MATH 256B: CONFORMALLY COMPACT SPACES

0.1. **Preliminaries.** We now show how the microlocal results of the previous sections give the meromorphic extension of the resolvent of the Laplacian on even conformally compact (asymptotically hyperbolic) spaces. We start by recalling the definition of manifolds with *even* conformally compact metrics. These are Riemannian metrics g_0 on the interior of an n-dimensional compact manifold with boundary X_0 such that near the boundary Y, with a product decomposition nearby and a defining function x, they are of the form

$$g_0 = \frac{dx^2 + h}{x^2} \tag{1}$$

where h is a family of metrics on $Y = \partial X_0$ depending on x in an even manner, i.e. only even powers of x show up in the Taylor series. (There is a much more natural way to phrase the evenness condition due to Guillarmou.) It is convenient to take x to be a globally defined boundary defining function. Then the dual metric is

$$G_0 = x^2(\partial_x^2 + H),$$

with H the dual metric family of h (depending on x as a parameter), and

$$|dg_0| = \sqrt{|\det g_0|}\, dx\, dy = x^{-n}\sqrt{|\det h|}\, dx\, dy$$

so

$$\Delta_{g_0} = (xD_x)^2 + \imath(n-1+x^2\gamma)(xD_x) + x^2\Delta_h, \tag{2}$$

with γ even, and Δ_h the x-dependent family of Laplacians of h on Y. Below we consider the spectral family

$$\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2$$

of the Laplacian.

In addition to working with finite σ, or σ in a compact set, we also want to consider $\sigma \to \infty$, mostly in strips, with $|\operatorname{Im}\sigma|$ bounded. In that case we should consider σ as a 'large parameter'. In general, on $\mathbb{R}^n$, the large parameter setting just means that one has a family of operators given by symbols a on $\mathbb{R}^n_z \times \mathbb{R}^n_{z'} \times \mathbb{R}^n_\zeta \times \Omega_\sigma$, $\Omega \subset \mathbb{C}$, satisfying, e.g. in the scattering setting, estimates

$$|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma D_\sigma^k a| \le C_{\alpha\beta\gamma,k}\langle z\rangle^{\ell_1-|\alpha|}\langle z'\rangle^{\ell_2-|\beta|}\langle(\zeta,\sigma)\rangle^{m-|\gamma|-k},$$

i.e. there is joint symbolic behavior in (ζ,σ), with differentiation in either giving rise to joint decay. This is natural, as a typical way such a family of operators arises is by Mellin the transforming normal operator of a b-operator; then σ is the b-dual of the boundary defining function. For instance, starting with the elliptic b-operator on $[0,\infty)_x \times X$

$$L = (xD_x)^2 + \Delta_g,$$

with g a Riemannian metric on X, the Mellin transform gives rise to the family $\sigma^2 + \Delta_g$, and the (joint symbolic) large parameter behavior is a consequence of the fact that the b-principal symbol of L is a symbol (concretely a quadratic polynomial).

Since σ is simply a parameter as far as the action of pseudodifferential operators is concerned, in order to have a well-behaved algebra (so that e.g. left and right quantization are equivalent), one simply needs to check that the various asymptotic expansions are now asymptotic in this large parameter sense (i.e. jointly in (ζ, σ), which is straightforward to check. Thus, the composition of order (m, ℓ) and (m, ℓ') large parameter operators is order $(m+m', \ell+\ell')$ as a large parameter operator, with principal symbol given by the product of the principal symbols, etc. Notice that the natural compactification on which the symbols are well behaved is not $\overline{\mathbb{R}^n_z} \times \overline{\mathbb{R}^n_\zeta} \times \overline{\mathbb{C}}$ (with the image of Ω inserted in to the last factor), but $\overline{\mathbb{R}^n_z} \times \overline{\mathbb{R}^n_\zeta \times \mathbb{C}}$, i.e. (ζ, σ) are jointly compactified. Correspondingly, in the manifold setting, one needs to take $T^*X \times \mathbb{C}$, best thought of as the vector bundle $(T^* \oplus \mathbb{C})X$ over X, and then radially compactifying the fibers to obtain $\overline{T^* \oplus \mathbb{C}}X$.

