ON THE LIGHT RAY TRANSFORM WITH WAVE CONSTRAINTS

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ABSTRACT. We study the light ray transform on Minkowski space-time and its small metric perturbations acting on scalar functions which are solutions to wave equations. We show that the light ray transform uniquely determines the function in a stable way. The problem is of particular interest because of its connection to inverse problems of the Sachs-Wolfe effect in cosmology.

1. INTRODUCTION

Let $M = [t_0, t_1] \times \mathbb{R}^3$ and $(t, x), t \in [t_0, t_1], x \in \mathbb{R}^3$ be the local coordinates. Let $g_M = -dt^2 + dx^2$ be the Minkowski metric on M. Consider the Lorentzian manifold (M, g_M). We denote the interior by $M^\circ = (t_0, t_1) \times \mathbb{R}^3$ and the boundaries by $\mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3$ and $\mathscr{S} = \{t_1\} \times \mathbb{R}^3$. See Figure 1.

Consider light-like geodesics on (M, g_M) which are straight lines. We parametrize the light rays as follows: let $x_0 \in \mathscr{S}_0$ and $v \in \mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. Then a light ray from x_0 in direction $(1, v)$ is $\gamma(\tau) = (t_0, x_0) + \tau(1, v), \tau \in [0, t_1 - t_0]$. See Figure 1. The light ray transform for scalar functions on (M, g_M) is defined by

$$(1.1) \qquad X_M(f)(\gamma) = \int_0^{t_1 - t_0} f(\gamma(\tau)) d\tau, \quad f \in C_0^\infty(M)$$

Of course, one can regard X_M as the restriction of the light ray transform $X_{\mathbb{R}^4}$ of the Minkowski spacetime $(\mathbb{R}^4, g_M)$ acting on functions supported in M. However, it is perhaps better to think of X_M as the compact version of the transform, which is similar to the geodesic ray transform on a compact Riemannian manifold with boundary.

In this work, we study X_M acting on scalar functions which are solutions to the Cauchy problem of wave equations on (M, g_M). Let $c > 0$ be a constant. Denote $\Box_c = \partial_t^2 + c^2 \Delta$ where Δ is the positive Laplacian on $\mathbb{R}^3$, namely $\Delta = \sum_{i=1}^3 D_{x_i}^2, D_{x_i} = -\sqrt{-1}\frac{\partial}{\partial x_i}$. Here, c is the wave speed. On (M, g_M), $c = 1$ is the speed of light. Consider the Cauchy problem

$$(1.2) \qquad \begin{aligned} \Box_c f &= 0 \quad \text{on } M \\ f = f_1, \ \partial_t f &= f_2, \quad \text{on } \mathscr{S}_0. \end{aligned}$$

The problem we address in this paper is the determination of f or equivalently f_1, f_2 from $X_M(f)$ with the constraint (1.2). Our main result is

Theorem 1.1. *Suppose $0 < c \leq 1$ is constant. Assume that $(f_1, f_2) \in \mathcal{N}^s \stackrel{\text{def}}{=} H_{\text{comp}}^{s+1}(\mathscr{S}_0) \times H_{\text{comp}}^s(\mathscr{S}_0), s \geq 0$, and f_1, f_2 are supported in a compact set $\mathscr{K}$ of $\mathscr{S}_0$. Then $X_M f$ uniquely determines f and f_1, f_2 which satisfy (1.2). Moreover, there exists $C > 0$ such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})} \text{ and } \|f\|_{H^{s+1}(M)} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})}$$

where $\mathscr{C}$ is the set of light rays on M, see Section 4.

We will prove stronger versions of the theorem including lower order terms in the wave equation in Section 8. However, for ease of presentation, we use the standard wave equation on Minkowski spacetime throughout the paper until the final sections where the necessary changes are indicated.
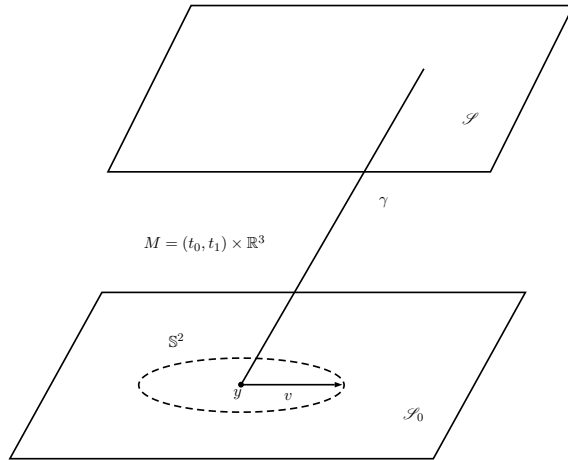

FIGURE 1. The setup of the problem for the Minkowski space-time.

We also consider metric perturbations $g_\delta = g_M + h$ where h satisfies assumptions (A1), (A2) in Section 9, which says that h is a suitably smooth small perturbation of the Minkowski spacetime. In this case, the light rays may not be straight lines. Let X_δ be the light ray transform on (M, g_δ) see (9.6). Let $\Box_{g_\delta}$ be the Laplace-Beltrami operator on (M, g_δ). Consider the Cauchy problem

$$\Box_{g_\delta} f = 0 \quad \text{on } M^\circ$$
(1.3)
$$f = f_1, \quad \partial_t f = f_2, \quad \text{on } \mathscr{S}_0.$$

We prove

Theorem 1.2. *Consider (M, g_δ) satisfying assumptions (A1), (A2) to be stated in Section 9. Assume that $(f_1, f_2) \in \mathcal{N}^s, s \geq 0$, and f_1, f_2 are supported in a compact set $\mathscr{K}$ of $\mathscr{S}_0$. For $\delta \geq 0$ sufficiently small, $X_\delta f$ uniquely determines f and f_1, f_2 which satisfy (1.3). Moreover, there exists $C > 0$ such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C \|X_\delta f\|_{H^{s+3/2}(\mathscr{C}_\delta)} \text{ and } \|f\|_{H^{s+1}(M)} \leq C \|X_\delta f\|_{H^{s+3/2}(\mathscr{C}_\delta)}$$

where $\mathscr{C}_\delta$ is the set of light rays on (M, g_δ), see Section 9.

Our motivation for this setup of the light ray transform comes from some inverse problems in cosmology. We are particularly interested in the determination of gravitational perturbations such as primordial gravitational waves from the anisotropies of the Cosmic Microwave Background (CMB), see for example [11, 2, 4]. The Sachs-Wolfe effect in their 1967 paper [18] manifests the connection of the CMB anisotropy and the light ray transform of the gravitational perturbations. We discuss the background in Section 2 and 3. Physically, $c < 1$ and $c = 1$ in Theorem 1.1 correspond to different Universe models driven by hydrodynamical perturbations and scalar field perturbations, respectively.

The reason that we are able to get a stable determination is due to the restriction of singularities of f. In general, it is known that time-like singularities in f are lost after taking the light ray transform, so one does not expect to determine all information of f, although the light ray

transform is injective on $C_0^\infty(M)$. In particular, we do not expect Theorem 1.1 to hold for $c > 1$. There is a fundamental difference in our treatment between the $c < 1$ and $c = 1$ cases. The former requires a good understanding of the normal operator $X_M^* X_M$ which was considered in [12] and further generalized in [13], while the latter relies on a thorough analysis of the operator $X_M E$ where E is the fundamental solution or parametrix for the Cauchy problem.

The paper is organized as follows. In Section 2 and 3, we discuss the (integrated) Sachs-Wolfe effects and explain how the inverse problem is related to our theorems. In Section 4, we review some properties of the light ray transform. Then we consider the Cauchy problem in Section 5. In Section 6 and 7, we construct the microlocal parametrix for the light ray transform with the wave constraint for $c < 1$ and $c = 1$ respectively. We prove Theorem 1.1 and the version including lower order terms in the wave equation in Section 8. Finally, we address the small metric perturbations of Minkowski space-time in Section 9.

2. The integrated Sachs-Wolfe effect

Consider the flat Friedman-Lemaîte-Robertson-Walker (FLRW) model for the cosmos:
$$\mathscr{M} = (0, \infty) \times \mathbb{R}^3, \quad g_0 = dt^2 - a^2(t)\delta_{ij} dx^i dx^j$$
where $(t, x), t \in (0, \infty), x \in \mathbb{R}^3$ are coordinates and $\delta_{ij} = 1$ if $i = j$ and otherwise 0. Here, the signature of g_0 is $(+, -, -, -)$ because we will refer to some results in [15] later. The factor $a(t)$ is assumed to be positive and smooth in t. It represents the rate of expansion of the Universe.

It is known that the actual Universe is not exactly isotropic. We assume that the actual cosmos is a metric perturbation $g = g_0 + \delta g$ on $\mathscr{M}$ where δg is a small perturbation compared to g_0. We introduce the conformal time s such that $ds = a^{-1}dt$. Then we get
$$g_0 = a^2(s)(ds^2 - \delta_{ij} dx^i dx^j) = a^2(s) g_M$$
where g_M is the Minkowski metric on $\mathscr{M} = (0, \infty)$. We write $g = a^2(s)(g_M + \delta g)$ where δg denotes the corresponding perturbation in conformal time. In the literature, the metric perturbations are classified to scalar, vector and tensor type. We consider the scalar type perturbations. In the longitudinal gauge, also called the conformal Newtonian gauge, the metric g is of the form

(2.1) $$g = a^2(s)[(1 + 2\Phi)ds^2 - (1 - 2\Psi)dx^2]$$

see [15, Section 2]. Here, Φ, Ψ are scalar functions on M. We remark that there is a gauge invariant formulation of cosmological perturbations. However, in the longitudinal gauge, the gauge invariant variables are equal to Φ, Ψ, see [15]. In this work, we fix the gauge and work with Φ, Ψ for simplicity. The physical meaning of Φ, Ψ in (2.1) is very clear. They are the magnitudes of metric perturbations.

Consider the Cosmic Microwave Background (CMB) measurement. Our main references are [2, 4, 18]. Let $\mathscr{S}_0 = \{s_0\} \times \mathbb{R}^3$ be the surface of last scattering. This is the moment after which photons stopped interaction and started to travel freely in $\mathscr{M}$. Let $\mathscr{S} = \{s_1\} \times \mathbb{R}^3$ be the surface where we make observation of the photons. Let $\gamma(\tau)$ be a light ray from $\mathscr{S}_0$ to $\mathscr{S}$. It represents the trajectory of photons in $\mathscr{M}$. Explicitly, we have
$$\gamma(\tau) = (s_0, x_0) + \tau(1, v), \quad (s_0, x_0) \in \mathscr{S}_0, v \in \mathbb{S}^2, \tau \in [0, s_1 - s_0].$$

Then we consider the photon energies observed at $\mathscr{S}_0, \mathscr{S}$ denoted by $E_0 = g_0(\dot\gamma(s_0), \partial_s), E = g_0(\dot\gamma(s_1), \partial_s)$. Here, the observer is represented by the flow of the vector field ∂_s. The redshift z is defined by
$$1 + z = E/E_0.$$

In [18], Sachs and Wolfe derived that to the first order linearization, $1 + z$ is represented by a light ray transform of the metric perturbations, see [18, equation (39)]. In cosmological literatures, one often connects this to the CMB temperature anisotropies. Let T be the temperature observed at $\mathscr{S}$ in the isotropic background g_0. Let δT be the temperature fluctuation from the isotropic background. One can compute $\delta T/T$ in terms of the energies E_0, E. One component of $\delta T/T$ is the integrated Sachs-Wolfe (ISW) effects

$$(2.2) \qquad (\frac{\delta T}{T})^{ISW} = \int_0^{s_1 - s_0} (\partial_s \Phi(\gamma(\tau)) + \partial_s \Psi(\gamma(\tau)) d\tau = X_M (\partial_s \Phi + \partial_s \Psi)$$

see [4, Section 2.5]. Note that this quantity depends on the light ray γ which indicates the anisotropy. We remark that another component of $\delta T/T$ is the ordinary Sachs-Wolfe effect (OSW) which only involves Φ, Ψ at $\mathscr{S}_0$. The integrated Sachs-Wolfe effect can be extracted from the CMB and other astrophysical data, see for example [14].

