

RUELLE RESONANCES FOR ANOSOV DIFFEOMORPHISMS (AFTER
FAURE, ROY, AND SJÖSTRAND)

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ABSTRACT: I will report on a recent paper by Faure, Roy, and Sjöstrand who used microlocal methods to give new (and to some of us simple) proofs of many known facts about Anosov diffeomorphisms on compact manifolds: mixing and the existence of Sinai-Ruelle-Bowen measures for (Lebesgue) measure preserving maps, the spectral properties of the Koopman operator, and the decay of correlations in terms of Ruelle resonances (originally established by Anosov, Ruelle, Baladi, Liverani...).

The point of view in the paper is inspired by the work on quantum resonances, and in particular the work of Helffer and Sjöstrand. Those resonances were defined as eigenvalues of an originally self-adjoint operator acting on new spaces, $H(\Lambda_G)$ (roughly $\mathcal{T}^{-1}L^2(T^*\mathbb{R}^n, e^{-2G(x,\xi)}dx\xi)$, where \mathcal{T} is the FBI transform), associated to dynamical “escape functions” G . The operator on the new space is no longer self-adjoint and has complex eigenvalues describing decay rates of semistable states. The same approach, in a technically much simpler fashion, works for the Koopman operator, $u \mapsto u \circ f$, for an Anosov map $f : M \rightarrow M$. An escape function is then defined by demanding that

$$G(x, (df_x)^t(\xi)) - G(f(x), \xi) < -a < 0, \quad \forall x \in M, \quad |\xi| > C,$$

which is the discrete analogue of the condition that G decreases uniformly along the Hamiltonian flow. For a suitable choice of G , the Helffer-Sjöstrand space, $H(M, \Lambda_G) := \exp(-G(x, D_x))L^2(M)$, can then be described using, for instance, a simple version of the Weyl calculus of Bony-Chemin. The Koopman operator gains some compactness properties and its properties are easy to analyse. As the authors stress similar spaces have naturally appeared in previous investigations but the systematic microlocal approach makes them easier to handle. The paper was inspired by an earlier paper by the first two authors who investigated the case of real analytic Anosov diffeomorphisms of the torus. The dilation analyticity used there followed the ideas of the complex scaling of Aguilar-Balslev-Combes, which was also the starting point for the Helffer-Sjöstrand theory. However unlike in those works analyticity plays no role in this investigation of Anosov diffeomorphisms.