Flow-through problem for the Navier-Stokes equations (the Jean Leray problem)

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We consider stationary Navier-Stokes equations of incompressible fluid in a bounded domain with the boundary, which has several connected components. The velocity vector is given on the boundary. Due to continuity equation, the total fluid flux through the boundary equals to zero. J. Leray (1933) proved solvability of this problem if the partial flux through each of boundary components equals to zero. His proof used reasoning by contradiction and did not contain *a priori* estimate of the Dirichlet integral for velocity vector. This estimate was obtained by E. Hopf (1941). Next step was made in 1961 independently by R. Finn and H. Fujita, who replaced zero partial fluxes condition by assumption of their smallness.

In a general case of large partial fluxes, the solvability of this problem is an open question up to now. We give survey of previous results, which deal with partial versions of the problem. In particular, H. Fujita and H. Morimoto considered the flows, which are close to potential ones (1997). Ch.J. Amick (1984) studied plane motions with a line of symmetry. His proof of existence theorem was obtained by *reduction ad absurdum.* L.I. Sazonov (1993) and independently H. Fujita (1997) introduced the notion of *virtual drain,* which gave possibility to obtain effective estimates of the Dirichlet integral. This provided solvability of plane symmetric flow-through problem without restrictions on partial fluxes.

Essential progress in the investigation of problem was reached in the cycle of works by M.V. Korobkov, K. Pileckas and R. Russo (2011-2014). Authors have proven solvability of the plane problem without symmetry conditions and general axially symmetric problem. The proof uses a generalization of the classical Morse-Sard theorem for functions belonging to the Sobolev spaces (J. Bourgain, M.V. Korobkov and J. Kristensen, 2013) and modification of J. Leray argument by contradiction in non-standard situations.

We construct a priori estimate of the Dirichlet integral for velocity vector in the case, when the 3D flow has two mutually perpendicular planes of symmetry, moreover, the line of their intersection intersects each component of the boundary. Having available this estimate, we prove the existence theorem for arbitrary partial fluxes. Besides, we consider flow in the domain similar to a spherical layer under assumption of flow symmetry with respect to a plane. In this case, the existence theorem will be valid if flux through the inner boundary of layer is positive.

Further, we consider singular analogs of flow-through problem. In first of them, boundary of plane flow domain consists of a Jordanian curve and a point inside of it, where the source or sink is placed. Corresponding solution of Navier-Stokes equations has an infinite Dirichlet integral. We prove solvability of this problem for small intensity of source or sink and give an effective estimate of Dirichlet integral for the regular component of solution. In second problem, flow is axially symmetric and the axis of symmetry is occupied by uniformly distributed sources or sinks. It is of interest that a one-sided constraint is imposed on the problem data in this case: the intensity of distributed sinks can have an arbitrary value.

We notice in conclusion that difficulties in study of flow-through problem for the Navier-Stokes equations do not arise in non-stationary case (O.A. Ladyzhenskaya, 1970). Time-periodic problem is an exclusion form general situation. T. Kobayashi (2010) proved solvability of plane time-periodic flow-through problem for flows having the line of symmetry. We generalize this result for the case of three dimensional flows with two planes of symmetry.