The inverse Sachs-Wolfe problem we study is to determine Φ, Ψ from $(\delta T/T)^{ISW}$. It is also interesting to determine Φ, Ψ at the initial surface $\mathscr{S}_0$. Before we proceed, we observe that there are natural obstructions to the unique determination from (2.2). If $\Phi + \Psi$ is a constant, then the integrated Sachs-Wolfe effect is always zero. So the goal is to determine Φ, Ψ up to such natural obstructions.

3. Dynamical equations for perturbations

For the Sachs-Wolfe problem, we should take into account that g satisfies the Einstein equations with certain source fields and initial perturbations at $\mathscr{S}_0$ from g_0. On the linearization level, this puts the perturbation δg under some wave equation constraint as we discuss in this section. The derivations of the equations for the perturbation take some amount of work and they are mostly done in the literature, see for example [2, Section 5.1] and [4]. We follow the presentation and the notations in [15, Section 4-6] closely. Instead of the gauge invariant approach, we choose to work in the longitudinal gauge for simplicity. It is not hard to transform back and forth and our analysis works for the gauge invariant formulation as well.

Let $R^\mu{}_\nu$ be the Ricci curvature tensor and R the scalar curvature on $(\mathscr{M}, g)$ (in conformal time). Let $T^\mu{}_\nu$ denote the stress-energy tensor of certain source fields. The Einstein equations are

$$G^\mu{}_\nu = 8\pi G T^\mu{}_\nu, \quad G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R$$

where G is Newton's gravitational constant. We assume that $T^\mu{}_\nu = {}^{(0)}T^\mu{}_\nu + \delta T^\mu{}_\nu$ where ${}^{(0)}T$ denotes the stress-energy tensor of the background field and δT denotes the perturbation. We also have $g = a^2(g_M + \delta g)$. Then we can write $G^\mu{}_\nu = {}^{(0)}G^\mu{}_\nu + \delta G^\mu{}_\nu + \cdots$. From the asymptotic expansion, one finds that the Einstein tensor for the background metric g_M are

$$ {}^{(0)}G_0{}^0 = 3a^{-2}H^2, \quad {}^{(0)}G^0{}_i = 0, \quad {}^{(0)}G^i{}_j = a^{-2}(2H' + H^2)\delta^i{}_j,$$

where $i, j = 1, 2, 3$, $H(s) = \partial_s a(s)/a(s)$, see [15, equation (4.2)]. Hereafter, $H' = \partial_s H$ denotes the derivative in the conformal time variable. We emphasize that we work with a flat Universe and we get the equation ${}^{(0)}G^\mu{}_\nu = 8\pi G^{(0)}T^\mu{}_\nu$.

For the first order perturbation term, we get $\delta G^\mu_{\ \nu} = 8\pi G \delta T^\mu_{\ \nu}$. After lengthy calculations, one obtains (see [15, equation (4.15)]) the following equations for Φ, Ψ

$$-3H(H\Phi + \Psi') + \Delta\Psi = 4\pi G a^2 \delta T^0_{\ 0}$$

(3.1)
$$\partial_i(H\Phi + \Psi') = 4\pi G a^2 \delta T^0_{\ i}$$

$$[(2H' + H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi' + \frac{1}{2}\Delta(\Phi - \Psi)]\delta^i_{\ j} - \frac{1}{2}\delta^{ik}(\Phi - \Psi)_{|kj} = -4\pi G a^2 \delta T^i_{\ j},$$

where $i, j = 1, 2, 3$, ∂_i denotes the ith component of the covariant derivative with respect to the background metric g_M, and Δ denotes the standard Laplacian on $\mathbb{R}^3$.

Now we need to specify the source field. We consider two important examples: the perfect fluid and the scalar field.

We first consider Universe dominated by perfect fluid sources. Let u be the four fluid velocity of a fluid source. The stress-energy tensor for a perfect fluid is

$$T^\alpha_{\ \beta} = (\epsilon + p)u^\alpha u_\beta - p\delta^\alpha_{\ \beta}$$

see [15, equation (5.2)], Here, ϵ is the energy density and p is the pressure of the fluid. We assume that $\epsilon = \epsilon_0 + \delta\epsilon, p = p_0 + \delta p$ where 0 denotes the quantity for the background and δ denotes the perturbations. For fluid source, from (3.1) one deduces that the perturbations $\Phi = \Psi$. In the case of adiabatic perturbations, Φ satisfies the following equation, called Bardeen's equation

(3.2) $$\Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2\Delta\Phi + [2H' + (1 + 3c_s^2)H^2]\Phi = 0,$$

see [15, equation (5.22)]. In general, the right hand side of the equation is a non-zero term related to the entropy perturbations. The fluid velocity u also satisfies a wave equation with speed c_s, see [15, equation (5.25)]. Here, $c_s < 1$ is the speed of sound. Prescribing Cauchy data of Φ at $\mathscr{S}_0$, one can solve the Cauchy problem of (3.2) to get Φ in $\mathscr{M}$. We formulate the inverse Sachs-Wolfe problem in this case as

Problem 3.1. *Determining Φ from* (2.2) *where Φ satisfies the Cauchy problem of* (3.2).

Commuting equation (3.2) with ∂_s, we see that $\partial_s\Phi$ also satisfies a wave equation. Hence, we arrived at the model problem we proposed in the introduction.

Next, let's consider Universe governed by a scalar field ϕ. The stress energy tensor is

$$T^\mu_{\ \nu} = \nabla^\mu\phi\nabla_\nu\phi - [\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)]\delta^\mu_{\ \nu}$$

see [15, equation (6.2)]. Here, V is the potential function for the scalar field ϕ. The field itself satisfies the Klein-Gordon equation $\Box\phi + \partial_\phi V(\phi) = 0$. Now assume that $\phi = \phi_0 + \delta\phi$ where ϕ_0 is the scalar field which drives the background model and $\delta\phi$ denotes the perturbation. Then we can split $T^\mu_{\ \nu} = {}^{(0)}T^\mu_{\ \nu} + \delta T^\mu_{\ \nu}$. Again, one finds that $\Phi = \Psi$ and it satisfies the equation

(3.3) $$\Phi'' + 2(H - \phi_0''/\phi_0')\Phi' - \Delta\Phi + 2(H' - H\phi_0''/\phi_0)\Phi = 0$$

see [15, equation (6.48)]. This is a damped wave equation with wave speed $c = 1$. We can formulate the inverse Sachs-Wolfe problem in this case as

Problem 3.2. *Determining Φ from* (2.2) *in which Φ satisfies the Cauchy problem of* (3.3).

Again, we arrived at the model problem in the introduction with $c = 1$. We do not need it but record that the scalar field perturbation also satisfies a wave equation, see [15, equation (6.47)].

Applying our main result of the paper, in particular Theorem 8.3 which allows lower order terms in the wave equation, we obtain the following result.

Corollary 3.3. *For the inverse Sachs-Wolfe effect Problems 3.1 and 3.2, one can uniquely determine Φ in $\mathcal{M}$ (and the initial conditions at $\mathscr{S}_0$) in the longitudinal gauge up to a constant in a stable way.*

4. THE LIGHT RAY TRANSFORM ON FUNCTIONS

We recall some facts about the light ray transform on scalar functions. Consider the Lorentzian manifold (M, g_M) and hereafter we change the signature of g_M to $(-, +, +, +)$. For $(t, x) \in M^\circ, t \in (t_0, t_1), x \in \mathbb{R}^3$, we use $\Xi = (\tau, \xi), \tau \in \mathbb{R}, \xi \in \mathbb{R}^3$ for the dual variables in $T_{(t,x)}M^\circ$. We divide the tangent vectors in $T_{(t,x)}M^\circ$ into time-like vectors $\Omega^-_{(t,x)}M^\circ = \{\Xi \in \mathbb{R}^4 : g_M(\Xi, \Xi) = -\tau^2 + |\xi| < 0\}$, space-like vectors $\Omega^+_{(t,x)}M^\circ = \{\Xi \in \mathbb{R}^4 : g_M(\Xi, \Xi) > 0\}$ and light-like vectors $L_{(t,x)}M^\circ = \{\Xi \in \mathbb{R}^4 : g_M(\Xi, \Xi) = 0\}$. We denote the corresponding vector bundles by $\Omega^- M^\circ, \Omega^+ M^\circ, LM^\circ$. The cotangent vectors can be classified similarly using the dual metric g_M^* on T^*M°. The corresponding bundles are denoted by $\Omega^{*,-}M^\circ, \Omega^{*,+}M^\circ, L^*M^\circ$.

From now on, *without loss of generality, we take $t_0 = 0$ in M*, which amounts to a translation in the t variable. Let $\mathscr{C}$ be the set of light rays on (M, g_M). As M has a global coordinate system, we can parametrize $\mathscr{C}$ as follows. Let $y \in \mathbb{R}^3, v \in \mathbb{S}^2 \overset{\text{def}}{=} \{z \in \mathbb{R}^3 : |z| = 1\}$ with $|\cdot|$ the Euclidean norm. We denote $\theta = (1, v)$ so that θ is a (future pointing) light-like vector. Then we have

$$\mathscr{C} = \{\gamma(\tau) : \gamma(\tau) = (\tau, y + \tau v), \quad \tau \in (0, t_1)\}$$

which is parametrized by $(y, v) \in \mathbb{R}^3 \times \mathbb{S}^2$. For $f \in C_0^\infty(M^\circ)$ and $y \in \mathbb{R}^3, v \in \mathbb{S}^2$, we have

$$(4.1) \qquad X_M f(y, v) = \int_0^{t_1} f(\tau, y + \tau v) d\tau = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((x-y)\cdot\eta + tv\cdot\eta)} f(t, x) dt dx d\eta$$

The Schwartz kernel of X_M is δ_Z the delta distribution on $\mathscr{C} \times M^\circ$ supported on the point-line relation Z defined by

$$Z = \{(\gamma, q) \in \mathscr{C} \times M^\circ : q \in \gamma\} = \{(y, v, (t, x)) \in \mathbb{R}^3 \times \mathbb{S}^2 \times M^\circ : x = y + tv\}.$$

We know (see e.g. [12]) that X_M is an Fourier integral operator of order $-3/4$ associated with the canonical relation $(N^*Z)'$, where N^*Z denotes the conormal bundle of Z minus the zero section. Hence $X_M : \mathcal{E}'(M^\circ) \to \mathcal{D}'(\mathscr{C})$ is continuous. Here, $\mathcal{D}'(M^\circ), \mathcal{E}'(M^\circ)$ denotes the space of distributions and compactly supported distributions on M°.

It is known that on $\mathbb{R}^4$, the light ray transform is injective on $C_0^\infty(\mathbb{R}^4)$, see [16, 10], but not injective on $\mathcal{S}(\mathbb{R}^4)$ (Schwartz functions on $\mathbb{R}^4$). It is proved in [10, Corollary 7] that the kernel of the transform consists of $\mathcal{S}(\mathbb{R}^4)$ functions whose Fourier transforms are supported in the time-like cone. One can obtain analogous results for X_M. The point is that after taking the light ray transform, time-like singularities in the functions are lost.

To see the difference in the treatment between space-like and light-like singularities, consider the normal operator $X_M^* X_M$. For the light ray transform on $\mathbb{R}^4$, the Schwartz kernel of the normal operator can be computed explicitly using Fourier transforms, see [16]. Let's look at the microlocal structure. The canonical relation $C = N^*Z'$ is

$$(4.2) \qquad \begin{aligned} C = \{&((y, v, \eta, w); (t, x, \tau, \xi)) \in (T^*\mathscr{C}\backslash 0) \times (T^*M^\circ\backslash 0) : y = x - tv, \quad \eta = \xi, \\ &w = t\xi|_{T_v\mathbb{S}^2}, \quad \tau = -\xi v, \quad y \in \mathbb{R}^3, v \in \mathbb{S}^2, \eta \in \mathbb{R}^3, (t, x) \in M^\circ\}, \end{aligned}$$

see [12, equation (39)]. In the expression of w, ξ is regarded as a co-tangent vector to $T_v\mathbb{S}^2$. If $\Xi = (\tau, \xi)$ is light-like, then $w = 0$, see [12, Lemma 10.1]. We look at the double fibration picture

$$
\begin{array}{ccc}
 & C & \\
\pi\swarrow & & \searrow\rho \\
T^*M & & T^*\mathscr{C}
\end{array}
$$

If ρ is an injective immersion, the double fibration satisfies the Bolker condition, and the normal operator $X_M^* \circ X_M$ belongs to the clean intersection calculus so that the normal operartor is a pseudo-differential operator, see for instance [6]. As shown in [12, Lemma 10.1], ρ fails to be injective on the set $\mathcal{L} \cap C$ where

$$\mathcal{L} = \{(y,v,\eta,w;t,x,\Xi) \in (T^*\mathscr{C}\backslash 0) \times (T^*M^\circ\backslash 0) : \Xi \text{ is light-like}\}.$$

In particular, the normal operator is an elliptic pseudo-differential operator when restricted to space-like directions, see [16] and [12]. In general, it is proved in [21] that the Schwartz kernel of the normal operator $X_M^* X_M$ is a paired Lagrangian distribution and a parametrix can be constructed within the framework of [5]. However, the picture near light-like directions is still not so clear. We remark that Guillemin [7] considered the structure of $X_M X_M^*$ for $2+1$ dimensional Minkowski spacetime.

