^2 z)^2),$$

so the eigenvalues of the linearization are $\lambda = \pm 3\sqrt{3}r_s z\big(1-\frac{9}{4}\Lambda r_s^2\big)^{-1/2}$, in agreement with the result of [61] when $\Lambda = 0$. The rest of the arguments concerning the flow in [61, Section 2] go through. In particular, when analyzing the flow *within* $\Gamma = \cup_{z>0}\Gamma_z$, the pull backs of both dp and $d\zeta$ are *exactly* as in the Schwarzschild setting (unlike the normal dynamics, which has different eigenvalues), so the arguments of

[71] By compactness considerations, $e^{C\tilde{F}}\mathsf{H}_{p_{\hbar,z}}\tilde{F}$ is an escape function outside Γ_z for $C > 0$ large, cf. Footnote 34, i.e. its $\mathsf{H}_{p_{\hbar,z}}$ is bounded below by a positive constant on compact subsets of $T^*X_0 \setminus \Gamma_z$. Correspondingly, in either the future or the past direction, bicharacteristics must either reach $r = r_\pm$ in finite time, or have a sequence of points tending to $L_\pm$ or Γ_z. In the latter two cases the whole bicharacteristic is easily seen to be tending to these in view of the local dynamics at $L_\pm$ (source/sink), and in case of Γ_z since $\tilde{F}\circ\gamma$ cannot have local maxima. Indeed, if there is a sequence, say, $t_n \to +\infty$ with $\gamma(t_n) \to \Gamma_z$ then $\tilde{F}(\gamma(t)) \to 0$ as $t \to +\infty$; if γ does not tend to Γ_z, then there is a sequence $s_n \to +\infty$ with $\gamma(s_n)$ bounded away from Γ_z, one can take a subsequence along which $\gamma(s_n)$ converges to a limit α, which is thus a point at which $\tilde{F} = 0$; then $\gamma|_{[s_n,s_n+1]}$ converges to the bicharacteristic through this point (along the subsequence), which is impossible since $\tilde{F}$ would have to be zero along this segment, but it can only have strict local minima away from Γ_z.

[72] A different way of phrasing the argument is to regard a compact interval I in $(r_-, \frac{3}{2}r_s) \cup (\frac{3}{2}r_s, r_+)$ as the gluing region, for sufficiently small a, for $r \in I$, $\mathsf{H}_{p_{\hbar,z}}r = 0$ still implies $\pm\mathsf{H}^2_{p_{\hbar,z}}r > 0$, and [15] is applicable. This concretely means that one uses Theorem 1 of [61] with an absorbing potential in the de Sitter-Schwarzschild setting with $K = \Gamma_z$ in the notation there, where it applies equally well given our observations regarding the dynamics, including normal hyperbolicity. If instead one works with compact subsets of $\{\tilde{\mu} > 0\} \setminus \Gamma_z$, one has non-trapping dynamics for a small, and the results still apply.

[61, Proof of Proposition 2.1] go through unchanged, giving normal hyperbolicity for small a by the structural stability.

We now check the hyperbolic nature of trapping for larger values of a. With $c = 0$, as above,

$$p_{\hbar,z} = -\tilde{\mu}\xi^2 \mp 2(1+\gamma)\big((r^2+a^2)z - a\zeta\big)\xi - \tilde{p}_{\hbar.z},\ \tilde{p}_{\hbar,z} = \kappa\eta^2 + \frac{(1+\gamma)^2}{\kappa\sin^2\theta}(-a\sin^2\theta z + \zeta)^2,$$

and in the region $\tilde{\mu} > 0$ this can be rewritten as

(6.19)

$$p_{\hbar,z} = -\tilde{\mu}\left(\xi \pm \frac{1+\gamma}{\tilde{\mu}}\big((r^2+a^2)z - a\zeta\big)\right)^2 + \frac{(1+\gamma)^2}{\tilde{\mu}}\big((r^2+a^2)z - a\zeta\big)^2 - \tilde{p}_{\hbar.z};$$

note that the first term would be just $-\tilde{\mu}\xi^2$ in the original coordinates (6.1) which are valid in $\tilde{\mu} > 0$. This explicitly shows that $(r^2 + a^2)z - a\zeta$ cannot vanish on $\Sigma_\hbar$ in $\tilde{\mu} > 0$, for if it did then $\tilde{p}_{\hbar,z}$ would vanish as well at the same point, thus $\zeta = a\sin^2\theta z$, hence $(r^2 + a^2)z - a^2\sin^2\theta z = 0$, which is impossible (as $z \neq 0$, $r > 0$). Further,

$$\tilde{\mu} > 0 \text{ and } p_{\hbar,z} = 0 \Rightarrow \big((r^2+a^2)z - a\zeta\big)^2 - \frac{\tilde{\mu}}{\kappa\sin^2\theta}(-a\sin^2\theta z + \zeta)^2 \geq 0, \tag{6.20}$$

so considering the last expression as a quadratic polynomial in ζ, we see that the inequality (ignoring the restriction $p_{\hbar,z} = 0$) can only hold between its roots if the coefficient $a^2 - \frac{\tilde{\mu}}{\kappa\sin^2\theta}$ of ζ^2 is negative, and outside the roots if the coefficient is positive, and has to be below ($az > 0$) or above ($az < 0$) the unique root if this coefficient vanishes (since the coefficient of ζ is the negative of that of az then). The roots of the polynomial (when $a^2 - \frac{\tilde{\mu}}{\kappa\sin^2\theta} \neq 0$) are[73]

$$\zeta_\pm = a\sin^2\theta z + \frac{(r^2 + a^2\cos^2\theta)z}{a \pm \frac{\sqrt{\tilde{\mu}}}{\sqrt{\kappa}\sin\theta}},$$

at which points

$$(r^2+a^2)z - a\zeta_\pm = \frac{(r^2 + a^2\cos^2\theta)z}{1 \mp \frac{a\sqrt{\kappa}\sin\theta}{\sqrt{\tilde{\mu}}}}; \tag{6.21}$$

so in particular when $\tilde{\mu} \geq a^2\kappa\sin^2\theta$ then $(r^2+a^2)z - a\zeta$ has the same sign as z on $\Sigma_\hbar$ as[74] it is in the interval between the two stated values of $(r^2+a^2)z - a\zeta$.

Next,

$$\mathsf{H}_{p_{\hbar.z}} r = -2\left(\tilde{\mu}\xi \pm (1+\gamma)\big((r^2+a^2)z - a\zeta\big)\right), \tag{6.22}$$

shows that

$$\tilde{\mu} \leq 0,\ \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow p_{\hbar,z} = \tilde{\mu}\xi^2 - \tilde{p}_{\hbar.z} \leq 0,$$

and equality on the right hand side implies $\zeta = a\sin^2\theta z$, so $\mathsf{H}_{p_{\hbar,z}} r = \mp 2(1+\gamma)(r^2 + a^2\cos^2\theta)z \neq 0$, a contradiction, showing that in $\tilde{\mu} \leq 0$, $\mathsf{H}_{p_{\hbar,z}} r$ cannot vanish on the characteristic set.

[73] Note that one of the roots tends to $-\infty$ if $az > 0$ and $\tilde{\mu} \searrow a^2\kappa\sin^2\theta$, and to $+\infty$ if $az < 0$ and $\tilde{\mu} \searrow a^2\kappa\sin^2\theta$.

[74] The other component, $\Sigma_{\hbar,-\operatorname{sgn} z}$ does intersect $\tilde{\mu} = a^2\kappa\sin^2\theta$, but only does so at fiber infinity which was already analyzed for the classical dynamics. This corresponds to the root of the quadratic polynomial in ζ that escaped to $\mp\infty$ and reemerges from $\pm\infty$ at $\tilde{\mu} = a^2\kappa\sin^2\theta$, depending on the sign of az.