Below we are interested in the setting of compact manifolds, which means that the behavior as $|z| \to \infty$ is irrelevant, so we could just as well use Ψ_∞-type estimates in our definition of the large parameter algebra. To make things concrete for differential operators, if

$$P(\sigma) = \sum_{|\alpha|+|\beta| \leq m} a_\alpha(z) \sigma^\beta D_z^\alpha$$

is an order m differential operator depending on a large parameter σ, then the large-parameter symbol is denoted by

$$\sigma_{\mathrm{full}}(P(\sigma)) = \sum_{|\alpha|+|\beta| = m} a_\alpha(z) \sigma^\beta \zeta^\alpha.$$

It is often convenient to relax the requirements a bit on the joint symbolic behavior, and consider the semiclassical operator algebra, which we at first simply pull out of a hat. We adopt the convention that $\hbar$ denotes semiclassical objects, while h is the actual semiclassical parameter. This algebra, $\cup_{m,\ell} \Psi_\hbar^{m,\ell}(\mathbb{R}^n)$, is given by

(3)

$$A_h = \mathrm{Op}_\hbar(a);\ \mathrm{Op}_\hbar(a)u(z) = (2\pi h)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(z-z')\cdot\zeta/h} a(z, \zeta, h)\, u(z')\, d\zeta\, dz',$$
$$u \in \mathcal{S}(\mathbb{R}^n),\ a \in \mathcal{C}^\infty([0,1)_h; S^{m,\ell}(\mathbb{R}^n_z; \mathbb{R}^n_\zeta));$$

its classical subalgebra, $\Psi_{\hbar,\mathrm{cl}}(\mathbb{R}^n)$ corresponds to $a \in \mathcal{C}^\infty([0,1)_h; S^{m,\ell}_{\mathrm{cl}}(\mathbb{R}^n_z; \mathbb{R}^n_\zeta))$. More generally, we write

$$I_\hbar(a)u(z) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta/h} a(z, z', \zeta, h)\, u(z')\, d\zeta\, dz',$$

$a \in \mathcal{C}^\infty([0,1)_h; S^{m,\ell_1,\ell_2}(\mathbb{R}^n_z, \mathbb{R}^n_{z'}; \mathbb{R}^n_\zeta))$. A straightforward computation gives, with $\iota : \mathbb{R}^n_z \times \mathbb{R}^n_\zeta \to \mathbb{R}^n_z \times \mathbb{R}^n_{z'} \times \mathbb{R}^n_\zeta$ the inclusion map as the diagonal ($z = z'$) in the first two factors, the left reduction formula

$$a_L \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \iota^* D_{z'}^\alpha (hD_\zeta)^\alpha a, \tag{4}$$

and the right reduction formula

$$a_R \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} \iota^* D_z^\alpha (hD_\zeta)^\alpha a,$$

where the factors of h arise in front of D_ζ since one has

$$I_\hbar((z-z')^\alpha a) = I_\hbar((hD_\zeta)^\alpha a)$$

as

$$(z-z')e^{i(z-z')\cdot\zeta/h} = hD_\zeta e^{i(z-z')\cdot\zeta}.$$

This gives that the αth term in (4) is in $h^{|\alpha|}S^{m-|\alpha|,\ell_1+\ell_2-|\alpha|}$, i.e. the terms not only become lower order as symbols, but also have higher order vanishing as $h\to 0$.

The semiclassical principal symbol is now $\sigma_{\hbar,m,\ell}(A) = a|_{h=0} \in S^{m,\ell}(\mathbb{R}^n\times\mathbb{R}^n)$; there is still the standard principal symbol $[a]\in\mathcal{C}^\infty([0,1)_h; S^{m,\ell}/S^{m-1,\ell-1})$ which is now a function depending on the parameter h. As usual, in the classical setting, which is best encoded in terms of a compactification (or bordification if $h=1$ is not added!)

$$[0,1)_h\times\overline{\mathbb{R}^n_z}\times\overline{\mathbb{R}^n_\zeta},$$

which has now three boundary hypersurfaces:

$$\{0\}_h\times\overline{\mathbb{R}^n_z}\times\overline{\mathbb{R}^n_\zeta},$$

carrying the semiclassical principal symbol,

$$[0,1)_h\times\partial\overline{\mathbb{R}^n_z}\times\overline{\mathbb{R}^n_\zeta},$$

carrying the scattering symbol at base-infinity (which is irrelevent when we transfer to compact manifolds without boundary) and

$$[0,1)_h\times\overline{\mathbb{R}^n_z}\times\partial\overline{\mathbb{R}^n_\zeta},$$

carrying the usual principal symbol at fiber infinity.