5. Solution of the Cauchy problem

We find a representation of the solution of the Cauchy problem in this section. Consider

(5.1)
$$\Box_c u = 0, \quad \text{on } M^\circ = (t_0, t_1) \times \mathbb{R}^3$$
$$u = f_1, \quad \partial_t u = f_2, \text{ on } \mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3.$$

The fundamental solution can be written down quite explicitly. However, it will be more convenient to look at its microlocal structure. For (5.1), all we need is the Fourier transform, see for example Trèves [19, Chapter VI, Section 1]. For general strictly hyperbolic equations, Duistermaat-Hörmander (see [3, Chaper 5]) constructed a parametrix for the Cauchy problem. So one can find a parametrix for (5.1) even when the equation contains lower order terms which will be used in Section 8.

Let $(\tau,\xi), \xi \in \mathbb{R}^3$ be the dual variables in T^*M° to $(t,x), x \in \mathbb{R}^3$. Take Fourier transform of (5.1) in x variable, we get (for $t_0 = 0$)

$$\partial_t^2 \hat{u}(t,\xi) + c^2|\xi|^2 \hat{u}(t,\xi) = 0,$$
$$\hat{u}(0,\xi) = \hat{f}_1(\xi), \quad \partial_t \hat{u}(0,\xi) = \hat{f}_2(\xi).$$

Solve this ODE, we get

$$\hat{u}(t,\xi) = \frac{1}{2}e^{itc|\xi|}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2) + \frac{1}{2}e^{-itc|\xi|}(\hat{f}_1 - \frac{1}{ic|\xi|}\hat{f}_2)$$

Taking the inverse Fourier transform, we get

(5.2)
$$u(t,x) = (2\pi)^{-3}\frac{1}{2}\int e^{i(x\cdot\xi+ct|\xi|)}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2)d\xi + (2\pi)^{-3}\frac{1}{2}\int e^{i(x\cdot\xi-tc|\xi|)}(\hat{f}_1 - \frac{1}{ic|\xi|}\hat{f}_2)d\xi$$
$$= (2\pi)^{-3}\int e^{i(x\cdot\xi+ct|\xi|)}\hat{h}_1(\xi)d\xi + (2\pi)^{-3}\int e^{i(x\cdot\xi-tc|\xi|)}\hat{h}_2(\xi)d\xi$$
$$= E_+ h_1 + E_- h_2,$$

where

$$\hat{h}_1 = \frac{1}{2}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2), \quad \hat{h}_2 = \frac{1}{2}(\hat{f}_1 - \frac{1}{ic|\xi|}\hat{f}_2)$$

We see that $E_\pm$ are represented by oscillatory integrals

$$(5.3) \qquad E_\pm(f) = (2\pi)^{-3} \int e^{i((x-y)\cdot\xi \pm ct|\xi|)} f(y) dy d\xi$$

The phase functions are $\phi_\pm(t,x,y,\xi) = (x-y)\cdot\xi \pm ct|\xi|$ and amplitude function $a(t,x,\xi) = 1$. In Hörmander's notation, we conclude that $E_\pm \in I^{-\frac{1}{4}}(\mathbb{R}^3 \times M^\circ; (C^\pm)')$ are Fourier integral operators where the canonical relations are

$$(5.4) \qquad C^\pm = \{(t,x,\zeta_0,\zeta';y,\xi) \in T^*M^\circ\backslash 0 \times T^*\mathbb{R}^3\backslash 0 : y = x - ct(\pm\xi/|\xi|), \zeta' = \xi, \zeta_0 = \pm c|\xi|\}.$$

It suffices to regard h_1, h_2 as the reparametrized initial conditions for the Cauchy problem and represent $u = E_+h_1 + E_-h_2$ in (5.2). Once we find h_1, h_2, we can easily find f_1, f_2 from

$$f_1 = h_1 + h_2, \quad f_2 = ic\Delta^{\frac{1}{2}}(h_1 - h_2)$$

6. The microlocal inversion: $c < 1$

In this case, it is important to observe that singularities (or the wave front set) of the solution u to (5.1) are all in space-like directions for (M, g_M). From the canonical relation $C^\pm$ in (5.4), we know that for u in (5.1)

$$\mathrm{WF}(u) \subset \{(t,x,\xi_0,\xi') \in T^*M^\circ\backslash 0 : \xi_0 = \pm c|\xi'|\},$$

and $|(\xi_0,\xi')|_{g_M^*} = -\xi_0^2 + |\xi'| = (-c^2+1)|\xi'| > 0$ for $c < 1$. For such (ξ_0,ξ'), the corresponding vector in TM° is time-like. So these singularities correspond to trajectories of particles moving slower than photons in (M, g_M).

Now we can use the fact that in space-like directions, the normal operator $X_M^* \circ X_M$ is actually a pseudo-differential operator as shown in [12]. The symbol of $\Box_c$ is $p_c(\xi_0,\xi') = -\xi_0^2 + c^2|\xi'|^2$. Let $\chi(t)$ be a smooth cut-off function with $\chi(t) = 1, |t| < 1$ and $\chi(t) = 0, |t| > 1/c^2$ for $c < 1$. Then we define

$$\chi_1(\xi_0,\xi') = \chi(\frac{\xi_0^2}{c^2|\xi'|^2})$$

so $\chi_1(\xi_0,\xi') = 1$ on $\{(\xi_0,\xi') \in \mathbb{R}^4 : p_c(\xi_0,\xi') > 0\}$ and $\chi_1(\xi_0,\xi') = 0$ on $\Omega^{*,-}M^\circ$. Let $\chi_1(D)$ be the pseudo-differential operator with symbol χ_1. We have

Lemma 6.1. $\chi_1(D)X_M^* \circ X_M\chi_1(D)$ *is a pseudo-differential operator of order* -1 *on* M°. *The principal symbol at* $(t,x,\xi_0,\xi') \in T^*M^\circ$ *is*

$$\frac{4\pi^2}{|\xi'|}\chi_1^2(\xi_0,\xi').$$

Proof. First of all, X_M is a properly supported FIO. It follows from the proof of Proposition 11.4 of [12] that $\chi_1(D)X_M^* \circ X_M\chi_1(D)$ is a pseudo-differential operator on M°. The symbol is

$$\sigma(t,x,\xi_0,\xi') = 2\pi\chi_1^2(\xi_0,\xi')(|\xi'|^2 - |\xi_0^2|)^{-\frac{1}{2}} \int_{S^1_{(\xi_0,\xi')}} \theta^0\theta^0\theta^0\theta^0 dv$$

$$= 2\pi\chi_1^2(\xi_0,\xi')(|\xi'|^2 - |\xi_0^2|)^{-\frac{1}{2}} \int_{S^1_{(\xi_0,\xi')}} dv$$

where $S^1_{(\xi_0,\xi')} = \{v \in \mathbb{S}^2 : \xi_0 + \xi'v = 0\}$ and $\theta = (1, v)$. We remark that in Proposition 11.4 of [12], a restricted version of the light ray transform was considered, but the calculation also works for the full transform on M that we are considering here. Also, in [12], the author studied the light

ray transform on symmetric two tensors. Here, we need the result on scalar functions and that corresponds to θ^0 in the integral. Finally, we use the calculation result in [12, Section 8, Lemma 8.1] to find the symbol in the statement of the lemma. $\qquad\square$

Now we show that

Lemma 6.2. *The normal operator* $E_+^* X_M^* \circ X_M E_+, E_-^* X_M^* \circ X_M E_-$ *are elliptic pseudo-differential operators of order* -1 *on* $\mathbb{R}^3$, *and* $E_+^* X_M^* \circ X_M E_-$ *and* $E_-^* X_M^* \circ X_M E_+$ *are smoothing operators on* $\mathbb{R}^3$.

Proof. First of all, we know that $(X_M^* \circ X_M)E_+ = (\chi_1(D)X_M^* \circ X_M \chi_1(D))E_+$ modulo a smoothing operator, thus $(X_M^* \circ X_M)E_+ \in I^{-\frac{5}{4}}(\mathbb{R}^3 \times M^\circ; (C^+)')$ from the composition of a pseudo-differential operator and an FIO. The principal symbol is non-vanishing. We also know that $E_+^* \in I^{-\frac{1}{4}}(M^\circ \times \mathbb{R}^3; (C^{+,-1})')$. To compose these two operators, we would like to apply the clean composition theorem [8, Theorem 25.2.3], however, the operators are not properly supported. But this can be justified using the oscillatory integral representation. We have (modulo a smoothing term)

$$E_+^*(X_M^* \circ X_M E_+)f(z) = (2\pi)^{-6}\int e^{i((z-x)\cdot\eta - ct|\eta|)}e^{i((x-y)\cdot\xi + ct|\xi|)}a(\xi)f(y)dyd\xi dxdtd\eta$$

$$= (2\pi)^{-6}\int e^{i(z\cdot\eta - y\cdot\xi + x(\xi-\eta) - ct|\eta| + ct|\xi|)}a(\xi)f(y)dyd\xi dxdtd\eta$$

$$= (2\pi)^{-3}\int e^{i(z\cdot\xi - y\cdot\xi)}a(\xi)f(y)dyd\xi dt$$

where $a(\xi)$ is the symbol of $\chi_1(D)X_M^* X_M \chi_1(D)$. This is a pseudo-differential operator of order -1 on $\mathbb{R}^3$. The same proof works for the minus sign.

To see that $E_+^* X_M^* \circ X_M E_-$ is smoothing, we just need to observe that the canonical relations C^+, C^- in (5.4) are disjoint. So a wave front analysis using e.g. [3, Theorem 1.3.7] tells that the operator is smoothing. $\qquad\square$

We finished the proof but we mention the following alternative argument. Essentially, we want to consider the operator E_+ for fixed t, denoted by $E_+(t)$. We know that $E_+(t): \mathcal{E}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3)$ is a Fourier integral operator

$$E_+(t)f(x) = (2\pi)^{-3}\int e^{i((x-y)\cdot\xi + ct|\xi|)}f(y)dyd\xi$$

with canonical relation $C_t = \{(y, \eta; x, \xi) \in T^*\mathbb{R}^3 \backslash 0 \times T^*\mathbb{R}^3 \backslash 0 : y = x + ct\xi/|\xi|, \xi = \eta\}$. Then $E_+(t) \in I^0(\mathbb{R}^3 \times \mathbb{R}^3; C_t')$ is properly supported. The canonical relation C_t is a graph of a symplectic transformation, thus the composition $E_+^*(t)E_+(t)$ is a pseudo-differential operator of order 0 on $\mathbb{R}^3$. In our case, $E_+^*(t)X_M^* X_M E_+(t)$ is a pseudo-differential operator of order -1 and the symbols are smooth in $t \in [t_0, t_1]$. Finally, integrating the symbols in t produces a symbol and we get the result.

Now we construct a parametrix for the transform.