We now turn to $\tilde\mu > 0$, where

$$\mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \mathsf{H}^2_{p_{\hbar,z}} r = -2\tilde\mu \mathsf{H}_{p_{\hbar,z}}\xi = 2\tilde\mu(1+\gamma)^2 \frac{\partial}{\partial r}\left(\tilde\mu^{-1}\big((r^2+a^2)z - a\zeta\big)^2\right).$$

Thus, we are interested in critical points of

$$\Phi = \tilde\mu^{-1}\big((r^2+a^2)z - a\zeta\big)^2$$

in $\tilde\mu > 0$ which lie at points with $p_{\hbar,z} = 0$ and $\mathsf{H}_{p_{\hbar,z}} r = 0$, and whether these are non-degenerate. Note that

$$\frac{\partial\Phi}{\partial r} = -\big((r^2+a^2)z - a\zeta\big)\tilde\mu^{-2} f, \qquad f = \big((r^2+a^2)z - a\zeta\big)\frac{\partial\tilde\mu}{\partial r} - 4r\tilde\mu z, \tag{6.23}$$

so as $(r^2+a^2)z - a\zeta \neq 0$ when $p_{\hbar,z}$ vanishes in $\tilde\mu > 0$,

$$\begin{aligned} &p_{\hbar,z} = 0 \text{ and } \tilde\mu > 0 \Rightarrow \Big(\frac{\partial\Phi}{\partial r} = 0 \Leftrightarrow f = 0\Big);\\ &\frac{\partial\Phi}{\partial r} = 0 \text{ and } p_{\hbar,z} = 0 \text{ and } \tilde\mu > 0 \Rightarrow \frac{\partial^2\Phi}{\partial r^2} = -\big((r^2+a^2)z - a\zeta\big)\tilde\mu^{-2}\frac{\partial f}{\partial r}. \end{aligned} \tag{6.24}$$

Also, from (6.23),

$$\tilde\mu > 0,\ f = 0 \Rightarrow \frac{\partial\tilde\mu}{\partial r} \neq 0,$$

and further, if $\frac{\partial\tilde\mu}{\partial r} < 0$ and $\tilde\mu > a^2\kappa\sin^2\theta$ then f cannot vanish since $(r^2+a^2)z - \zeta$ and z have the same sign. Thus $\mathsf{H}^2_{p_{\hbar,z}} r$ cannot vanish on $\Sigma_\hbar$ when $\tilde\mu > a^2\kappa\sin^2\theta$, $\frac{\partial\tilde\mu}{\partial r} < 0$ and $\mathsf{H}_{p_{\hbar,z}} r = 0$. Now

$$\frac{\partial f}{\partial r} = \big((r^2+a^2)z - a\zeta\big)\frac{\partial^2\tilde\mu}{\partial r^2} - 4\tilde\mu z - 2rz\frac{\partial\tilde\mu}{\partial r}, \tag{6.25}$$

and

$$\frac{\partial\Phi}{\partial r} = 0 \Rightarrow (r^2+a^2)z - a\zeta = \frac{4r\tilde\mu z}{\frac{\partial\tilde\mu}{\partial r}},$$

so substituting into (6.25),

$$\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r} = 4r\tilde\mu z\frac{\partial^2\tilde\mu}{\partial r^2} - 4z\tilde\mu\frac{\partial\tilde\mu}{\partial r} - 2rz\left(\frac{\partial\tilde\mu}{\partial r}\right)^2. \tag{6.26}$$

Thus,

$$\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r} = 2z\left(2\tilde\mu\left(r\frac{\partial^2\tilde\mu}{\partial r^2} - 3\frac{\partial\tilde\mu}{\partial r}\right) - (r\frac{\partial\tilde\mu}{\partial r} - 4\tilde\mu)\frac{\partial\tilde\mu}{\partial r}\right),$$

so taking into account

$$\begin{aligned} r\frac{\partial\tilde\mu}{\partial r} - 4\tilde\mu &= -2\left(1 - \frac{\Lambda a^2}{3}\right)r^2 + 3r_s r - 4a^2,\\ r\frac{\partial^2\tilde\mu}{\partial r^2} - 3\frac{\partial\tilde\mu}{\partial r} &= -4\left(1 - \frac{\Lambda a^2}{3}\right)r + 3r_s,\\ r\frac{\partial^2\tilde\mu}{\partial r^2} - 3\frac{\partial\tilde\mu}{\partial r} &= \frac{2}{r}\left(r\frac{\partial\tilde\mu}{\partial r} - 4\tilde\mu\right) - 3r_s + \frac{8a^2}{r}, \end{aligned}$$

we obtain

$$\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r} = 2z\left(-\frac{1}{r}\left(r\frac{\partial\tilde\mu}{\partial r} - 4\tilde\mu\right)^2 - \frac{2\tilde\mu}{r}(3r_s r - 8a^2)\right).$$

We claim that if $|a| < r_s/2$ then $r_- > r_s/2$. To see this, note that for $r = r_s/2$,

$$\tilde\mu(r) = \Big(\frac{r_s^2}{4} + a^2\Big)\Big(1 - \frac{\Lambda r_s^2}{12}\Big) - \frac{r_s^2}{2} < 0;$$

since at $a = 0$, $r_- > r_s/2$, we deduce that $r_- > r_s/2$ for $|a| < r_s/2$. Making the slightly stronger assumption,

$$|a| < \frac{\sqrt{3}}{4} r_s, \tag{6.27}$$

we obtain that for $\tilde\mu > 0$, $r > r_-$, $3r_s r - 8a^2 > \frac{3}{2}r_s^2 - 8a^2 > 0$, so, when $\frac{\partial\Phi}{\partial r} = 0$,

$$z\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r} < 0.$$

Thus, when $\frac{\partial\Phi}{\partial r} = 0$, using (6.24),

$$\frac{\partial^2\Phi}{\partial r^2} = -\big((r^2+a^2)z - a\zeta\big)\tilde\mu^{-2}\frac{\partial f}{\partial r} = -\frac{4r}{\tilde\mu\Big(\frac{\partial\tilde\mu}{\partial r}\Big)^2} z\frac{\partial\tilde\mu}{\partial r}\frac{\partial f}{\partial r} > 0, \tag{6.28}$$

so critical points of Φ are all non-degenerate and are minima. Correspondingly, as $\Phi \to +\infty$ as $\tilde\mu \to 0$ in $\tilde\mu > 0$, the critical point r_c of Φ exists and is unique in (r_-, r_+) (when ζ is fixed), depends smoothly on ζ, and $\frac{\partial\Phi}{\partial r} > 0$ if $r > r_c$, and $\frac{\partial\Phi}{\partial r} < 0$ if $r < r_c$. Thus,

$$\tilde\mu > 0,\ \pm(r - r_c) > 0,\ \mathsf{H}_{p_{\hbar,z}} r = 0 \Rightarrow \pm\mathsf{H}^2_{p_{\hbar,z}} r > 0,$$

giving the natural generalization of (6.17), allowing the application of the results of [15]. Since $\mathsf{H}_{p_{\hbar,z}} r$ cannot vanish in $\tilde\mu \le 0$ (apart from fiber infinity, which is understood already), we conclude that r gives rise to an escape function, as in Footnote 34, away from

$$\Gamma_z = \{\varpi :\ \frac{\partial\Phi}{\partial r}(\varpi) = 0,\ (\mathsf{H}_{p_{\hbar,z}} r)(\varpi) = 0,\ p_{\hbar,z}(\varpi) = 0\},$$

which is a smooth submanifold as the differentials of the defining functions are linearly independent on it in view of (6.28), (6.22), and the definition of $\tilde p_{\hbar,z}$ (as the latter is independent of r and ξ).

The linearization of the Hamilton flow at Γ_z is

$$\begin{aligned}&\begin{bmatrix} r - r_c \\ \tilde\mu\xi \pm (1+\gamma)\big((r^2+a^2)z - a\zeta\big)\end{bmatrix}' \\ &\quad= \begin{bmatrix} 0 & -2 \\ -\tilde\mu(1+\gamma)^2\frac{\partial^2\Phi}{\partial r^2} & 0\end{bmatrix}\begin{bmatrix} r - r_c \\ \tilde\mu\xi \pm (1+\gamma)\big((r^2+a^2)z - a\zeta\big)\end{bmatrix} \\ &\qquad + \mathcal{O}\Big((r-r_c)^2 + \big(\tilde\mu\xi \pm (1+\gamma)\big((r^2+a^2)z - a\zeta\big)\big)^2\Big),\end{aligned}$$

so by (6.28), the linearization is non-degenerate, and is indeed hyperbolic. This suffices for the resolvent estimates of [61] for exact Kerr-de Sitter space, but for stability one also needs to check normal hyperbolicity. While it is quite straightforward to check that the only degenerate location is $\eta = 0$, $\theta = \frac{\pi}{2}$, the computation of the Morse-Bott non-degeneracy in the spirit of [61, Proof of Proposition 2.1], where it is done for Kerr spaces with small angular momentum, is rather involved, so we do not pursue this here (for small angular momentum in Kerr-de Sitter space, the de Sitter-Schwarzschild calculation above implies normal hyperbolicity already).