In the setting of a general manifold X, $\mathbb{R}^n\times\mathbb{R}^n$ is replaced by T^*X. Correspondingly, $\mathrm{WF}'_\hbar(A)$ and $\mathrm{Ell}_\hbar(A)$ are subsets of T^*X. We can add an extra parameter $\lambda\in O$, so $a\in\mathcal{C}^\infty([0,1)_h; S^m(\mathbb{R}^n\times O;\mathbb{R}^n_\zeta))$; then in the invariant setting the principal symbol is $a|_{h=0}\in S^m(T^*X\times O)$.

In order to motivate the definition a posteriori, notice that differential operators now take the form

$$A_{h,\lambda} = \sum_{|\alpha|\le m} a_\alpha(z,\lambda;h)(hD_z)^\alpha. \tag{5}$$

Indeed, if a is actually a polynomial in ζ, depending on a parameter λ, in (3), namely

$$a_{h,\lambda} = \sum_{|\alpha|\le m} a_\alpha(z,\lambda;h)\zeta^\alpha.$$

then letting $\tilde\zeta = \zeta/h$, we have

$$\begin{aligned}(\mathrm{Op}_\hbar(a)u)(z) &= (2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(z-z')\cdot\tilde\zeta}\sum_{|\alpha|\le m} a_\alpha(z,\lambda;h)(h\tilde\zeta)^\alpha u(z')\,dz' \\ &= \sum_{|\alpha|\le m} a_\alpha(z)((hD_z)^\alpha u)(z)\end{aligned}$$

The two principal symbols (ignoring base-infinity) are the standard one (but taking into account the semiclassical degeneration, i.e. based on $(hD_z)^\alpha$ rather than D_z^α), which depends on h and is homogeneous, and the semiclassical one, which is at $h=0$, and is not homogeneous:

$$\sigma_m(A_{h,\lambda}) = \sum_{|\alpha|=m} a_\alpha(z,\lambda;h)\zeta^\alpha,$$
$$\sigma_\hbar(A_{h,\lambda}) = \sum_{|\alpha|\le m} a_\alpha(z,\lambda;0)\zeta^\alpha.$$

However, the restriction of $\sigma_m(A_{h,\lambda})$ to $h=0$ is the principal symbol of $\sigma_\hbar(A_{h,\lambda})$. In the special case in which $\sigma_m(A_{h,\lambda})$ is independent of h (which is true in the setting considered below), one can simply regard the usual principal symbol as the principal part of the semiclassical symbol.

In terms of the fiber compactification of T^*X, in the semiclassical context one considers $\overline{T^*}X\times[0,1)$, and notes that 'classical' semiclassical operators of order 0 are given locally by $\mathrm{Op}_\hbar(a)$ with a extending to be smooth up to the boundaries of this space, with semiclassical symbol given by restriction to $\overline{T^*}X\times\{0\}$, and standard symbol given by restriction to $S^*X\times[0,1)$. Thus, the claim regarding the limit of the semiclassical symbol at infinity is simply a matching statement of the two symbols at the corner $S^*X\times\{0\}$ in this compactified picture.

We can convert a large parameter operator into a semiclassical one (but not conversely) as follows. If $P(\sigma)=\sum_{|\alpha|+|\beta|\le m} a_\alpha(z)\sigma^\beta D_z^\alpha$ is an order m differential operator depending on a large parameter σ, then letting $\sigma=h^{-1}\lambda$, where $h^{-1}\sim\sigma$ (e.g. one may take $h^{-1}=|\sigma|$, but this is often too restrictive), so λ is in a compact set, and where we restrict to $|\sigma|>1$, say,

$$P_{\hbar,\lambda}=h^m P(\sigma)=\sum_{|\alpha|+|\beta|\le m} h^{m-|\alpha|-|\beta|}a_\alpha(z)\lambda^\beta(hD_z)^\alpha$$

is a semiclassical differential operator with semiclassical symbol

$$\sigma_\hbar(P_{\hbar,\lambda})=\sum_{|\alpha|+|\beta|=m} a_\alpha(z)\lambda^\beta\zeta^\alpha.$$

Note that the full large-parameter symbol and the semiclassical symbol are 'the same', i.e. they are simply related to each other.