Proposition 6.3. *For* $c < 1$, *there exist operators* A_1, A_2 *such that*

$$A_1 X_M f = f_1 + R_1 f_1 + R_1' f_2, \quad A_2 X_M f = f_2 + R_2 f_1 + R_2' f_2$$

where R_1, R_2, R_1', R_2' *are smoothing operators and* $A_i = \widetilde{A}_i \circ X_M^*, i = 1, 2$ *in which* $\widetilde{A}_i$ *are Fourier integral operators.*

Proof. First, we represent $f = E_+h_1 + E_-h_2$ and write

(6.1) $$X_M f = X_M E_+ h_1 + X_M E_- h_2.$$

We apply $E_+^* X_M^*$ to get

$$E_+^* X_M^* X_M f = E_+^* X_M^* X_M E_+ h_1 + E_+^* X_M^* X_M E_- h_2 = E_+^* X_M^* X_M E_+ h_1 + R_1 h_2.$$

Since $E_+^* X_M^* X_M E_+$ is an elliptic pseudo-differential operator of order -1, we can find a parametrix B_+ which is a pseudo-differential operator of order 1 on $\mathbb{R}^3$ and

$$B_+ \circ E_+^* X_M^* X_M f = h_1 + R_1 h_1 + R_1' h_2$$

where R_1, R_1' are smoothing. We repeat the argument for the minus sign. Apply $E_-^* X_M^*$ to (6.1), we get

$$E_-^* X_M^* X_M f = E_-^* X_M^* X_M E_+ h_1 + E_-^* X_M^* X_M E_- h_2 = E_-^* X_M^* X_M E_- h_2 + R_2 h_2.$$

Apply the parametrix B_- for $E_-^* X_M^* X_M E_-$ and we get

$$B_- \circ E_-^* X_M^* X_M f = h_2 + R_2 h_1 + R_2' h_2$$

Finally, we get

$$f_1 + R_1 f_1 + R_2 f_2 = (B_+ \circ E_+^* + B_- \circ E_-^*) X_M^* X_M f$$

$$\text{and } f_2 + R_1' f_1 + R_2' f_2 = ic\Delta^{\frac{1}{2}}(B_+ \circ E_+^* + B_- \circ E_-^*) X_M^* X_M f$$

as claimed. We set $\widetilde{A}_1 = B_+ \circ E_+^* + B_- \circ E_-^*$ which is a sum of two FIOs in $I^{3/4}(M^\circ \times \mathbb{R}^3; (C^{+,-1})')$ and $I^{3/4}(M^\circ \times \mathbb{R}^3; (C^{-,-1})')$, and $\widetilde{A}_2 = ic\Delta^{\frac{1}{2}}(B_+ \circ E_+^* + B_- \circ E_-^*)$ which is a sum of two FIOs in $I^{7/4}(M^\circ \times \mathbb{R}^3; (C^{+,-1})')$ and $I^{7/4}(M^\circ \times \mathbb{R}^3; (C^{-,-1})')$. This completes the proof. $\square$

For convenience, we formulate a microlocal inversion result for determining f.

Corollary 6.4. *For $c < 1$, there exist operators A such that*

$$AX_M f = f + R_1 f_1 + R_2 f_2,$$

where R_1, R_2 are smoothing operators.

Proof. Again, we simply solve the wave equation (5.1) using the parametrix. In fact, it is easier to use h_1, h_2.

$$f = E_+ h_1 + E_- h_2$$

$$= E_+ B_+ \circ E_+^* X_M^* X_M f + E_- B_- \circ E_-^* X_M^* X_M f + \text{ smoothing operators acting on } h_1, h_2$$

$$= (E_+ B_+ \circ E_+^* + E_- B_- \circ E_-^*) X_M^* X_M f + R_1 f_1 + R_2 f_2$$

as claimed, where R_1, R_2 are smoothing operators and $A = (E_+ B_+ \circ E_+^* + E_- B_- \circ E_-^*) X_M^*$. $\square$

7. THE MICROLOCAL INVERSION: $c = 1$

In this case, the singularities of the solutions of (5.1) are all in light-like directions. Then the Schwartz kernel of the normal operator $X_M^* \circ X_M$ is a paired Lagrangian distribution and the previous argument doesn't work. Now, we start with the composition $X_M \circ E_\pm$ and show that this is a Fourier integral operator. We consider the plus sign case as the other case is identical. We recall from (4.1) that

$$X_M f(y, v) = (2\pi)^{-3} \int e^{i((x-y)\cdot\eta + tv\cdot\eta)} f(t, x) \, dt \, dx \, d\eta$$

and from Section 5 that

$$E_+(f)(t,x) = (2\pi)^{-3} \int e^{i((x-z)\cdot\xi + ct|\xi|)} f(z) dz d\xi$$

Here, we recall that $M = [t_0, t_1] \times \mathbb{R}^3$ and $t_0 = 0$. The canonical relations are parametrized as

$$C = \{((y, v, \eta, w); (t, x, \zeta_0, \zeta')) \in (T^*\mathscr{C}\backslash 0) \times (T^*M^\circ\backslash 0) : y = x - tv, \ \ \eta = \zeta',$$
$$w = t\zeta'|_{T_v\mathbb{S}^2}, \ \ \zeta_0 = -\zeta'v, \ \ y \in \mathbb{R}^3, v \in \mathbb{S}^2, \eta \in \mathbb{R}^3\},$$

$$C^+ = \{(t, x, \zeta_0, \zeta'; z, \xi) \in T^*M^\circ\backslash 0 \times T^*\mathbb{R}^3\backslash 0 : x = z - t\xi/|\xi|, \zeta' = \xi, \zeta_0 = |\xi|\}$$

We want to compose them as two Fourier integral operators. First of all, we prove

Lemma 7.1. *The composition of $C^\pm$ and C is clean with excess one.*

Proof. We consider the plus sign case and the minus sign case is identical. We look at

$$\mathscr{X} = C \times C^+, \ \ \mathscr{Y} = T^*\mathscr{C} \times \mathrm{diag}(T^*M^\circ) \times T^*\mathbb{R}^3$$

and we need to show that $T_p(\mathscr{X} \cap \mathscr{Y}) = T_p\mathscr{X} \cap T_p\mathscr{Y}, p \in \mathscr{X} \cap \mathscr{Y}$. We use variables $(v, t, x, \zeta') \in \mathscr{A} \overset{\text{def}}{=} \mathbb{S}^2 \times (t_0, t_1) \times \mathbb{R}^3 \times \mathbb{R}^3$ to parametrize C and let $\pi : \mathscr{A} \to C, \pi(v, t, x, \zeta') = (y, v, \eta, w, t, x, \zeta_0, \zeta')$. We use variables $(t, x, \zeta') \in \mathscr{B} \overset{\text{def}}{=} (t_0, t_1) \times \mathbb{R}^3 \times \mathbb{R}^3$ to parametrize C^+ and let $\pi_+ : \mathscr{B} \to C^+, \pi_+(t, x, \zeta') = (t, x, \zeta_0, \zeta', z, \xi)$. Then we compute the Jacobian

$$d\pi = \begin{bmatrix} -t & -v & \mathrm{Id} & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ * & \zeta'|_{T_v\mathbb{S}^2} & 0 & t\,\mathrm{Id}\,|_{T_v\mathbb{S}^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ -\zeta'|_{T_v\mathbb{S}^2} & 0 & 0 & -v \\ 0 & 0 & 0 & \mathrm{Id} \end{bmatrix}$$

where $*$ denotes a term we didn't compute explicitly. Let $(\delta v, \delta t, \delta x, \delta\zeta') \in T\mathscr{A}$ be a tangent vector at $(v, t, x, \zeta') \in \mathscr{A}$, then the tangent vector to C is $d\pi(\delta v, \delta t, \delta x, \delta\zeta')$. Next, we compute

$$d\pi_+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \zeta'/|\zeta'| \\ 0 & 0 & \mathrm{Id} \\ \zeta'/|\zeta'| & \mathrm{Id} & td_{\zeta'}(\zeta'/|\zeta'|) \\ 0 & 0 & \mathrm{Id} \end{bmatrix}$$

Similarly, if $(\delta t, \delta x, \delta\zeta') \in T\mathscr{B}$ is a tangent vector at $(t, x, \zeta') \in \mathscr{B}$, then $d\pi_+(\delta t, \delta x, \delta\zeta')$ will be the tangent vector to C^+. So we find parametrizations of $T_p\mathscr{X}$ at $p = (\pi(v, t, x, \zeta'), \pi_+(\widetilde{t}, \widetilde{x}, \widetilde{\zeta}'))$. For $p \in \mathscr{X} \cap \mathscr{Y}$, we must have $(t, x, \zeta') = (\widetilde{t}, \widetilde{x}, \widetilde{\zeta}')$ and the tangent vector in $T_p\mathscr{X}$ is given by
(7.1)
$$(d\pi(\delta v, \delta t, \delta x, \delta\zeta'), d\pi_+(\delta\widetilde{t}, \delta\widetilde{x}, \delta\widetilde{\zeta}'))$$
$$= (-t(\delta v) - v(\delta t) + \delta x, \ \ \delta t, \ \ \delta\zeta', \ \ [(*)(\delta v) + \zeta'(\delta t) + t(\delta\zeta')]|_{T_v\mathbb{S}^2}, \ \ \delta t, \ \ \delta x, \ \ -\zeta'(\delta v) - v(\delta\zeta'), \ \ \delta\zeta',$$
$$\delta\widetilde{t}, \ \ \delta\widetilde{x}, \ \ \zeta'/|\zeta'|(\delta\widetilde{\zeta}'), \ \ \delta\widetilde{\zeta}', \ \ \zeta'/|\zeta'|(\delta\widetilde{t}) + \delta\widetilde{x} + td_{\zeta'}(\zeta'/|\zeta'|(\delta\widetilde{\zeta}'), \ \ \delta\widetilde{\zeta}')$$

So tangent vectors in $T_p \mathscr{X} \cap T_p \mathscr{Y}$ must satisfy $\delta t = \delta \widetilde{t}, \delta x = \delta \widetilde{x}, \delta \zeta = \delta \widetilde{\zeta}$ and furthermore

(7.2)
$$-\zeta'(\delta v) - v(\delta \zeta') = \zeta'/|\zeta'|(\delta \zeta').$$

However, in $\mathscr{X} \cap \mathscr{Y}$, we have $-\zeta' v = |\zeta'|$ so that $v = -\zeta'/|\zeta'|$. We also get $-\zeta'(\delta v) = 0$ because $\zeta = (\zeta^0, \zeta')$ is light-like. Thus (7.2) is automatically satisfied. The tangent vectors in $T_p \mathscr{X} \cap T_p \mathscr{Y}$ consist of vectors in (7.1) where all the tildes are removed.

The intersection $\mathscr{X} \cap \mathscr{Y}$ is parametrized by (v, t, x, ζ') and the map from these variables to $\mathscr{X} \cap \mathscr{Y}$ is $\widetilde{\pi} \overset{\text{def}}{=} (\pi, \pi_+)$. Thus, $T_p(\mathscr{X} \cap \mathscr{Y})$ is spanned by

$$(d\pi(\delta v, \delta t, \delta x, \delta \zeta'), d\pi_+(\delta t, \delta x, \delta \zeta'))$$

This agrees with (7.1) when the tildes are removed. Therefore, $T_p(\mathscr{X} \cap \mathscr{Y}) = T_p \mathscr{X} \cap T_p \mathscr{Y}$ and we proved that the intersection is clean.

Now we let $\widetilde{C}^+ \overset{\text{def}}{=} C \circ C^+$ and find that

$$\widetilde{C}^+ = \{((y, v, \eta, w); (z, \xi)) \in (T^* \mathscr{C} \backslash 0) \times (T^* \mathbb{R}^3 \backslash 0) : y = z, \quad \eta = \xi,$$
$$w = 0, \quad v = -\xi/|\xi|, \quad y \in \mathbb{R}^3, v \in \mathbb{S}^2, \eta \in \mathbb{R}^3, x \in \mathbb{R}^3\}.$$

To find the excess, we consider $p_0 \overset{\text{def}}{=} ((y, v, \eta, w); (z, \xi)) \in \widetilde{C}^+$. The fiber over p_0 in $C \times C^+$ is simply (t, x, ζ_0, ζ') where

$$x = z - t(\xi/|\xi|), \zeta' = \xi, \zeta_0 = |\xi|, \quad t \in (t_0, t_1)$$

which is one dimensional. This shows that the excess is one and we note that the fiber is connected.

The minus sign case is the same. For later reference, we record that

$$\widetilde{C}^- = \{((y, v, \eta, w); (z, \xi)) \in (T^* \mathscr{C} \backslash 0) \times (T^* \mathbb{R}^3 \backslash 0) : y = z, \quad \eta = \xi,$$
$$w = 0, \quad v = \xi/|\xi|, \quad y \in \mathbb{R}^3, v \in \mathbb{S}^2, \eta \in \mathbb{R}^3, x \in \mathbb{R}^3\}.$$

$\square$

Before we proceed, we remark that the union $\widetilde{C}^+ \cup \widetilde{C}^-$ actually is the twisted conormal bundle of the point line relation $\widetilde{Z} = \{(\gamma, z) \in \mathscr{C} \times \mathscr{S}_0 : z \in \gamma\}$ and $\widetilde{C}^+ \cap \widetilde{C}^- = \emptyset$. Comparing with the previous point line relation Z, we see that here we only consider points on $\mathscr{S}_0$ instead of all points on M.