One has the following analogue of (6.18) in the Kerr-de Sitter setting, using that $\mathsf{H}_{p_{\hbar,z}}\zeta = 0$, $r_c = r_c(\zeta)$,

$$\tilde{F} = (r - r_c)^2 \Rightarrow \mathsf{H}_{p_{\hbar,z}}\tilde{F} = 2(r - r_c)\mathsf{H}_{p_{\hbar,z}}r,\ \mathsf{H}^2_{p_{\hbar,z}}\tilde{F} = 2(\mathsf{H}_{p_{\hbar,z}}r)^2 + 2(r - r_c)\mathsf{H}^2_{p_{\hbar,z}}r.$$

All the arguments after (6.18) apply, showing that indeed Kerr-de Sitter space satisfies the dynamical parts of the semiclassical mild trapping property (namely (i) of the local definition plus the global flow assumptions stated after this).

In addition, in view of an overall sign difference between our convention and that of [61] for the operator we are considering, [61] requires the positivity of $z\frac{\partial}{\partial z}p_{\hbar,z}$ for $z \neq 0$. (Note that the notation for z is also different; our z is $1 + z$ in the notation of [61], so our z being near 1 corresponds to the z of [61] being near 0.) Unlike the flow, whose behavior is independent of c when z is real, this fact does depend on the choice of c. Note that in the high energy version, this corresponds to the positivity of $\sigma\frac{\partial}{\partial\sigma}p_{\mathrm{full}}$. Now, $p_{\mathrm{full}} = \langle\sigma\,\frac{d\tau}{\tau} + \varpi, \sigma\,\frac{d\tau}{\tau} + \varpi\rangle_G$, with $\varpi \in \Pi$, the 'spatial' hyperplane, identified with T^*X in ${}^{\mathrm{b}}T^*\bar{M}$, so

$$\begin{aligned}\sigma\partial_\sigma p_{\mathrm{full}} &= 2\langle\sigma\,\frac{d\tau}{\tau}, \sigma\,\frac{d\tau}{\tau}\rangle_G + 2\langle\sigma\,\frac{d\tau}{\tau}, \varpi\rangle_G\\ &= \sigma^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G + \langle\sigma\,\frac{d\tau}{\tau} + \varpi, \sigma\,\frac{d\tau}{\tau} + \varpi\rangle_G - \langle\varpi, \varpi\rangle_G.\end{aligned}$$

Thus, if non-zero elements of Π are space-like and $\frac{d\tau}{\tau}$ is time-like, $\sigma\partial_\sigma p_{\mathrm{full}} > 0$ for $\sigma \neq 0$ on the characteristic set of p_{full}. If c is such that $c = -\tilde{\mu}^{-1}(1+\gamma)(r^2+a^2)$ near the projection of Γ_z to the base space X, which as we mentioned can be arranged, and which corresponds to undoing our change of coordinates in (6.5), then directly from (6.1) the time-like statement holds; the space-like statement holds outside the ergoregion, i.e. when $\tilde{\mu} > a^2\kappa\sin^2\theta$ ($d\phi$ ceases to be spacelike in the ergoregion). For small $|a|$, Γ_z does not intersect the ergoregion, and thus the hypotheses of [61] are satisfied. Thus, to complete checking the hypotheses of [61] in general we need to show that even in $0 < \tilde{\mu} \leq a^2\kappa\sin^2\theta$, $z\partial_z p_{\hbar,z} > 0$ at Γ_z. Now,

$$\partial_z p_{\hbar,z} = 2(1+\gamma)^2\Big(\frac{r^2 + a^2}{\tilde{\mu}}\big((r^2 + a^2)z - a\zeta\big) + \frac{a}{\kappa}(\zeta - a\sin^2\theta z)\Big).$$

Further, on $\Sigma_\hbar$ (in $\tilde{\mu} > 0$) by (6.20),

$$\big|(r^2 + a^2)z - a\zeta\big| \geq \frac{\sqrt{\tilde{\mu}}}{\sqrt{\kappa}\sin\theta}|\zeta - a\sin^2\theta z|,$$

so, with the first inequality below due to $(r^2 + a^2)^2 > \tilde{\mu}a^2$ by the definition of $\tilde{\mu}$,

$$\frac{r^2 + a^2}{\tilde{\mu}} > \frac{|a|}{\sqrt{\tilde{\mu}}} \geq \frac{|a|}{\sqrt{\tilde{\mu}}\sqrt{\kappa}}\sin\theta$$

shows that $\partial_z p_{\hbar,z}$ has the same sign as $(r^2 + a^2)z - a\zeta$ in $\tilde{\mu} > 0$. Notice that by (6.23) $z((r^2 + a^2)z - a\zeta)$ and $\frac{\partial\tilde{\mu}}{\partial r}$ have the same sign on Γ_z; thus we only need to show that $\frac{\partial\tilde{\mu}}{\partial r}$ cannot be negative[75] on Γ_z. In view of (6.21), the negativity of $z((r^2 + a^2)z - a\zeta)$, together with $\tilde{\mu} \leq a^2\kappa\sin^2\theta$ and being on $\Sigma_\hbar$ would mean that

$$z((r^2 + a^2)z - a\zeta) < \frac{(r^2 + a^2\cos^2\theta)z^2}{1 - \frac{|a|\sqrt{\kappa}\sin\theta}{\sqrt{\tilde{\mu}}}}.$$

[75]We already remarked this outside the ergoregion, but here we need to consider the ergoregion.

Using (6.23) to substitute in for $(r^2+a^2)z-a\zeta$, we get

$$\frac{4r\tilde{\mu}}{\frac{\partial\tilde{\mu}}{\partial r}}<\frac{(r^2+a^2\cos^2\theta)}{1-\frac{|a|\sqrt{\kappa}\sin\theta}{\sqrt{\tilde{\mu}}}},$$

or multiplying through by $\frac{\partial\tilde{\mu}}{\partial r}(1-\frac{|a|\sqrt{\kappa}\sin\theta}{\sqrt{\tilde{\mu}}})>0$,

$$4r\tilde{\mu}(1-\frac{|a|\sqrt{\kappa}\sin\theta}{\sqrt{\tilde{\mu}}})<(r^2+a^2\cos^2\theta)\frac{\partial\tilde{\mu}}{\partial r},$$

or equivalently

$$\begin{aligned}0&<4r\sqrt{\tilde{\mu}}|a|\sqrt{\kappa}\sin\theta+r^2\frac{\partial\tilde{\mu}}{\partial r}-4r\tilde{\mu}+a^2\cos^2\theta\frac{\partial\tilde{\mu}}{\partial r}\\&=4r\sqrt{\tilde{\mu}}|a|\sqrt{\kappa}\sin\theta+r^6\frac{\partial}{\partial r}(r^{-4}(\tilde{\mu}-a^2))-4ra^2+a^2\cos^2\theta\frac{\partial\tilde{\mu}}{\partial r}\\&=r^6\frac{\partial}{\partial r}(r^{-4}(\tilde{\mu}-a^2))-4r|a|(|a|-\sqrt{\tilde{\mu}}\sqrt{\kappa}\sin\theta)+a^2\cos^2\theta\frac{\partial\tilde{\mu}}{\partial r}.\end{aligned}$$

As soon as we show that $\frac{\partial}{\partial r}(r^{-4}(\tilde{\mu}-a^2))<0$ in the ergoregion where $\frac{\partial\tilde{\mu}}{\partial r}<0$, we conclude that the right hand side is negative, providing a contradiction. To see this final claim we notice that $r^{-4}(\tilde{\mu}-a^2)$ has two zeros by assumption[76] in (r_-,r_+), hence its derivative has a zero between these zeros, which is thus outside the ergoregions[77]. Further, this derivative is $-2r^{-3}(1-\frac{\Lambda a^2}{3})+3r_sr^{-4}$ which has a single root, $r=\frac{3r_s}{2(1-\frac{\Lambda a^2}{3})}$, which thus lies outside the ergoregions, and thus in the ergoregion near r_+, where $\frac{\partial\tilde{\mu}}{\partial r}<0$, we have $\frac{\partial}{\partial r}(r^{-4}(\tilde{\mu}-a^2))<0$ (since it cannot change sign), providing the desired contradiction. This shows that $z\partial_z p_{\hbar,z}>0$ on Γ_z, completing our checking of the hypotheses of [61]. Wunsch and Zworski add a complex absorbing potential supported away from the projection of the trapped set, Γ_z, in the statement of Theorem 1; we can do so similarly[78] or we can use a pseudodifferential absorber, which is elliptic outside a neighborhood of Γ_z, including at fiber infinity. In summary, the result of Wunsch and Zworski is applicable for Kerr-de Sitter space-times with angular momenta satisfying (6.27).