0.2. **From the Laplacian to the extended operator.** We show now that if we change the smooth structure on X_0 by declaring that only even functions of x are smooth, i.e. introducing $\mu=x^2$ as the boundary defining function, then after a suitable conjugation and division by a vanishing factor the resulting operator smoothly and non-degenerately continues across the boundary, i.e. continues to $X_{-\delta_0}=(-\delta_0,0)_\mu\times Y\sqcup X_{0,\mathrm{even}}$, where $X_{0,\mathrm{even}}$ is the manifold X_0 with the new smooth structure. At the level of the principal symbol, i.e. the dual metric, the conjugation is irrelevant, so we can easily see what happens: changing to coordinates (μ,y), $\mu=x^2$, as $x\partial_x=2\mu\partial_\mu$,

$$G_0=4\mu^2\partial_\mu^2+\mu H=\mu(4\mu\partial_\mu^2+H),$$

so after dividing by μ, we obtain

$$\mu^{-1}G_0=4\mu\partial_\mu^2+H.$$

This is a quadratic form that is positive definite for $\mu>0$, is Lorentzian for $\mu<0$, and has a transition at $\mu=0$ that as we shall see involves radial points.

To see that the full spectral family of the Laplacian is well behaved, first, changing to coordinates (μ, y), $\mu = x^2$, we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\imath(n-1+\mu\gamma)(\mu D_\mu) + \mu\Delta_h. \tag{6}$$

Now we conjugate by $\mu^{-\imath\sigma/2+(n+1)/4}$ to obtain

$$\begin{aligned}
&\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4} \\
&= 4(\mu D_\mu - \sigma/2 - \imath(n+1)/4)^2 + 2\imath(n-1+\mu\gamma)(\mu D_\mu - \sigma/2 - \imath(n+1)/4) \\
&\qquad + \mu\Delta_h - \frac{(n-1)^2}{4} - \sigma^2 \\
&= 4(\mu D_\mu)^2 - 4\sigma(\mu D_\mu) + \mu\Delta_h - 4\imath(\mu D_\mu) + 2\imath\sigma - 1 \\
&\qquad + 2\imath\mu\gamma(\mu D_\mu - \sigma/2 - \imath(n+1)/4).
\end{aligned}$$

Next we multiply by $\mu^{-1/2}$ from both sides to obtain

$$\begin{aligned}
&\mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2} \\
&= 4\mu D_\mu^2 - \mu^{-1} - 4\sigma D_\mu - 2\imath\sigma\mu^{-1} + \Delta_h - 4\imath D_\mu + 2\mu^{-1} + 2\imath\sigma\mu^{-1} - \mu^{-1} \\
&\qquad + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4) \\
&= 4\mu D_\mu^2 - 4\sigma D_\mu + \Delta_h - 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4).
\end{aligned} \tag{7}$$

This operator is in $\mathrm{Diff}^2(X_{0,\mathrm{even}})$, and now it continues smoothly across the boundary, by extending h and γ in an arbitrary smooth manner. This form suffices for analyzing the problem for σ in a compact set, or indeed for σ going to infinity in a strip near the reals. However, it is convenient to modify it as we would like the resulting operator to be semiclassically elliptic when σ is away from the reals. We achieve this via conjugation by a smooth function, with exponent depending on σ. The latter would make no difference even semiclassically in the real regime as it is conjugation by an elliptic semiclassical FIO. However, in the non-real regime (where we would like ellipticity) it does matter; the present operator is not semiclassically elliptic at the zero section. So finally we conjugate by $(1+\mu)^{\imath\sigma/4}$ to obtain

$$\begin{aligned}
P_\sigma = 4(1+a_1)\mu D_\mu^2 - 4(1+a_2)\sigma D_\mu - (1+a_3)\sigma^2 + \Delta_h \\
- 4\imath D_\mu + b_1\mu D_\mu + b_2\sigma + c_1
\end{aligned} \tag{8}$$

with a_j smooth, real, vanishing at $\mu = 0$, b_j and c_1 smooth. In fact, we have $a_1 \equiv 0$, but it is sometimes convenient to have more flexibility in the form of the operator since this means that we do not need to start from the relatively rigid form (2).