At this point, it is natural to apply the clean intersection FIO composition theorem. However, we know that X_M is properly supported but $E_\pm$ are not. If we add a smooth cut-off function $\chi(t, x) = \chi(t)$ which is positive and compactly supported in $t \in (t_0, t_1)$, then χE_+ is properly supported. We can then apply the clean composition theorem [8, Theorem 25.2.3] to conclude that $X_M \circ \chi E_+ \in I^{-\frac{1}{2}}(\mathscr{S}_0 \times \mathscr{C}; (\widetilde{C}^+)')$. The principal symbol of $X_M \circ \chi E_+$ at $\gamma \in \widetilde{C}^+$ is

$$a = \int_{C_\gamma} a_1 a_2$$

where a_1, a_2 are the principal symbols of X_M and χE_+ respectively and the integral is over the fiber C_γ of γ in $C \circ C^+$. Since a_1, a_2 are non-zero constant and χ is assumed to be positive we conclude that a is non-zero hence $X_M \circ \chi E_+$ is an elliptic FIO.

To justify the composition without introducing the cut-off, we examine the proof of the clean composition theorem [8, Theorem 25.2.3]. From the oscillatory integral representations, we have

(7.3)
$$X_M E_+ f(y, v) = (2\pi)^{-6} \int e^{i((x-y)\cdot\eta + tv\cdot\eta + (x-z)\cdot\xi + t|\xi|)} f(z) dz d\xi dt dx d\eta$$

Here, the amplitude function is a constant. When there are lower order terms in the wave equation, the amplitude would be $a(t, x, \xi)$, but this will not change the argument below. We should have introduced partition of unities for $\mathscr{C}, M, \mathscr{S}_0$ and considered the amplitudes locally but we omitted this step for simplicity. Consider the phase function

$$\phi(y, v, z; \xi, \eta, x, t) = (x - y) \cdot \eta + tv \cdot \eta + (x - z) \cdot \xi + t|\xi|$$

where $y, z, x \in \mathbb{R}^3, \xi, \eta \in \mathbb{R}^3, v \in \mathbb{S}^2, t \in (t_0, t_1)$. We know from the previous lemma that this is a clean phase function with excess one, see [8, Proposition 25.2.2]. We split the parameters $\theta = (\xi, \eta, x, t)$ to $\theta' = (\xi, \eta)$ and $\theta'' = (x, t)$. We remark that $t \in (t_0, t_1)$ is treated as a parameter. Because the amplitude is compactly supported, we can take t in $\mathbb{R}$.

The critical set of the phase function $\{(y, v, z; \theta) : d_\theta \phi = 0\}$ is defined by the following equations

$$\phi_\xi = x - z + t\xi/|\xi| = 0, \quad \phi_\eta = x - y + tv = 0, \quad \phi_x = \eta + \xi = 0, \quad \phi_t = v \cdot \eta + |\xi| = 0.$$

We deduce that

(7.4) $$\eta = -\xi, \quad \xi/|\xi| = v, \quad y = z, \quad x = y + tv.$$

For fixed (x, t), we compute the Hessian of ϕ

(7.5) $$\text{Hess}(\phi) \overset{\text{def}}{=} \phi''_{(y,v,z;\xi,\eta)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\text{Id} \\ 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & -\text{Id} & 0 \\ 0 & 0 & -\text{Id} & td_\xi(\frac{\xi}{|\xi|}) & 0 \\ -\text{Id} & * & 0 & 0 & 0 \end{pmatrix}$$

in which $*$ denotes terms that we haven't computed yet. To compute these terms, we introduce coordinates on $\mathbb{S}^2$ to represent v, that is

$$v = (\cos \alpha, \sin \alpha \cos \beta, \sin \alpha \sin \beta), \quad \alpha \in [0, \pi], \beta \in [0, 2\pi).$$

Then we have

$$d_v \phi = d_{(\alpha,\beta)} \phi = d_{(\alpha,\beta)}(tv \cdot \eta) = t \begin{pmatrix} -\eta_1 \sin \alpha + \eta_2 \cos \alpha \cos \beta + \eta_3 \cos \alpha \sin \beta \\ -\eta_2 \sin \alpha \sin \beta + \eta_3 \sin \alpha \cos \beta \end{pmatrix}$$

Next,

$$d_\eta(\frac{\partial \phi}{\partial v}) = t \begin{pmatrix} -\sin \alpha & \cos \alpha \cos \beta & \cos \alpha \sin \beta \\ 0 & \sin \alpha \sin \beta & \sin \alpha \cos \beta \end{pmatrix}$$

and

$$d_v(\frac{\partial \phi}{\partial v}) = t \begin{pmatrix} -\eta_1 \cos \alpha - \eta_2 \sin \alpha \cos \beta - \eta_3 \sin \alpha \sin \beta & -\eta_2 \cos \alpha \sin \beta + \eta_3 \cos \alpha \cos \beta \\ -\eta_2 \cos \alpha \sin \beta + \eta_3 \cos \alpha \cos \beta & -\eta_2 \sin \alpha \cos \beta - \eta_3 \sin \alpha \sin \beta \end{pmatrix}$$

$$= t|\xi| \begin{pmatrix} 1 & 0 \\ 0 & -\sin^2 \alpha \end{pmatrix}$$

Here, we used the fact that on the critical set, $\eta = -\xi$ and $\eta = -|\xi|v$. The $*$ in the last row is $d_{\eta v}^2 \phi = (d_{v\eta}^2 \phi)^T$ where T denotes transpose.

Now we see that $\text{Hess}(\phi)$ is non-degenerate. At $\alpha = 0, \pi$ and $t = 0$, there is a coordinate singularity and we should use spherical coordinates relative to a different axis, that is using an orthogonal transformation. Now according to [8, Proposition 25.1.5'], the principal symbol of $X_M E_+$ should be given by

(7.6) $$C \int e^{(\pi i/4)\text{sgn}(\text{Hess}(\phi))} |\det \text{Hess}(\phi)(y, v, z; \theta)|^{-\frac{1}{2}}$$

provided that the integral is well-defined. Here, C is a non-zero constant and the integration is over the fiber over $d_{(y,v,z)}\phi$. From (7.4), the fiber is given by $(x,t) \in \mathbb{R}^3 \times (t_0, t_1)$ such that $x = y + tv$, so the fiber is connected, one-dimensional but not compact. The non-compactness is the consequence of E_+ not being properly supported. But it follows from the expression (7.5) and the argument in the previous paragraph that $\mathrm{Hess}(\phi)$ is smooth and non-degenerate on $[t_0, t_1]$. The fiber is contained in $\{(x,t) : x = y + tv, x \in \mathbb{R}^3, t \in [t_0, t_1]\}$ which is compact and $|\det \mathrm{Hess}(\phi)|^{-\frac{1}{2}}$ is smooth there. Therefore, the integral (7.6) is indeed well-defined. This justified that

Lemma 7.2. $X_M \circ E_\pm \in I^{-\frac{1}{2}}(\mathscr{S}_0 \times \mathscr{C}; (\widetilde{C}^\pm)')$ *are elliptic Fourier integral operators.*

Now we prove

Lemma 7.3. *The normal operators* $E_+^* X_M^* \circ X_M E_+, E_-^* X_M^* \circ X_M E_-$ *are elliptic pseudo-differential operators of order* -1 *on* $\mathbb{R}^3$, *and* $E_+^* X_M^* X_M E_-, E_-^* X_M^* X_M E_+$ *are smoothing operators on* $\mathbb{R}^3$.

Proof. We consider the composition $E^{+,*} X_M^* \circ X_M E^+$. If we look at the double fibration picture

$$
\begin{array}{ccc}
 & \widetilde{C}^+ & \\
\pi \swarrow & & \searrow \rho \\
T^*\mathscr{S}_0 & & T^*\mathscr{C}
\end{array}
$$

(7.7)

we realize that ρ is an injective immersion. Moreover, the other projection π is also an injective immersion. Thus, the composition of $C^{+,-1}$ and C^+ is clean with excess 0 so by [8, Theorem 25.2.3] again, we get that $E^{+,*} X_M^* \circ X_M E^+$ is a pseudo-differential operator of order -1 on $\mathscr{S}_0$. The principal symbol is non-vanishing so it is elliptic. For the mixed sign case, because $\widetilde{C}^+, \widetilde{C}^-$ are disjoint, the operators are smoothing. $\square$

We obtain parallel results about the microlocal inversion as to Proposition 6.3 for the $c < 1$ case. The proofs are identical hence omitted here.

Proposition 7.4. *For $c = 1$, there exist operators A_1, A_2 such that*

$$A_1 X_M f = f_1 + R_1 f_1 + R_1' f_2, \quad A_2 X_M f = f_2 + R_2 f_1 + R_2' f_2$$

where R_1, R_2, R_1', R_2' are smoothing operators and $A_i = \widetilde{A}_i \circ X_M^*$ with $\widetilde{A}_i$ Fourier integral operators.

Corollary 7.5. *For $c = 1$, there exist operators A such that*

$$AX_M f = f + R_1 f_1 + R_2 f_2,$$

where R_1, R_2 are smoothing operators.

8. THE STABLE DETERMINATION

We prove Theorem 1.1, starting with the injectivity of the light ray transform. It is known, see for instance [16, 10], that the light ray transform on $\mathbb{R}^{n+1}$ is injective on C_0^∞ functions. This also holds for L^1_{comp} functions and the proof is similar, see [16].

Theorem 8.1. *Suppose $f \in L^1_{\mathrm{comp}}(\mathbb{R}^{n+1}), n \geq 2$ and $X_{\mathbb{R}^{n+1}} f = 0$. Then $f = 0$.*

Proof. For $f \in L^1_{\text{comp}}(\mathbb{R}^{n+1})$, the Fourier transform $\hat{f}$ is analytic. Let $\theta \in \mathbb{S}^{n-1}$ and $\Theta = (1, \theta)$ be a light-like vector. Let $z = (s, y + s\theta) \in \mathbb{R}^{n+1}, s \in \mathbb{R}, y \in \mathbb{R}^n$. We parametrize the light ray transform as

$$X_{\mathbb{R}^{n+1}} f(z, \Theta) = \int_{\mathbb{R}} f(t, y + t\theta) dt.$$

From the standard Fourier Slice Theorem for geodesic ray transforms on $\mathbb{R}^{n+1}$, we get

$$\hat{f}(\zeta) = \int_{\Theta^\perp} e^{-iy \cdot \zeta} X_{\mathbb{R}^{n+1}} f(z, \Theta) dS_z$$

where the integration is over a plane $\Theta^\perp$ perpendicular to Θ with respect to the Euclidean inner product in $\mathbb{R}^{n+1}$ and $\zeta = (\tau, \xi) \in \mathbb{R}^{n+1}, \xi \in \mathbb{R}^n, \xi \neq 0$ is perpendicular to Θ. We notice that if $|\tau| \leq |\xi|$, then there is a null vector $(1, \theta)$ which is Euclidean orthogonal to ζ. Actually, $\tau + \theta \cdot \xi = 0$ so $\theta \cdot (\xi/|\xi|) = -\tau/|\xi| \in [-1, 1]$ and we can find $\theta \in \mathbb{S}^{n-1}$. We conclude that $\hat{f}(\zeta) = 0$ for $|\tau| \leq |\xi|$. By analyticity, $\hat{f} = 0$ and thus $f = 0$. $\qquad \square$

Corollary 8.2. *Suppose $X_M f = 0$ where f satisfies the wave equation constraint (1.2) in which $f_1 \in H^{s+1}_{\text{comp}}(\mathbb{R}^3), f_2 \in H^s_{\text{comp}}(\mathbb{R}^3), s \geq 0$ are compactly supported. Then $f = f_1 = f_2 = 0$.*