6.5. **Complex absorption.** The final step of fitting P_σ into a general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. This section is *almost completely parallel* to Subsection 4.7 in the de Sitter case; the only change is that absorption needs to be added at the trapped set as well.

We thus consider a compact manifold without boundary X for which X_δ is identified as an open subset with smooth boundary; we can again take X to be the double of X_δ. As in the de Sitter case, we discuss the 'classical' and 'semiclassical' cases separately, for in the former setting trapping does not matter, while in the latter it does.

[76]By (6.13) there are at least two zeros; in view of $\tilde{\mu}$ having a single critical point between (r_-,r_+), there are exactly two zeros.

[77]Since in the ergoregions $r^{-4}(\tilde{\mu}-a^2)<0$.

[78]There is a slight complication if the projection of Γ_z enters the ergoregion as the operator ceases to be elliptic, though the latter is assumed by Wunsch and Zworski; in this case one needs a pseudodifferential absorber, which however barely affects their arguments.

We then introduce a complex absorbing operator $Q_\sigma \in \Psi^2_{\mathrm{cl}}(X)$ with principal symbol q, such that $h^2 Q_{h^{-1}z} \in \Psi^2_{\hbar,\mathrm{cl}}(X)$ with semiclassical principal symbol $q_{\hbar,z}$, and such that $p \pm \imath q$ is elliptic near ∂X_δ, i.e. near $\tilde\mu = \tilde\mu_0$, the Schwartz kernel of Q_σ is supported in $\tilde\mu < \tilde\mu_0 + \epsilon'$ for some sufficiently small $\epsilon' > 0$, and which satisfies that $\pm q \geq 0$ on $\Sigma_\mp$. Having done this, we extend P_σ and Q_σ to X in such a way that $p \pm \imath q$ are elliptic near $X \setminus X_\delta$; the region we added is thus irrelevant. In particular, as the event horizon is characteristic for the wave equation, the solution in the exterior of the event horizons is *unaffected* by thus modifying P_σ, i.e. working with P_σ and $P_\sigma - \imath Q_\sigma$ is equivalent for this purpose.

As in de Sitter space, an alternative to this extension is adding a boundary at $\tilde\mu = \tilde\mu_0$; this is easy to do since this is a space-like hypersurface, see Remark 2.6.

For the semiclassical problem, when z is real we need to increase the requirements on Q_σ. As in the de Sitter setting, discussed in Subsection 4.7, we need in addition, in the semiclassical notation, semiclassical ellipticity near $\tilde\mu = \tilde\mu_0$, i.e. that $p_{\hbar,z} \pm \imath q_{\hbar,z}$ are elliptic near ∂X_δ, i.e. near $\tilde\mu = \tilde\mu_0$, and which satisfies that $\pm q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\mp}$. Following the general prescription of Subsection 3.2, as well as the discussion of Subsection 4.7, this can be achieved by taking Q_σ the (standard) quantization of

$$-\langle \xi\,dr + \sigma\frac{d\tau}{\tau} + \eta\,d\theta + \zeta\,d\phi, \frac{d\tau}{\tau}\rangle_G\,(\|\xi\,dr + \eta\,d\theta + \zeta\,d\phi\|^{2j}_{\tilde H} + \sigma^{2j} + C^{2j})^{1/2j}\chi(\mu), \tag{6.29}$$

where $\tilde H$ is a *Riemannian* dual metric on X, $\chi \geq 0$ as in Subsection 4.7 supported near $\tilde\mu = \tilde\mu_0$, $C > 0$ is chosen suitably large, and the branch of the $2j$th root as in Subsection 4.7. One can again combine p with a Riemannian metric function $\|.\|^2_{\tilde H}$, to replace p by $\chi_1 p - \chi_2 \hat p_{\hbar,z}$, $\hat p_{\hbar,z} = (\|.\|^{2j}_{\tilde H} + z^{2j})^{1/j}$, as in Subsection 4.7.

We recall that in the proof of Theorem 2.17 one also need to arrange semiclassical ellipticity (i.e. define an appropriate Q'_σ) for an appropriate perturbation of $p_{\hbar,z}$ at the trapped set (for z real, as usual), which is in X_+; we now make this more explicit. To achieve this, we want $q'_{\hbar,z}$ elliptic on the trapped set; since this is in $\Sigma_{\hbar,\mathrm{sgn}\,z}$, we need $q'_{\hbar,z} \leq 0$ there. To do so, we simply add a microlocal absorbing term Q'_σ supported microlocally near the trapping with $h^2 Q_{h^{-1}z}$ having semiclassical principal symbol $q'_{\hbar,z}$. We *do not* need to arrange that Q'_σ is holomorphic in σ; thus simply quantizing a $q'_{\hbar,z}$ of compact support on T^*X and with smooth dependence on $z \in \mathbb{R}\setminus\{0\}$ suffices. Then with Q_σ as above (so we do not change Q_σ) $P_\sigma - \imath Q_\sigma$ is holomorphic, and its inverse is meromorphic, with non-trapping large energy estimates in closed cones disjoint from $\mathbb{R}$ in the upper half plane, corresponding to the semiclassical estimates for non-real z given in Theorem 7.3. To see that large energy estimates also hold for $(P_\sigma - \imath Q_\sigma)^{-1}$ near $\mathbb{R}$, namely in $\operatorname{Im}\sigma > -C$, one considers $(P_\sigma - \imath(Q_\sigma + Q'_\sigma))^{-1}$, which enjoys such estimates but is not holomorphic, and then use the semiclassical resolvent gluing of Datchev-Vasy [15] together with the semiclassical normally hyperbolic trapping estimates of Wunsch-Zworski [61], to conclude the same estimates for $(P_\sigma - \imath Q_\sigma)^{-1}$. For a as in (6.27), the dynamics (away from the radial points) has only the hyperbolic trapping (and for small a, it is normally hyperbolic); however, our results apply more generally, as long as the dynamics has the same non-trapping character (so a might be even larger as (6.27) may not be optimal). Note also that since the trapping is in a compact subset of $X_+ = \{\tilde\mu > 0\}$, we arranged that the complex absorption $Q_\sigma + Q'_\sigma$ is the sum of two terms: one supported near the trapping in X_+, the other in $\tilde\mu < 0$; this is useful for relating our construction to that of Dyatlov [20] in the appendix.

This completes the setup. Now all of the results of Section 2 are applicable, proving all the theorems stated in the introduction on Kerr-de Sitter spaces, Theorems 1.1-1.4. Namely, Theorem 1.1 follows from Theorem 2.14, Theorem 1.2 follows from Theorem 7.3, Theorem 1.3 follows from Theorem 2.17. Further, $d\tilde\mu$ is time-like in $\tilde\mu < 0$ (since p_{full}, considered as a quadratic form, evaluated at $\xi = 1$, $\sigma = 0$, $\zeta = 0$, $\eta = 0$, is positive then), so Proposition 3.9 is applicable. This, together with Theorem 1.3, the Mellin transform result, Corollary 3.10, in the Kerr-de Sitter setting, or the appropriately modified, as indicated in Remark 3.11, version of Proposition 3.5 for general b-perturbations (so $\partial_{\tilde t}$ may no longer be Killing, and the space-time may no longer be stationary), keeping in mind Footnote 63 for the zero-resonance, finally proves Theorem 1.4.

7. Large $\mathrm{Im}\,\sigma$

In this section we discuss the extensions of the results to large $\mathrm{Im}\,\sigma$, i.e. uniform estimates when $\mathrm{Im}\,\sigma > C_- > 0$, which only enter in the semiclassical context.

7.1. Semiclassical estimates.

We now assume $\mathrm{Im}\,\sigma > C_- > 0$; as before we translate this into a semiclassical problem, i.e. obtain families of operators $P_{h,z}$, with $h = |\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. As usual, we multiply through by h^k for convenient notation when we define $P_{h,z}$:

$$P_{h,z} = h^k P_{h^{-1}z} \in \Psi^k_{\hbar,\mathrm{cl}}(X).$$

Now z need not be real in the limit $h \to 0$ since we dropped the upper bound on $\mathrm{Im}\,\sigma$. We still have $p_{\hbar,z}, q_{\hbar,z}$, $z \in O \subset \mathbb{C}$, $0 \notin \overline{O}$ compact, real at $S^*X = \partial\overline{T}^*X$, but we do not assume reality on T^*X. However, we assume that $p_{\hbar,z}$ and $q_{\hbar,z}$ are real when z is real. We write the semiclassical characteristic set of $\mathrm{Re}\,p_{\hbar,z}$ as $\Sigma_{\hbar,z}$, and sometimes drop the z dependence and write $\Sigma_\hbar$ simply; assume that

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-},\ \Sigma_{\hbar,+} \cap \Sigma_{\hbar,-} = \emptyset,$$

$\Sigma_{\hbar,\pm}$ are relatively open in $\Sigma_\hbar$, and

$$\pm\,\mathrm{Im}\,p_{\hbar,z} \geq 0 \text{ and } \mp\,\mathrm{Re}\,q_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm}.$$

The microlocal elliptic, real principal type and complex absorption (in which one considers the bicharacteristics of $\mathrm{Re}\,p_{\hbar,z}$) estimates apply. The radial point estimates need a bit more care, however.

Proposition 7.1. *For all N, for $s \geq m > (k-1)/2 - \beta\,\mathrm{Im}\,\sigma$, $\sigma = h^{-1}z$, and for all $A, B, G \in \Psi^0_\hbar(X)$ such that $\mathrm{WF}'_h(G) \cap \mathrm{WF}'_h(Q_\sigma) = \emptyset$, A elliptic at $L_\pm$, and forward (or backward) bicharacteristics from $\mathrm{WF}'_h(B)$ tend to $L_\pm$, with closure in the elliptic set of G, one has estimates*

$$Au \in H^m_h \Rightarrow \|Bu\|_{H^s_h} \leq C(h^{-1}\|GP_{h,z}u\|_{H^{s-k+1}_h} + h\|u\|_{H^{-N}_h}), \tag{7.1}$$

where, as usual, $GP_{h,z}u \in H^{s-k+1}_h$ and $u \in H^{-N}_h$ are assumptions implied by the right hand side.

Proposition 7.2. *For $s < (k-1)/2 + \beta\,\mathrm{Im}\,\sigma$, for all N, $\sigma = h^{-1}z$, and for all $A, B, G \in \Psi^0_\hbar(X)$ such that $\mathrm{WF}'_h(G) \cap \mathrm{WF}'_h(Q_\sigma) = \emptyset$, B, G elliptic at $L_\pm$, and* *forward (or backward) bicharacteristics from* $\mathrm{WF}'_h(B) \setminus L_\pm$ *reach* $\mathrm{WF}'_h(A)$*, while remaining in the elliptic set of* G*, one has estimates*

$$\|Bu\|_{H^s_\hbar} \le C(h^{-1}\|GP^*_{h,z}u\|_{H^{s-k+1}_\hbar} + \|Au\|_{H^s_\hbar} + h\|u\|_{H^{-N}_\hbar}). \tag{7.2}$$

Proof. For the sake of definiteness, we consider the proof of Proposition 7.1; the changes relative to Propositions 2.10-2.11 in the two cases are completely analogous.

We follow Propositions 2.10-2.11; in particular the commutant family C_ϵ, which is bounded $\Psi_\hbar^{s-(k-1)/2}(X)$, and its principal symbol c_ϵ, built using a functions ϕ (on which we make a further assumption below, shrinking its support further) and ϕ_0, are defined as there; in the case of ϕ_0 we take $\phi_0(\rho^k \operatorname{Re} p_{\hbar,z})$ instead of $\phi_0(\rho^k p_{\hbar,z})$ now. The difference in the rest of the proof is that $\frac{1}{2\imath}(P_{h,z} - P^*_{h,z}) \in \Psi_\hbar^{k-1}(X)$ is not necessarily lower order in the semiclassical sense than $P_{h,z}$, i.e. the semiclassical principal symbol of $P_{h,z}$ is not real, though at $S^*X = \partial\overline{T}^*X$ it is, so in the differential order sense it is still lower order than $P_{h,z}$. However, if $\operatorname{Im} z > C'h$, $C' > 0$ fixed (to be determined later), which we may assume in view of Propositions 2.10-2.11, we combine the argument of the proofs of Propositions 2.3-2.4 with that of Subsection 2.5, writing

$$\imath(P^*_{h,z}C^*_\epsilon C_\epsilon - C^*_\epsilon C_\epsilon P_{h,z}) = 2\frac{1}{2\imath}(P_{h,z} - P^*_{h,z})C^*_\epsilon C_\epsilon + \imath[P_{h,z}, C^*_\epsilon C_\epsilon], \tag{7.3}$$

with the first term in $\Psi_\hbar^{k-1}(X)$, the second in $h\Psi_\hbar^{k-1}(X)$. Denoting the semiclassical principal symbol of $\frac{1}{2\imath}(P_{h,z} - P^*_{h,z})$ by $p^\flat_{\hbar,z}$, we have $p^\flat_{\hbar,z} = \pm\tilde\beta\beta_0(\operatorname{Im} z)\tilde\rho^{-k+1}$ at L_+ by (2.5), so $\pm p^\flat_{\hbar,z}$ is a positive elliptic multiple of $\operatorname{Im} z$ at L_+. But $p_{\hbar,z}$ is real for z real, and thus $\pm p^\flat_{\hbar,z} = (\operatorname{Im} z)\tilde p^\flat_{\hbar,z}$, so $\tilde p^\flat_{\hbar,z}$ is *positive* elliptic at L_+ and thus nearby. Since the standard principal symbol of $\frac{1}{2\imath}(P_{h,z} - P^*_{h,z})$ is (2.5), we can write

$$\begin{gathered}\frac{1}{2\imath}(P_{h,z} - P^*_{h,z}) = \pm(\operatorname{Im} z)\tilde B^*_{h,z}\tilde B_{h,z} + hR_{h,z} + T_{h,z},\\ B_{h,z} \in \Psi_\hbar^{(k-1)/2}(X),\ R_{h,z} \in \Psi_\hbar^{k-2}(X),\ T_{h,z} \in \Psi_\hbar^{k-1}(X),\ \mathrm{WF}'_h(T) \cap L_+ = \emptyset,\end{gathered}$$

with

$$\sigma_{\hbar,(k-1)/2}(B_0) = \sqrt{\tilde p^\flat_{\hbar,z}}.$$

Now we write the right hand side of (7.3) as

$$\begin{aligned}&\pm 2(\operatorname{Im} z - C'h)\tilde B^*_{h,z}\tilde B_{h,z}C^*_\epsilon C_\epsilon + 2T_{h,z}C^*_\epsilon C_\epsilon\\ &\qquad + \Big(h(\pm 2C'\tilde B^*_{h,z}\tilde B_{h,z} + R_{h,z})C^*_\epsilon C_\epsilon + \imath[P_{h,z}, C^*_\epsilon C_\epsilon]\Big).\end{aligned} \tag{7.4}$$