Proof. Let $K = \text{supp } f_1 \cup \text{supp } f_2 \subset \mathbb{R}^3$. Let $I_c^+(K)$ be the chronological future of K with respect to the Lorentzian metric induced by c. We know that there is a unique solution $f \in H^{s+1}(M)$ of (1.2). By finite speed of propagation (or strong Huygens principle), the solution f is supported in $I_c^+(K) \cap M$. Now we extend f trivially to $\tilde{f} \in L^1_{\text{comp}}(\mathbb{R}^4)$ and we regard X_M as the light ray transform $X_{\mathbb{R}^4}$ on $\mathbb{R}^4$. We still have $X_{\mathbb{R}^4} \tilde{f} = 0$. By Theorem 8.1, we conclude that $f = 0$ on $\mathbb{R}^4$ so that $f = 0$ on M and $f_1 = f_2 = 0$ on $\mathscr{S}_0$. $\qquad \square$

Proof of Theorem 1.1. The uniqueness is done in Corollary 8.2. From Proposition 6.3 and 7.4, we know that for $c \in (0, 1]$, there are operators A_1, A_2 such that

$$A_1 X_M f = f_1 + R_1 f_1 + R_1' f_2, \quad A_2 X_M f = f_2 + R_2 f_1 + R_2' f_2$$

and $R_i, R_i', i = 1, 2$ are all smoothing operators. We denote

$$T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \text{Id} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + K \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad K = \begin{pmatrix} R_1 & R_1' \\ R_2 & R_2' \end{pmatrix}$$

We consider T acting on $\mathcal{N}^s, s \geq 0$. Then K is compact from $\mathcal{N}^s$ to $\mathcal{N}^{s-\rho}, \rho \in \mathbb{R}$. So we have the estimate

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq \|A_1 X_M f\|_{H^{s+1}(\mathbb{R}^3)} + \|A_2 X_M f\|_{H^s(\mathbb{R}^3)} + C_\rho \|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}}$$

for some constant C_ρ. We recall from Proposition 6.3 (the same holds for Proposition 7.4) that $A_1 = B_+ \circ (X_M \circ E_+)^* + B_- (X_M \circ E_-)^*$ and $A_2 = ic\Delta^{\frac{1}{2}} (B_+ \circ (X_M \circ E_+)^* + B_- \circ (X_M \circ E_-)^*)$. We know from Lemma 7.2 that $X_M \circ E_\pm \in I^{-\frac{1}{2}}(\mathscr{S}_0 \times \mathscr{C}; (\tilde{C}^\pm)')$. The normal operator is a pseudo-differential operator of order -1. By the L^2 estimate of pseudo-differential operators, we conclude that $X_M \circ E_\pm : H^s_{\text{comp}}(\mathbb{R}^3) \to H^{s+\frac{1}{2}}_{\text{loc}}(\mathscr{C})$ is bounded. Next, $(X_M \circ E_\pm)^* \in I^{-\frac{1}{2}}(\mathscr{C} \times \mathscr{S}_0; (\tilde{C}^{\pm,-1})')$. We consider the double fibration (7.7). The two projections π, ρ are both injective, so the normal operator of $(X_M \circ E_\pm)^*$ is an elliptic pseudo-differential operator of order -1 as well but on $\mathscr{C}$. Thus, we know that $(X_M \circ E_\pm)^* : H^s_{\text{comp}}(\mathscr{C}) \to H^{s+\frac{1}{2}}_{\text{loc}}(\mathbb{R}^3)$ is bounded. Therefore, $A_1 : H^{s+\frac{1}{2}}_{\text{comp}}(\mathscr{C}) \to$

$H^s_{\text{loc}}(\mathbb{R}^3)$ and $A_2 : H^{s+\frac{1}{2}}_{\text{comp}}(\mathscr{C}) \to H^{s-\frac{1}{2}}_{\text{loc}}(\mathbb{R}^3)$ are bounded. For $(f_1, f_2) \in \mathscr{N}^s$, we know from (5.2) that $X_M f = X_M E_+ h_1 + X_M E_- h_2$ and $h_1, h_2 \in H^{s+1}(\mathbb{R}^3)$. Thus, $X_M f \in H^{s+3/2}(\mathscr{C})$ so we get

$$(8.1) \qquad \|(f_1, f_2)\|_{\mathbb{N}^s} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})} + C_\rho \|(f_1, f_2)\|_{\mathbb{N}^{s-\rho}}$$

where $C_\rho > 0$ is a constant depending on ρ.

Finally, we get rid of the last term. Let $\mathscr{K}$ be a compact subset of $\mathbb{R}^3$ and denote by $\mathbb{N}^s(\mathscr{K})$ the function space consisting of $(f_1, f_2) \in \mathbb{N}^s$ supported in $\mathscr{K}$. Then the inclusion of $\mathbb{N}^s(\mathscr{K})$ into $\mathbb{N}^{s-\rho}(\mathscr{K}), \rho > 0$ is compact. We claim that

$$\|(f_1, f_2)\|_{\mathbb{N}^s(\mathscr{K})} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})}$$

for some $C > 0$. We argue by contradiction. Assume the estimate without the error term is not true. We can get a sequence $(f_1^{(j)}, f_2^{(j)}), j = 1, 2, \cdots$ with unit norm in $\mathbb{N}^s(\mathscr{K})$ such that $X_M f^{(j)}$ goes to 0 in $H^{s+3/2}(\mathscr{C})$. By (8.1) (for (f_1, f_2) supported in $\mathscr{K}$), we conclude that $1 = \|(f_1^{(j)}, f_2^{(j)})\|_{\mathbb{N}^s(\mathscr{K})} \leq C_\rho \|(f_1^{(j)}, f_2^{(j)})\|_{\mathbb{N}^{s-\rho}(\mathscr{K})}$. This gives a weak limit (f_1, f_2) in $\mathbb{N}^s(\mathscr{K})$ along a subsequence, which thus converges strongly in $\mathbb{N}^{s-\rho}(\mathscr{K})$. Therefore, $\|(f_1, f_2)\|_{\mathbb{N}^{s-\rho}(\mathscr{K})}$ is bounded below by $1/C_\rho$, thus non-zero. However, $X_M f = 0$ which contradicts the injectivity of X_M. This finishes the proof. $\qquad\square$

At last, we prove a stronger version of Theorem 1.1 which allows lower order terms in the wave equation. We consider differential operators of the form

$$P(x, t, D_x, \partial_t) = \partial_t^2 + c^2 \sum_{i=1}^{3} D_{x_i}^2 + P_1(x, t, D_x, \partial_t) + P_0(x, t)$$

where P_1 is a first order differential operator with real valued smooth coefficients and P_0 is smooth. Then we consider the Cauchy problem

$$(8.2) \qquad \begin{aligned} P(x, t, D_x, \partial_t)f &= 0 \quad \text{on } M^\circ \\ f = f_1, \ \partial_t f &= f_2, \quad \text{on } \mathscr{S}_0. \end{aligned}$$

We remark that the equations for Φ in Section 3 are of this type. We prove

Theorem 8.3. *Under the same assumptions as in Theorem 1.1, $X_M f$ uniquely determines f and f_1, f_2 which satisfy (8.2). Moreover, there exists a $C > 0$ such that*

$$\|(f_1, f_2)\|_{\mathbb{N}^s} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})} \text{ and } \|f\|_{H^{s+1}(M)} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})}$$

where $\mathscr{C}$ is the set of light rays on M.

Proof. The proof follows the same arguments as for Theorem 1.1. So we just point out what needs to be modified. When the wave equation contains lower order terms, one can construct parametrices $E_\pm$ for the Cauchy problem, see [3, Chapter 5]. These are Fourier integral operators and can be represented by oscillatory integrals. So the construction in Section 5 works through, and the microlocal structure of $X_M E_\pm$ is the same as the standard wave equation case. However, we do need to justify the ellipticity of the involved operators in Lemma 6.2, 7.2 and 7.3. We remark that ellipticity of the solution itself is standard, and follows simply from the principal symbol satisfying a transport equation, but that only implies the ellipticity of the normal operator if the integral computing its symbol still gives an elliptic result, typically ensured by showing that there can be no cancellations. We follow the parametrix construction in Trèves [19, Section 1, Chapter VI] to check this in a transparent manner.

We look for operators $E_j, j = 0, 1$ such that

$$P(x, t, D_x, \partial_t)E_j = 0 \quad \text{on } M^\circ$$

$$\partial_t^k E_j = \delta_{kj}, k = 0, 1, \quad \text{on } \mathscr{S}_0.$$

Here, for $j = 0, 1$ we have

$$E_j f(x) = (2\pi)^{-3} \int e^{i\phi_0(x,t,\xi)} a_{j0}(x,t,\xi)\hat{f}(\xi)d\xi + (2\pi)^{-3} \int e^{i\phi_1(x,t,\xi)} a_{j1}(x,t,\xi)\hat{f}(\xi)d\xi + R_j(t)f(x)$$

where R_j are smoothing operators, see [19, (1.37)]. The phase functions are

$$\phi_0(x, t, \xi) = x \cdot \xi + ct|\xi|, \quad \phi_1(x, t, \xi) = x \cdot \xi - ct|\xi|.$$

The amplitude can be written as $a_{jk}(x, t, \xi) = \sum_{l=0}^{\infty} a_{jkl}(x, t, \xi)$ and each a_{jkl} is homogeneous of degree $-j - l$ for $|\xi|$ large. Before we look into the structures that we need of the amplitude, we find the initial values of the leading order term a_{jk0} at $t = t_0$. They satisfy (see [19, (1.53)])

$$a_{000}(x, t, \xi) = \frac{1}{2}, \quad a_{010}(x, t, \xi) = \frac{1}{2}, \quad a_{100}(x, t, \xi) = \frac{1}{2ic|\xi|}, \quad a_{110}(x, t, \xi) = -\frac{1}{2ic|\xi|}.$$

The amplitudes satisfy first order equations which are deduced from (see [19, (1.39)])

$$P(x, t, D_x + \partial_x\phi_k, \partial_t + i\partial_t\phi_k)a_{jk}(x, t, \xi) = 0.$$

For the leading order term, we get

$$(8.3) \quad \partial_\tau P_2(x, t, \partial_x\phi_k, i\partial_t\phi_k)\partial_t a_{jk0} + \sum_{\nu=1}^{3} \partial_{\xi_\nu} P_2(x, t, \partial_x\phi_k, i\partial_t\phi_k)D_{x^\nu} a_{jk0} + C(\phi_k; x, t, \xi)a_{jk0} = 0$$

and the C term in this case is (the sub-principal symbol of P)

$$C(\phi_k; x, t, \xi) = P_1(x, t, \partial_x\phi_k, i\partial_t\phi_k)$$

Dividing by $i = \sqrt{-1}$, equation (8.3) is a first order linear equation with real valued coefficients. Solving the equation amounts to solving a ODE along the integral curve and the solution a_{jk0} will be positive scalar multiples of the initial conditions hence not only non-vanishing, but is real or purely imaginary depending on its initial value.

Finally, we can represent the solution to (8.2) as

$$f(x, t) = E_0 f_1 + E_1 f_2 = E_+ h_1 + E_- h_2$$

where

$$(8.4) \quad \begin{aligned} E_+ h &= (2\pi)^{-3} \int e^{i(x\cdot\xi+ct|\xi|)}(a_{00}(x, t, \xi) + 2ic|\xi|a_{10}(x, t, \xi))\hat{h}(\xi)d\xi \\ &= (2\pi)^{-3} \int e^{i(x\cdot\xi+ct|\xi|)} a_+(x, t, \xi)\hat{h}(\xi)d\xi \\ E_- h &= (2\pi)^{-3} \int e^{i(x\cdot\xi-ct|\xi|)}(a_{01}(x, t, \xi) - 2ic|\xi|a_{11}(x, t, \xi))\hat{h}(\xi)d\xi \\ &= (2\pi)^{-3} \int e^{i(x\cdot\xi-ct|\xi|)} a_-(x, t, \xi)\hat{h}(\xi)d\xi \end{aligned}$$

and

$$h_1 = f_1 + \frac{1}{2ic}\Delta^{-\frac{1}{2}}f_2, \quad h_2 = f_1 - \frac{1}{2ic}\Delta^{-\frac{1}{2}}f_2.$$

We see that the leading order terms of a_+, a_- are all positive. From these oscillatory integral representations, it is easy to see that Lemma 6.2 holds for this case. For Lemma 7.2, we see that

the integral of the principal symbol over the fiber is non-vanishing so the operators $X_M E_\pm$ are elliptic. Then the ellipticity of the normal operators in Lemma 7.3 is justified. The rest of the proof is the same as in Theorem 1.1. $\qquad\square$

9. Small perturbations of the Minkowski spacetime

We consider metric perturbations $g_\delta = g_M + h$ with $h = \sum_{i,j=0}^{3} h_{ij} dx^i dx^j$. We assume that

(A1) h is a symmetric two tensor smooth on M;

(A2) for $\delta > 0$ small, the seminorm $\|h_{ij}\|_{C^3} = \sup_{(t,x)\in M} \sum_{|\alpha|\leq 3} |\partial^\alpha h_{ij}(t,x)| < \delta$, $i,j = 0,1,2,3$.