The $\mp$ of the first term in (7.4) is then negative when $\operatorname{Im} z > C'h$, the second term is in $h^\infty\Psi_\hbar^{-\infty}(X)$ if ϕ and thus c_ϵ have sufficiently small support by the wave front set property of $T_{h,z}$ (and is thus negligible), while the third term is in $h\Psi_\hbar^{2s}(X)$ uniformly. Now, $\mp$ of the third term has principal symbol in $h\Psi_\hbar^{k-1}(X)$, modulo terms where ϕ or ϕ_0 is differentiated (and which behave just as in the classical and real z semiclassical settings), is

$$2h\left(-\tilde\beta\beta_0 C' \pm r_{\hbar,z} + \beta_0\left(-s + \frac{k-1}{2}\right) + \delta\beta_0\frac{\epsilon}{\tilde\rho + \epsilon}\right)\phi^2\phi_0^2\tilde\rho^{-2s}(1 + \epsilon\tilde\rho^{-1})^{-2\delta}.$$

Choosing $C' > 0$ sufficiently large, all other terms in the parentheses are dominated by it on $\operatorname{supp}\phi$, and the argument can be finished as in Proposition 2.3. $\square$

Without assuming semiclassical non-trapping (which is a real z property), but under extra assumptions, giving semiclassical ellipticity for $\operatorname{Im} z$ bounded away from 0, we now show non-trapping estimates. So assume that for $|\operatorname{Im} z| > \epsilon > 0$, $p_{\hbar,z}$ is semiclassically elliptic on T^*X (but not necessarily at $S^*X = \partial\overline{T}^*X$, where the standard principal symbol p already describes the behavior). In addition, assume that $\pm \operatorname{Im} p_{\hbar,z} \geq 0$ near the classical characteristic set $\Sigma_{\hbar,\pm} \subset S^*X$. Assume also that $p_{\hbar,z} - \imath q_{\hbar,z}$ is elliptic[79] on T^*X for $|\operatorname{Im} z| > \epsilon > 0$, $h^{-1}z \in \Omega$, and $\mp \operatorname{Re} q_{\hbar,z} \geq 0$ near the classical characteristic set $\Sigma_{\hbar,\pm} \subset S^*X$. Then the semiclassical version of the classical results (with ellipticity in T^*X making these trivial except at S^*X) apply. Let H^s_h denote the usual semiclassical function spaces. Then, on the one hand, for any $s \geq m > (k-1)/2 - \beta \operatorname{Im} z/h$, $h < h_0$,

$$\|u\|_{H^s_h} \leq Ch^{-1}(\|(P_{h,z} - \imath Q_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H^m_h}), \tag{7.5}$$

and on the other hand, for any N and for any $s < (k-1)/2 + \beta \operatorname{Im} z/h$, $h < h_0$,

$$\|u\|_{H^s_h} \leq Ch^{-1}(\|(P^*_{h,z} + \imath Q^*_{h,z})u\|_{H_h^{s-k+1}} + h^2\|u\|_{H_h^{-N}}). \tag{7.6}$$

The h^2 term can be absorbed in the left hand side for sufficiently small h, so we automatically obtain invertibility of $P_{h,z} - \imath Q_{h,z}$.

In particular, $P_{h,z} - \imath Q_{h,z}$ is invertible for $h^{-1}z \in \Omega$ with $\operatorname{Im} z > \epsilon > 0$ and h small, i.e. $P_\sigma - \imath Q_\sigma$ is such for $\sigma \in \Omega$ in a cone bounded away from the real axis with $\operatorname{Im}\sigma$ sufficiently large, proving the meromorphy of $P_\sigma - \imath Q_\sigma$ under these extra assumptions[80].

Theorem 7.3. *Let P_σ, Q_σ, β, $\mathbb{C}_s$ be as above, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in* (2.22)*. Then, for $\sigma \in \mathbb{C}_s \cap \Omega$,*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

has a meromorphic inverse

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s.$$

Moreover, for all $\epsilon > 0$ there is $C > 0$ such that it is invertible in $\operatorname{Im}\sigma > C + \epsilon|\operatorname{Re}\sigma|$, $\sigma \in \Omega$, and non-trapping estimates *hold:*

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-k+1}\|f\|_{H^{s-1}_{|\sigma|^{-1}}}.$$

Remark 7.4. We emphasize again that the large $\operatorname{Im}\sigma$ behavior of $P_\sigma - \imath Q_\sigma$ does not matter for our main results, except the support conclusion of the existence part of Lemma 3.1, and the analogous statement in its consequences, Proposition 3.5 and Corollary 3.10. In particular, when the solution is known to exist in a weighted b-Sobolev space, the large $\operatorname{Im}\sigma$ behavior is not used at all; for existence the only loss would be that the solution would not have the stated support property (which is desirable to have in the wave equation setting).

Theorem 2.14 has a generalization now under the non-trapping assumptions of Definition 2.12, extending beyond a strip to a half-space:

[79] We need to assume this since $p_{\hbar,z}, q_{\hbar,z}$ are not real, so the ellipticity of $p_{\hbar,z}$ does not imply this. Arranging this in the Lorentzian setting is the reason for an extended argument starting with the paragraph of (3.17).

[80] Recall that we are not assuming semiclassical non-trapping here, which is the reason we cannot simply quote the relevant part of Theorem 2.14 for the meromorphy.

Theorem 7.5. *Let P_σ, Q_σ, $\mathbb{C}_s$, β be as in Theorem 2.14, in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in (2.22). Let $C > 0$. Then there exists σ_0 such that*

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is holomorphic in $\{\sigma \in \Omega : \ \operatorname{Im}\sigma > -C, \ |\operatorname{Re}\sigma| > \sigma_0\}$, assumed to be a subset of $\mathbb{C}_s$, and non-trapping estimates *hold:*

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-k+1}\|f\|_{H^{s-k+1}_{|\sigma|^{-1}}}.$$

Remark 7.6. An advantage of the present theorem over Theorem 2.14 is that while the former ensures that only finitely many poles can lie in any strip $C_- < \operatorname{Im}\sigma < C_+$, there is no need for this statement to hold if we allow $\operatorname{Im}\sigma > -C$. Since, for the application to the wave equation, $\operatorname{Im}\sigma$ depends on the a priori growth rate of the solution u which we are Mellin transforming, this would mean that depending on the a priori growth rate one could get more (faster growing) terms in the expansion of u if one relaxes the growth condition on u.

Theorems 2.15 and 2.17 also have analogous extensions to a half-space.

7.2. **Lorentzian metrics.** As discussed in Subsection 3.2, whose notation we adopt, the semiclassical principal symbol of $P_{\hbar,z} = h^2 P_{h^{-1}z}$ is the dual metric G on the complexified cotangent bundle ${}^{\mathrm{b},\mathbb{C}}T^*_m\bar{M}$, $m = (x,\tau)$, evaluated on covectors $\varpi + z\,\frac{d\tau}{\tau}$, where ϖ is in the (real) span Π of the 'spatial variables' $T^*_x X$; thus Π and $\frac{d\tau}{\tau}$ are linearly independent. In general, by (3.15), for $\operatorname{Im} z \neq 0$, the vanishing of the imaginary part of this principal symbol states that $\langle \varpi + \operatorname{Re} z\,\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G = 0$; the real part is the first two terms on the right hand side of (3.15).

In the setting of Subsection 7.1 we want that when $\operatorname{Im} z \neq 0$ and $\operatorname{Im} p_{\hbar,z}$ vanishes then $\operatorname{Re} p_{\hbar,z}$ does not vanish, i.e. that on the orthocomplement of the span of $\frac{d\tau}{\tau}$ the metric should have the opposite sign as that of $\langle \frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$. For a Lorentzian metric this is only possible if $\frac{d\tau}{\tau}$ is time-like (note that $\varpi + \operatorname{Re} z\,\frac{d\tau}{\tau}$ spans the whole fiber of the b-cotangent bundle as $\operatorname{Re} z$ and $\varpi \in \Pi$ vary), when, however, this is automatically the case, namely the metric is negative definite on this orthocomplement. *From now on we always assume that $\frac{d\tau}{\tau}$ is time-like for G.*

Recalling (3.16), when $\operatorname{Im} z \geq 0$, the sign of the imaginary part of $p_{\hbar,z}$ on $\Sigma_{\hbar,\pm}$ is given by $\pm \operatorname{Im} p_{\hbar,z} \geq 0$, as needed for the propagation of estimates: in $\Sigma_{\hbar,+}$ we can propagate estimates backwards, in $\Sigma_{\hbar,-}$ we can propagate estimates forward. For $\operatorname{Im} z \leq 0$, the direction of propagation is reversed.

We also need information about $p_{\hbar,z} - \imath q_{\hbar,z}$, i.e. when the complex absorption has been added, with $q_{\hbar,z}$ defined for z in an open set $\tilde{\Omega} \subset \mathbb{C}$. Here we need to choose $q_{\hbar,z}$ in such a way as to ensure that $p_{\hbar,z} - \imath q_{\hbar,z}$ does not vanish when $\operatorname{Im} z > 0$, but for real $z \neq 0$, $q_{\hbar,z}$ is real and for z sufficiently close to $\mathbb{R}$ with $\operatorname{Im} z \geq 0$, $\mp \operatorname{Re} q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$, and we also need ellipticity $p_{\hbar,z} - \imath q_{\hbar,z}$ where $q_{\hbar,z}$ is to act as absorption. We follow (3.17) and take

$$q_{\hbar,z} = -\chi f_z \langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G, \qquad \operatorname{Re} f_z \geq 0,\ z \in \mathbb{R} \Rightarrow f_z \text{ is real}, \qquad \chi \geq 0,\ \text{independent of } z; \tag{7.7}$$

recall from the discussion after (3.17) that if in addition f_z is bounded away from 0 when z is bounded away from 0 in $\mathbb{R}$, then the above conditions for real z are automatically satisfied, including ellipticity of $p_{\hbar,z} - \imath q_{\hbar,z}$ for z real where $\chi > 0$, which includes ellipticity in the classical sense – the latter thus holds for all z since the standard principal symbol is independent of z.

Thus, if we ensure that for $\operatorname{Im} z > 0$,

$$\operatorname{Im}\left(\langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^{-1} p_{\hbar,z}\right) > 0, \tag{7.8}$$

then $\operatorname{Im}(\imath f) \geq 0$ shows that $p_{\hbar,z} - \imath q_{\hbar,z} \neq 0$ as desired. We note that the imaginary part of $\langle \varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$ is $\operatorname{Im} z\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G$, and thus is non-zero as $\frac{d\tau}{\tau}$ is time-like and $\operatorname{Im} z > 0$. But the expression inside the imaginary part on the left hand side of (7.8) is a positive multiple of $\langle \varpi + \overline{z}\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G p_{\hbar,z}$, so it suffices to consider the latter, whose imaginary part is, with $\beta = \varpi + \operatorname{Re} z\frac{d\tau}{\tau}$,

$$\begin{aligned} &-\operatorname{Im} z\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G\left(\langle\beta,\beta\rangle_G - (\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G\right) + \langle\beta, \frac{d\tau}{\tau}\rangle_G(2\operatorname{Im} z)\langle\beta, \frac{d\tau}{\tau}\rangle_G \\ &= \operatorname{Im} z\left(-\langle\beta,\beta\rangle_G\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G + 2\langle\beta, \frac{d\tau}{\tau}\rangle_G^2 + (\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^2\right). \end{aligned} \tag{7.9}$$

As $\frac{d\tau}{\tau}$ time-like, the first two terms inside the parentheses on the right hand side give twice the positive definite stress-energy tensor; the positive definite character is checked by writing $\beta = \gamma + \lambda\frac{d\tau}{\tau}$ with[81] $\langle\gamma, \frac{d\tau}{\tau}\rangle_G = 0$, for then these terms give

$$\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G\left(-\langle\gamma,\gamma\rangle_G + \lambda^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G\right), \tag{7.10}$$

and the Lorentzian character of G then implies that $-G$ is positive definite on the orthocomplement of the span of $\frac{d\tau}{\tau}$. In view of the third term, $(\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^2$, on the right hand side of (7.9), we actually conclude that (7.8) holds (i.e. the inequality is strict) when $\operatorname{Im} z > 0$.

In extending the discussion around (3.19) regarding the ellipticity of the extended operator, $\tilde{p}_{\hbar,z} - \imath q_{\hbar,z}$, to complex z, it remains to consider finite ϖ and non-real z, and show that $\tilde{p}_{\hbar,z} - \imath q_{\hbar,z}$ does not vanish then; we have a priori that for $K \subset \mathcal{D}_j$ compact (thus disjoint from the branch cuts), there exist $C > 0$ and $\delta > 0$ such that for $z \in K$ either one of $\|\varpi\|_{\tilde{H}} \geq C$, resp. $|\operatorname{Im} z| \leq \delta$, implies non-vanishing of $\tilde{p}_{\hbar,z} - \imath q_{\hbar,z}$. To see the remaining cases, i.e. when both $\operatorname{Im} z > \delta$ and $\|\varpi\|_{\tilde{H}} < C$, first assume that $\chi_2 = 1$, so $\chi_1 = 0$. Then

$$\tilde{p}_{\hbar,z} - \imath q_{\hbar,z} = f_z\left(-\chi_2 f_z + \imath\chi\langle\varpi + z\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G\right),$$

and the real part of the second factor on the right hand side is

$$-\chi_2 \operatorname{Re} f_z - (\operatorname{Im} z)\chi\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G,$$

which is < 0 as $\chi_2 > 0$ and $\operatorname{Im} z \geq 0$, showing ellipticity.

Now, if neither χ_1 nor χ_2 vanish, then $\chi > 0$. First, suppose that, with $\beta = \varpi + \operatorname{Re} z\frac{d\tau}{\tau}$ as above, $\langle\beta, \frac{d\tau}{\tau}\rangle_G = 0$, and thus $\langle\beta,\beta\rangle \leq 0$, with the inequality strict if $\beta \neq 0$. Then

$$\tilde{p}_{\hbar,z} - \imath q_{\hbar,z} = \chi_1\langle\beta,\beta\rangle_G - \chi_1(\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G - \chi_2 f_z^2 - f_z\chi(\operatorname{Im} z)\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G,$$

[81]This is possible for $\frac{d\tau}{\tau}$ time-like. Note further that typically γ is *not* in the 'spatial' slice T_x^*X; the latter need even not be space-like.

so if $\operatorname{Re} f_z^2 > 0$, the real part is negative, thus showing ellipticity. On the other hand, if $\langle\beta, \frac{d\tau}{\tau}\rangle_G \neq 0$, then we compute

$$\langle\varpi + \overline{z}\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G(\tilde{p}_{\hbar,z} - \imath q_{\hbar,z}).$$

For the $p_{\hbar,z}$ part this is the computation performed in (7.9) (with an extra factor, χ_1, now); on the other hand,

$$\begin{aligned}\langle\varpi + \overline{z}\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G(-\chi_2\hat{p}_{\hbar,z} - \imath q_{\hbar,z}) = &- \chi_2 f_z^2 \left(\langle\beta, \frac{d\tau}{\tau}\rangle_G - \imath \operatorname{Im} z\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G\right)\\ &+ \imath\chi f_z \left(\langle\beta, \frac{d\tau}{\tau}\rangle_G^2 + (\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^2\right).\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{Im}&\left(\langle\varpi + \overline{z}\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G(-\chi_2\hat{p}_{\hbar,z} - \imath q_{\hbar,z})\right)\\ &= \chi_2 \operatorname{Re}(f_z^2) \operatorname{Im} z\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G - \chi_2 \operatorname{Im}(f_z^2)\langle\beta, \frac{d\tau}{\tau}\rangle_G\\ &\quad + \chi \operatorname{Re} f_z \left(\langle\beta, \frac{d\tau}{\tau}\rangle_G^2 + (\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^2\right)\end{aligned}$$

Combining this with (7.9) we note that the only term in

$$\operatorname{Im}\left(\langle\varpi + \overline{z}\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G(\tilde{p}_{\hbar,z} - \imath q_{\hbar,z})\right)$$

that is not automatically non-negative provided $\operatorname{Im} z \geq 0$, $\operatorname{Re} f_z > 0$, $\operatorname{Re}(f_z^2) > 0$ is $-\chi_2 \operatorname{Im}(f_z^2)\langle\beta, \frac{d\tau}{\tau}\rangle_G$. Thus, if we arrange that

$$2|\chi_2|\,|\operatorname{Im} f_z|\,|\langle\beta, \frac{d\tau}{\tau}\rangle_G| < \frac{F}{2}(\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^2,$$

which is guaranteed by choosing $F > 0$ sufficiently large as $\operatorname{Im} z > \delta$ and β is in a compact set, then in view of the actual positivity of $\chi \operatorname{Re} f_z(\operatorname{Im} z)^2\langle\frac{d\tau}{\tau}, \frac{d\tau}{\tau}\rangle_G^2$, this imaginary part does not vanish, completing the proof of ellipticity.