Without loss of generality, we can assume that h is extended to some larger manifold $\widetilde{M} = (\widetilde{t}_0, \widetilde{t}_1) \times \mathbb{R}^3$ such that $M \subset \widetilde{M}$ and (A2) holds on $\widetilde{M}$. In this section, we study the inverse problem on (M, g_δ) for δ sufficiently small. Note that in this case, light rays may not follow straight lines and the injectivity of the light ray transform on scalar functions is not known. We will show that by using a perturbation argument on the Fourier integral operator level, one can obtain the same determination result as for the Minkowski case.

We start with the light like geodesics on (M, g_δ) and their parametrizations. Let $\gamma(s)$ denote a light like geodesic from $\mathscr{S}_0$. It satisfies

$$
(9.1) \qquad
\begin{aligned}
\partial_s^2 \gamma^k(s) + \Gamma_{ij}^k \partial_s \gamma^i(s) \partial_s \gamma^j(s) &= 0 \\
\gamma(0) = (0, y), \partial_s \gamma(0) &= (\beta, v)
\end{aligned}
$$

where Γ_{ij}^k is the Christoffel symbol for g_δ, $v \in \mathbb{S}^2$ and β is such that $g_\delta(\beta, v) = 0$ and (β, v) future pointing. It is known, see for example [1], that (9.1) is equivalent to a first order system on T^*M. Here, M is regarded as a submanifold of $\widetilde{M}$. We use (t,x) and (τ, ξ) for the local coordinates on T^*M. Consider the Hamiltonian

$$
p(t,x,\tau,\xi) = \frac{1}{2} g_\delta^*(\tau,\xi) = \frac{1}{2} g_M^*(\tau,\xi) + H(t,x,\tau,\xi) = \frac{1}{2}\left(-|\tau|^2 + \sum_{i=1}^{3} |\xi_i|^2\right) + H(t,x,\tau,\xi).
$$

Let $\Xi = (\tau, \xi)$. Here, $H(t,x,\Xi) = \sum_{i,j=0,1,2,3} H_{ij}(t,x)\Xi_i\Xi_j$ is homogeneous of degree two in Ξ and the seminorm $\|H_{ij}\|_{C^3} < C\delta$ for some constants C. We denote the Hamilton vector field by H_p. Let $(t(s), x(s), \tau(s), \xi(s))$ be an integral curve of H_p in the characteristic set $\Sigma_p = \{(t,x,\tau,\xi) \in T^*M : p(t,x,\tau,\xi) = 0\}$, called null-bicharacteristics. With $\gamma(s) = (t(s), x(s))$, (9.1) can be converted to

$$
(9.2) \qquad
\begin{aligned}
\frac{dt}{ds} = \frac{\partial p}{\partial \tau} = -\tau + \partial_\tau H(t,x,\tau,\xi); &\quad \frac{dx_i}{ds} = \frac{\partial p}{\partial \xi_i} = \xi_i + \partial_{\xi_i} H(t,x,\tau,\xi) \\
\frac{d\tau}{ds} = -\partial_t H(t,x,\tau,\xi); &\quad \frac{d\xi_i}{ds} = -\partial_{x_i} H(t,x,\tau,\xi), \quad i = 1,2,3 \\
t(0) = t_0 = 0, \quad x_i(0) = y_i, &\quad \tau(0) = \tau_0, \quad \xi_i(0) = \xi_{0,i}.
\end{aligned}
$$

Here, (τ_0, ξ_0) is the cotangent vector obtained from (β, v) using g_δ and we also denote it by $(\tau_0, \xi_0) = (\beta, v)^\flat$. If we consider the system for the Minkowski metric namely $H = 0$, then $\beta = 1$ and the covector $(\tau_0, \xi_0) = (-1, v)$. (9.2) becomes

$$
(9.3) \qquad
\begin{aligned}
\frac{dt}{ds} = -\tau, \quad \frac{dx_i}{ds} = \xi_i, \quad \frac{d\tau}{ds} = 0, \quad \frac{d\xi_i}{ds} = 0, \quad i = 1,2,3 \\
t(0) = 0, \quad x_i(0) = y_i, \quad \tau(0) = -1, \quad \xi_i(0) = v_i.
\end{aligned}
$$

We see that $x(s) = (s, y + sv), t(s) = s$, which agrees with our parametrization used previously. Now we have the following result.

Lemma 9.1. *For $\delta > 0$ sufficiently small, the set of light rays on (M, g_δ) is given by $\mathscr{C}_\delta = \{\gamma = (t, x(t, y, v)) : (y, v) \in \mathscr{S}_0 \times \mathbb{S}^2, t \in [t_0, t_1]\}$, where x is a smooth function of t, y, v. Moreover, we have*

$$\|x(t, y, v) - (y + tv)\|_{C^2} < C\delta$$

for some constant C.

Proof. For $v \in \mathbb{S}^2$, the co-vectors $(\tau_0, \xi_0) = (\beta, v)^\flat$ are in a bounded set of $\mathbb{R}^4$. We assume that $|(\tau_0, \xi_0)| < M_1$. We also notice that τ_0 is away from zero, say $|\tau_0| > M_0 > 0$. Then we consider (τ, ξ) such that $|(\tau, \xi) - (\tau_0, \xi_0)| < M_0/2$ so that $|(\tau, \xi)| < M \doteq M_1 + M_0/2$ and $|\tau| > M_0/2$. Consider the system (9.2). Because H is homogeneous of degree two in (τ, ξ), for $|(\tau, \xi)| < M$ and for $\delta > 0$ sufficiently small, we see that $\frac{dt}{ds} \neq 0$. Therefore, we can take t as the parameter and convert (9.2) to

$$\frac{ds}{dt} = \frac{1}{-\tau + \partial_\tau H(t, x, \tau, \xi)}; \quad \frac{dx_i}{dt} = \frac{\xi_i + \partial_{\xi_i} H(t, x, \tau, \xi)}{-\tau + \partial_\tau H(t, x, \tau, \xi)}$$

(9.4)
$$\frac{d\tau}{dt} = \frac{-\partial_t H(t, x, \tau, \xi)}{-\tau + \partial_\tau H(t, x, \tau, \xi)}; \quad \frac{d\xi_i}{dt} = \frac{-\partial_{x_i} H(t, x, \tau, \xi)}{-\tau + \partial_\tau H(t, x, \tau, \xi)}, \quad i = 1, 2, 3$$
$$s(0) = 0, \quad x_i(0) = y_i, \quad \tau(0) = \tau_0, \quad \xi_i(0) = \xi_{0,i}.$$

The system corresponding to (9.3) is

$$\frac{ds}{dt} = \frac{1}{-\tau}; \quad \frac{dx_i}{dt} = \frac{\xi_i}{-\tau}, \quad \frac{d\tau}{dt} = 0; \quad \frac{d\xi_i}{dt} = 0, \quad i = 1, 2, 3$$

(9.5)
$$s(0) = 0, \quad x_i(0) = y_i, \quad \tau(0) = -1, \quad \xi_i(0) = v.$$

Let $(\widetilde{t}, \widetilde{x}, \widetilde{\tau}, \widetilde{\xi})$ be the solution of (9.5) and (t, x, τ, ξ) satisfy (9.4). Then let $u = (t - \widetilde{t}, x - \widetilde{x}, \tau - \widetilde{\tau}, \xi - \widetilde{\xi})$. We see that u satisfies the system

$$\frac{du}{ds} = F(u)$$
$$u(0) = u_0,$$

where F is smooth and $|F(u)| < C\delta$, $|u_0| < C\delta$ for generic constant C. Now it follows from standard ODE theorems, see for instance [9, Theorem 1.2.3] that for δ sufficiently small, there is a unique C^∞ solution u on $[t_0, t_1]$ and $|u| \leq C\delta$. Higher order estimates can be obtained similarly. This finishes the proof. $\qquad\square$

Now we consider the light ray transform X_δ on (M, g_δ). The parametrization of the light rays is not unique, although all choices give rise to equivalent analysis for our purpose. Perhaps the most natural parameterization is to use the cosphere bundle on $\mathscr{S}_0$ of the induced metric. Let $\bar{g}_\delta$ be the induced Riemannian metric of g_δ on $\mathscr{S}_0$. For $y \in \mathscr{S}_0$, let $\mathbb{S}^2_{\delta,y} = \{v \in T\mathscr{S}_0 : \bar{g}_\delta(v, v) = 1\}$. For $v \in \mathbb{S}^2_{\delta,y}$, there is a unique future pointing light like vector (v_0, v) at y. In particular, v_0 is close to 1 for δ small. Then the light ray from $(0, y)$ in direction (v_0, v) is parametrized by $\gamma_{y,v}(s) = \exp_{(0,y)} s(v_0, v), s \in [0, s_1]$ where s is the affine parameter such that $\gamma_{y,v}(0) = (0, y) \in \mathscr{S}_0$ and $\gamma_{y,v}(s_1) \in \mathscr{S}_1$. In this parametrization, we can write

(9.6)
$$X_\delta f(y, v) = \int_0^{s_1} f(\gamma_{y,v}(s)) ds.$$

Now we can identify $\mathbb{S}^2_{\delta,y}$ with $\mathbb{S}^2_y$ via a diffeomorphism. By the above Lemma 9.1, s is a smooth function of y, t and $v \in \mathbb{S}^2$ so we can use t variable to parametrize the light rays. We have

$$X_\delta f(y, v) = \int_0^{t_1} w(y, v, t) f(t, x(t, y, v)) dt, \quad y \in \mathscr{S}_0, v \in \mathbb{S}^2,$$

where w is a weight coming from the change of variables. In fact, w is smooth and close to 1 for δ sufficiently small. w only mildly affects the argument, changing the elliptic principal symbol of the final operator $X_\delta \circ E_+$ in (9.13), thus maintaining ellipticity. For simplicity, we will ignore it in the follows and take

$$(9.7) \qquad X_\delta f(y, v) = \int_0^{t_1} f(t, x(t, y, v)) dt = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((x(t,y,v)-z)\cdot\eta)} f(t, z) dt dz d\eta$$

This is the parametrization of X_δ we work with in the rest of this section. The Schwartz kernel of X_δ is the delta distribution on $\mathscr{C} \times M^\circ$ supported on the point-line relation Z_δ defined by

$$Z_\delta = \{(\gamma, q) \in \mathscr{C} \times M^\circ : q \in \gamma\} = \{(y, v, (t, x)) \in \mathbb{R}^3 \times \mathbb{S}^2 \times M^\circ : x = x(t, y, v)\}.$$

Next, let $\Box_{g_\delta}$ be the Laplace-Beltrami operator on (M, g_δ) and we consider the second order operator

$$P_\delta(x, t, D_x, \partial_t) = \Box_{g_\delta} + P_1(x, t, D_x, \partial_t) + P_0(x, t)$$

where P_1 is a first order differential operator with real valued smooth coefficients and P_0 is smooth. Then we consider the Cauchy problem

$$(9.8) \qquad \begin{aligned} P_\delta(x, t, D_x, \partial_t) f &= 0 \quad \text{on } M^\circ \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathscr{S}_0. \end{aligned}$$

We remark that for sufficiently small metric perturbations, the operators $\Box_{g_\delta}$ and P_δ are both strictly hyperbolic with respect to $\mathscr{S}_0$. Therefore, as in previous sections, the parametrix construction of Duistermaat-Hörmander can be applied. In general, the parametrix does not have a global oscillatory integral representation on M. However, we show below that for sufficiently small perturbations of the Minkowski spacetime, this is possible.