In summary, we have shown that *if* $\frac{d\tau}{\tau}$ *is time-like* then the assumptions on imaginary part of $p_{\hbar,z}$, as well as on the ellipticity of $p_{\hbar,z} - \imath q_{\hbar,z}$ for non-real z, in Section 2 are automatically satisfied in the Lorentzian setting if $q_{\hbar,z}$ is given by (3.17). Further, if we extend $p_{\hbar,z}$ to a new symbol, $\tilde{p}_{\hbar,z}$ across a hypersurface, $\mu = \mu_1$, in the manner (3.18), then with χ, χ_1 and χ_2 as discussed there, $\tilde{p}_{\hbar,z} - \imath q_{\hbar,z}$ satisfies the requirements for $p_{\hbar,z} - \imath q_{\hbar,z}$, and in addition it is elliptic in the extended part of the domain. We usually write $p_{\hbar,z} - \imath q_{\hbar,z}$ for this extension. Thus, these properties need not be checked individually in specific cases.

Appendix A. Comparison with cutoff resolvent constructions

By Semyon Dyatlov[82]

In this appendix, we will first examine the relation of the resolvent considered in the present paper to the cutoff resolvent for slowly rotating Kerr–de Sitter metric constructed in [20] using separation of variables and complex contour deformation

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near the event horizons. Then, we will show how to extract information on the resolvent beyond event horizons from information about the cutoff resolvent.

First of all, let us list some notation of [20] along with its analogues in the present paper:

Present paper	[20]	Present paper	[20]
X_+	M	r_s	$2M_0$
γ	α	$\tilde{\mu}$	Δ_r
κ	Δ_θ	$F_\pm$	$A_\pm$
$\tilde{t}, \tilde{\phi}$	t, φ	t, ϕ	t^*, φ^*
ω	σ	$e^{-i\sigma h(r)} P_\sigma e^{i\sigma h(r)}$	$-P_g(\sigma)$
K_δ	M_K		

The difference between $P_g(\omega)$ and P_σ is due to the fact that $P_g(\omega)$ was defined using Fourier transform in the $\tilde{t}$ variable and P_σ is defined using Fourier transform in the variable $t = \tilde{t} + h(r)$. We will henceforth use the notation of the present paper.

We assume that $\delta > 0$ is small and fixed, and α is small depending on δ. Define

$$K_\delta = (r_- + \delta, r_+ - \delta)_r \times \mathbb{S}^2.$$

Then [20, Theorem 2] gives a family of operators

$$R_g(\sigma) : L^2(K_\delta) \to H^2(K_\delta)$$

meromorphic in $\sigma \in \mathbb{C}$ and such that $P_g(\sigma) R_g(\sigma) f = f$ on K_δ for each $f \in L^2(K_\delta)$.

Proposition A.1. *Assume that the complex absorbing operator Q_σ satisfies the assumptions of Section 6.5 in the 'classical' case and furthermore, its Schwartz kernel is supported in $(X \setminus X_+)^2$. Let $R_g(\sigma)$ be the operator constructed in* [20] *and $R(\sigma) = (P_\sigma - iQ_\sigma)^{-1}$ be the operator defined in Theorem 1.2 of the present paper. Then for each $f \in C_0^\infty(K_\delta)$,*

$$-e^{i\sigma h(r)} R_g(\sigma) e^{-i\sigma h(r)} f = R(\sigma) f|_{K_\delta}. \tag{A.1}$$

Proof. The proof follows [20, Proposition 1.2]. Denote by u_1 the left-hand side of (A.1) and by u_2 the right-hand side. Without loss of generality, we may assume that f lies in the kernel $\mathcal{D}'_k$ of the operator $D_\phi - k$, for some $k \in \mathbb{Z}$; in this case, by [20, Theorem 1], u_1 can be extended to the whole X_+ and solves the equation $P_\sigma u_1 = f$ there. Moreover, by [20, Theorem 3], u_1 is smooth up to the event horizons $\{r = r_\pm\}$. Same is true for u_2; therefore, the difference $u = u_1 - u_2$ solves the equation $P_\sigma(u) = 0$ and is smooth up to the event horizons.

Since both sides of (A.1) are meromorphic, we may further assume that $\operatorname{Im}\sigma > C_e$, where C_e is a large constant. Now, the function $\tilde{u}(t, \cdot) = e^{-it\sigma} u(\cdot)$ solves the wave equation $\Box_g \tilde{u} = 0$ and is smooth up to the event horizons in the coordinate system (t, r, θ, ϕ); therefore, if C_e is large enough, by [20, Proposition 1.1] $\tilde{u}$ cannot grow faster than $\exp(C_e t)$. Therefore, $u = 0$ as required. □

Now, we show how to express the resolvent $R(\sigma)$ on the whole space in terms of the cutoff resolvent $R_g(\sigma)$ and the nontrapping construction in the present paper. Let Q_σ be as above, but with the additional assumption of semiclassical ellipticity near ∂X_δ, and $Q'_\sigma \in \Psi_\hbar^{-\infty}$ be an operator satisfying the assumptions of Section 6.5 in the 'semiclassical' case on the trapped set. Moreover, we require that the semiclassical wavefront set of $|\sigma|^{-2} Q'_\sigma$ be compact and $Q'_\sigma = \chi Q'_\sigma = Q'_\sigma \chi$, where $\chi \in C_0^\infty(K_\delta)$. Such operators exist for α small enough, as the trapped set is compact and located $O(\alpha)$ close to the photon sphere $\{r = 3r_s/2\}$ and thus is far from the event horizons. Denote $R'(\sigma) = (P_\sigma - iQ_\sigma - iQ'_\sigma)^{-1}$; by Theorem 2.14 applied in the case of Section 6.5, for each C_0 there exists a constant σ_0 such that for s large enough, $\operatorname{Im}\sigma > -C_0$, and $|\operatorname{Re}\sigma| > \sigma_0$,

$$\|R'(\sigma)\|_{H^{s-1}_{|\sigma|^{-1}} \to H^s_{|\sigma|^{-1}}} \leq C|\sigma|^{-1}.$$

We now use the identity

$$R(\sigma) = R'(\sigma) - R'(\sigma)(iQ'_\sigma + Q'_\sigma(\chi R(\sigma)\chi)Q'_\sigma)R'(\sigma). \tag{A.2}$$

(To verify it, multiply both sides of the equation by $P_\sigma - iQ_\sigma - iQ'_\sigma$ on the left and on the right.) Combining (A.2) with the fact that for each N, Q'_σ is bounded $H^{-N}_{|\sigma|^{-1}} \to H^N_{|\sigma|^{-1}}$ with norm $O(|\sigma|^2)$, we get for σ not a pole of $\chi R(\sigma)\chi$,

$$\|R(\sigma)\|_{H^{s-1}_{|\sigma|^{-1}} \to H^s_{|\sigma|^{-1}}} \leq C(1 + |\sigma|^2 \|\chi R(\sigma)\chi\|_{L^2(K_\delta) \to L^2(K_\delta)}). \tag{A.3}$$

Also, if σ_0 is a pole of $R(\sigma)$ of algebraic multiplicity j, then we can multiply the identity (A.2) by $(\sigma - \sigma_0)^j$ to get an estimate similar to (A.3) on the function $(\sigma - \sigma_0)^j R(\sigma)$, holomorphic at $\sigma = \sigma_0$.

The discussion above in particular implies that the cutoff resolvent estimates of [5] also hold for the resolvent $R(\sigma)$. Using the Mellin transform, we see that the resonance expansion of [5] is valid for any solution u to the forward time Cauchy problem for the wave equation on the whole M_δ, with initial data in a high enough Sobolev class; the terms of the expansion are defined and the remainder is estimated on the whole M_δ as well.

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