The parametrix construction is the same as in the previous section. We look for operators $E_j, j = 0, 1$ such that

$$\begin{aligned} P_\delta(x, t, D_x, \partial_t) E_j &= 0 \quad \text{on } M^\circ \\ \partial_t^k E_j &= \delta_{kj}, k = 0, 1, \quad \text{on } \mathscr{S}_0. \end{aligned}$$

For $j = 0, 1$ we have

$$E_j f(x) = (2\pi)^{-3} \int e^{i\phi_+(x,t,\xi)} a_{j,+}(x, t, \xi) \hat{f}(\xi) d\xi + (2\pi)^{-3} \int e^{i\phi_-(x,t,\xi)} a_{j,-}(x, t, \xi) \hat{f}(\xi) d\xi + R_j(t) f(x)$$

where R_j are smoothing operators, see [19, (1.37)]. We follow Trèves [19] to find the phase functions $\phi(t, x, \xi)$ for $(t, x) \in (t_0, t_1) \times \mathbb{R}^3$, $\eta \in \mathbb{R}^3$. The phase function should satisfy the eikonal equation

$$p(\nabla \phi) = -|\partial_t \phi|^2 + |\partial_x \phi|^2 + H(\partial_t \phi, \partial_x \phi) = 0$$

By the strict hyperbolicity, there are two solutions for $\partial_t \phi$ denoted by $\partial_t \phi = \lambda_\pm(t, x, \partial_x \phi)$ and $\lambda_\pm$ are smooth functions and homogeneous of degree one in $\partial_x \phi$. We take initial conditions

$\partial_t \phi = x \cdot \eta, \eta \in \mathbb{R}^3$ at $t = 0$. Below, we consider λ_+. The treatment for λ_- is identical. We consider the Hamilton-Jacobi equation

$$(9.9) \qquad \frac{dx}{dt} = -\partial_\xi \lambda_+(t, x, \xi), \quad \frac{d\xi}{dt} = \partial_x \lambda_+(t, x, \xi)$$

$$x(0) = y, \quad \xi(0) = \eta, \quad y \in \mathbb{R}^3, \eta \in \mathbb{R}^3 \backslash 0.$$

We denote the solution by $x(t, y, \eta), \xi(t, y, \eta)$. Then the phase function is

$$(9.10) \qquad \phi_+(t, x, \eta) = x \cdot \eta + \int_0^t \lambda_+(s, x, \xi(s, y, \eta))ds$$

Here, one can express y in terms of x, see [19, Section 2, Chapter VI] for more details. For the Minkowski spacetime, we know $\lambda_+ = |\xi|$ so that (9.9) becomes

$$(9.11) \qquad \frac{dx}{dt} = -\xi/|\xi|, \quad \frac{d\xi}{dt} = 0$$

$$x(0) = y, \quad \xi(0) = \eta.$$

The solution is simply $x(t) = y - t\eta/|\eta|, \xi(t) = \eta$ and the phase function is $\phi_0(t, x, \eta) = x \cdot \eta + t|\eta|$. Using the same argument as for Lemma 9.1, we get

Lemma 9.2. *For $\delta > 0$ sufficiently small, there is a unique smooth solution $(x(t, y, \eta), \xi(t, y, \eta))$ to (9.9) for $t \in [t_0, t_1], y \in \mathbb{R}^3, \eta \in \mathbb{R}^3 \backslash 0$, and they satisfy*

$$\|x(t, y, \eta) - (y - t\eta/|\eta|)\|_{C^2} < C\delta, \quad \|\xi(t, y, \eta) - \eta\|_{C^2} < C\delta.$$

for some constant $C > 0$. It follows that the phase function ϕ_+ in (9.10) is also smooth and satisfies

$$\|\phi_+(t, x, \eta) - (x \cdot \eta + t|\eta|)\|_{C^2} < C\delta|\eta|, \quad \eta \in \mathbb{R}^3 \backslash 0.$$

We remark that similar argument was used in [17] for a backscattering problem. Using this lemma, we can represent the solution to (9.8) as

$$f(x, t) = E_0 f_1 + E_1 f_2 = E_+ h_1 + E_- h_2$$

where

$$(9.12) \qquad E_+ h = (2\pi)^{-3} \int e^{i\phi_+(t, x, \xi)} a_+(x, t, \xi) \hat{h}(\xi) d\xi$$

$$E_- h = (2\pi)^{-3} \int e^{i\phi_-(t, x, \xi)} a_-(x, t, \xi) \hat{h}(\xi) d\xi$$

The $a_\pm$ and h_1, h_2 are the same as in (8.4).

With these preparations, we now state and prove our main result in this section.

Theorem 9.3. *Consider (M, g_δ) which satisfy the assumptions (A1), (A2) in the beginning of this section. Assume that $(f_1, f_2) \in \mathcal{N}^s, s \geq 0$, and f_1, f_2 are supported in a compact set $\mathcal{K}$ of $\mathcal{S}_0$. For $\delta \geq 0$ sufficiently small, $X_\delta f$ uniquely determines f and f_1, f_2 which satisfy (9.8). Moreover, there exists $C > 0$ such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_\delta f\|_{H^{s+3/2}(\mathscr{C}_\delta)} \text{ and } \|f\|_{H^{s+1}(M)} \leq C\|X_\delta f\|_{H^{s+3/2}(\mathscr{C}_\delta)}$$

where $\mathscr{C}_\delta$ is the set of light rays on (M, g_δ).

Proof. We examine the arguments in Section 7 and Section 8 and point out what needs to be modified. We consider the composition of X_δ and E_+ defined in (9.12). We have

$$X_\delta f(y,v) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((x(t,y,v)-z)\cdot\eta)} f(t,z) dt dz d\eta$$

and

$$E_+(f)(t,x) = (2\pi)^{-3} \int e^{i(\phi_+(t,x,\xi)-z\cdot\xi)} a_+(t,x,\xi) f(z) dz d\xi$$

The canonical relations can be described as follows,

$$C_\delta = \{((y,v,\eta,w);(t,x,\zeta_0,\zeta')) \in (T^*\mathscr{C}_\delta \backslash 0) \times (T^*M^\circ \backslash 0) : x = x(t,y,v), \quad \eta = (\partial_y x(t,y,v))\zeta',$$
$$w = d_v x(t,y,v)\zeta'|_{T_v\mathbb{S}^2}, \quad \zeta_0 = d_t x(t,y,v)\zeta', \quad y \in \mathbb{R}^3, v \in \mathbb{S}^2, t \in (t_0,t_1), \zeta' \in \mathbb{R}^3\}.$$

Here, we used the fact that this is the twisted conormal bundle of the point-line relation. We also have

$$C_\delta^+ = \{(t,x,\zeta_0,\zeta';z,\kappa) \in T^*M^\circ \backslash 0 \times T^*\mathbb{R}^3 \backslash 0 : (t,x,\zeta_0,\zeta') \text{ and } (0,z,\kappa_0,\kappa) \text{ are on the same}$$
$$\text{null-bicharacteristics where } \xi_0 \text{ is such that } p(\kappa_0,\kappa) = 0 \text{ and } (\kappa_0,\kappa) \text{ is future pointing.}\}$$

We first show that the composition $C_\delta \circ C_\delta^+ = \widetilde{C}_\delta^+$ is a canonical relation. Then we show that the phase function of the composition is a clean phase function using perturbation argument.

We notice that (t,x,ζ_0,ζ') in C_δ^+ are solutions of the Hamiltonian system (9.2) with initial conditions

$$t(0) = 0, \quad x(0) = z, \quad \zeta_0(0) = \kappa_0, \quad \zeta'(0) = \kappa.$$

For (t,x,ζ_0,ζ') in C_δ, they are solutions of the Hamiltonian system (9.2) with initial conditions

$$t(0) = 0, \quad x(0) = y, \quad \zeta_0(0) = \iota_0, \quad \zeta'(0) = \iota$$

where $(\iota_0,\iota) = (1,v)^\flat$. By the uniqueness of solutions of (9.2), in the composition $\widetilde{C}_\delta^+$, we must have $z = y$, $(\kappa_0,\kappa) = (\iota_0,\iota)$. In particular, $\kappa = \kappa(v), \kappa_0 = \kappa_0(v)$ are functions of v. Let $x(t,y,v)$ be the unique light ray satisfying (9.2) with the above initial condition and $\xi(t,y,v)$ be the cotangent vector to x. In C_δ, we have $\eta = (d_y x(t,y,v))\kappa, w = d_v x(t,y,v)\kappa$ hence these are determined by the value at $t = 0$ which are $\eta = \kappa(v), w = d_v x(0,y,v)\kappa(v) = 0$. Finally, we get

$$\widetilde{C}_\delta^+ = \{((y,v,\eta,w);(z,\kappa)) \in (T^*\mathscr{C}_\delta \backslash 0) \times (T^*\mathbb{R}^3 \backslash 0) : y = z, \quad \eta = \kappa(v),$$
$$w = 0, \quad y \in \mathbb{R}^3, v \in \mathbb{S}^2, z \in \mathbb{R}^3\}.$$

In particular, this is the twisted conormal bundle of the point-line relation

$$\widetilde{Z}_\delta = \{(z,q) \in \mathscr{S}_0 \times \mathscr{C}_\delta : z \in q\}$$

thus $\widetilde{C}_\delta^+$ is a canonical relation.

Next, using the oscillatory integral representations, we have

$$X_\delta \circ E_+ f(y,v) = (2\pi)^{-6} \int e^{i(x(t,y,v)\cdot\eta - z\cdot\eta + \phi_+(t,x,\xi) - z\cdot\xi)} a(t,x,\xi) f(z) dz d\xi dt dx d\eta$$

(9.13)
$$= (2\pi)^{-6} \int e^{i((x-y)\cdot\eta + tv\cdot\eta + (x-z)\cdot\xi + t|\xi| + \psi(t,x,y,z,\xi,\eta))} a(t,x,\xi) f(z) dz d\xi dt dx d\eta$$

in which ψ is a smooth function in (t,x,y,z) and homogeneous of degree one in ξ, η. We denote the phase function by $\Phi = \phi + \psi$ in which

$$\phi(y,v,z;\xi,\eta,x,t) = (x-y)\cdot\eta + tv\cdot\eta + (x-z)\cdot\xi + t|\xi|$$

where $y, z, x \in \mathbb{R}^3, \xi, \eta \in \mathbb{R}^3, v \in \mathbb{S}^2, t \in (t_0, t_1)$. So Φ is a small perturbation of ϕ. We checked in Lemma 7.1 that ϕ is a clean phase function and $\mathrm{Hess}(\phi)$ in (7.5) is non-degenerate. We repeat the same calculation for Φ. For δ sufficiently small, we use the estimate in Lemma 9.2 to see that $\mathrm{Hess}(\Phi)$ is also non-degenerate hence Φ is also a clean phase function. This shows that the composition $X_\delta \circ E_+$ is a Fourier integral operator associated with $\widetilde{C}_\delta^+$, in particular, $X_\delta \circ E_+ \in I^{-1/2}(\mathscr{S}_0 \times \mathscr{C}_\delta; (\widetilde{C}_\delta^+)')$ is an elliptic FIO.

Now, the proof of Theorem 1.1 in Section 8 go through line by line, except the injectivity of X_δ. In particular, we have the estimate as (8.1)

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C \|X_\delta f\|_{H^{s+3/2}(\mathscr{C}_\delta)} + C_\rho \|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}}$$

where C_ρ is a constant depending on ρ. To get rid of the last term, we use the following argument, see [20, Section 2.7]. Notice that given s, ρ and for some fixed small δ_0, if we consider all metric g such that $\|g - g_M\|_{C^3} \leq \delta_0$, then the above estimate is uniform (a fixed constant C_ρ works for all such metrics) by the uniformity of the construction. Now suppose there is no δ such that for all metrics within δ of the Minkowski metric g_M (in the Fréchet space sense) the transform is injective. Let $F^j = (f_1^j, f_2^j), j = 1, 2, \cdots$ be in the null-space of $X_{g_j} = X_j$ and $\|F^j\|_{\mathcal{N}^s} = 1$, with g_j within $1/j$ of the Minkowski metric. By the above inequality, $1 \leq C_\rho \|F^j\|_{\mathcal{N}^{s-\rho}}$. Now, F^j has a $\mathcal{N}^s$-weakly convergent subsequence, not shown in notation, to some $F \in \mathcal{N}^s$, which thus strongly converges in $\mathcal{N}^{s-\rho}$. By the above inequality, $F \neq 0$. But $0 = X_j F^j$ converges to $X_M F$ e.g. in the sense of distributions. So $X_M F = 0$, contradicting the injectivity of X_M and $F \neq 0$. This shows the injectivity of X_δ and finishes the proof of Theorem 9.3. $\qquad\square$

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