Writing covectors as

$$\xi\, d\mu + \eta\, dy,$$

the principal symbol of $P_\sigma \in \mathrm{Diff}^2(X_{-\delta_0})$, including in the high energy sense ($\sigma \to \infty$), is

$$p_{\mathrm{full}} = 4(1+a_1)\mu\xi^2 - 4(1+a_2)\sigma\xi - (1+a_3)\sigma^2 + |\eta|^2_{\mu,y}, \tag{9}$$

and is real for σ real. The Hamilton vector field is

$$(10)\quad \begin{aligned}\mathsf{H}_{p_{\mathrm{full}}} &= 4(2(1+a_1)\mu\xi - (1+a_2)\sigma)\partial_\mu + \tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}}\\ &\quad - \Big(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 - 4\frac{\partial a_2}{\partial\mu}\sigma\xi + \frac{\partial a_3}{\partial\mu}\sigma^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}\Big)\partial_\xi\\ &\quad - \Big(4\frac{\partial a_1}{\partial y}\mu\xi^2 - 4\frac{\partial a_2}{\partial y}\sigma\xi - \frac{\partial a_3}{\partial y}\sigma^2\Big)\partial_\eta,\end{aligned}$$

where $\tilde{\mathsf{H}}$ indicates that this is the Hamilton vector field in T^*Y, i.e. with μ considered a parameter. Correspondingly, the standard, 'classical', principal symbol is

$$(11)\quad p = \sigma_2(P_\sigma) = 4(1+a_1)\mu\xi^2 + |\eta|^2_{\mu,y},$$

which is real, independent of σ, while the Hamilton vector field is

$$(12)\quad \begin{aligned}\mathsf{H}_p &= 8(1+a_1)\mu\xi\partial_\mu + \tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}}\\ &\quad - \Big(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}\Big)\partial_\xi - 4\frac{\partial a_1}{\partial y}\mu\xi^2\partial_\eta.\end{aligned}$$

It is useful to keep in mind that as $\Delta_{g_0} - \sigma^2 - (n-1)^2/4$ is formally self-adjoint relative to the metric density $|dg_0|$ for σ real, so the same holds for $\mu^{-1/2}(\Delta_{g_0} - \sigma^2 - (n-1)^2/4)\mu^{-1/2}$ (as μ is real), and indeed for its conjugate by $\mu^{-\imath\sigma/2}(1+\mu)^{\imath\sigma/4}$ for σ real since this is merely unitary conjugation. As for f real, A formally self-adjoint relative to $|dg_0|$, $f^{-1}Af$ is formally self-adjoint relative to $f^2|dg_0|$, we then deduce that for σ real, P_σ is formally self-adjoint relative to

$$\mu^{(n+1)/2}|dg_0| = \frac{1}{2}|dh|\,|d\mu|,$$

as $x^{-n}\,dx = \frac{1}{2}\mu^{-(n+1)/2}\,d\mu$. Note that $\mu^{(n+1)/2}|dg_0|$ thus extends to a $\mathcal{C}^\infty$ density to $X_{-\delta_0}$, and we deduce that with respect to the extended density, $\sigma_1(\frac{1}{2\imath}(P_\sigma - P_\sigma^*))|_{\mu\geq 0}$ vanishes when $\sigma\in\mathbb{R}$. Since in general $P_\sigma - P_{\mathrm{Re}\,\sigma}$ differs from $-4\imath(1+a_2)\,\mathrm{Im}\,\sigma D_\mu$ by a zeroth order operator, we conclude that

$$(13)\quad \sigma_1\Big(\frac{1}{2\imath}(P_\sigma - P_\sigma^*)\Big)\Big|_{\mu=0} = -4(\mathrm{Im}\,\sigma)\xi.$$

We still need to check that μ can be appropriately chosen in the interior away from the region of validity of the product decomposition (1) (where we had no requirements so far on μ). This only matters for semiclassical purposes, and (being smooth and non-zero in the interior) the factor $\mu^{-1/2}$ multiplying from both sides does not affect any of the relevant properties (semiclassical ellipticity and possible non-trapping properties), so can be ignored — the same is true for σ-independent powers of μ.

Thus, near $\mu = 0$, but μ bounded away from 0, the only semiclassically non-trivial action we have done was to conjugate the operator by $e^{-\imath\sigma\phi}$ where $e^\phi = \mu^{1/2}(1+\mu)^{-1/4}$; we need to extend ϕ into the interior. But the semiclassical principal symbol of the conjugated operator is, with $\sigma = z/h$,

$$(14)\quad (\zeta - z\,d\phi, \zeta - z\,d\phi)_{G_0} - z^2 = |\zeta|^2_{G_0} - 2z(\zeta, d\phi)_{G_0} - (1 - |d\phi|^2_{G_0})z^2.$$

For z non-real this is elliptic if $|d\phi|_{G_0} < 1$. Indeed, if (14) vanishes then from the vanishing imaginary part we get

$$(15)\quad 2\,\mathrm{Im}\,z((\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\,\mathrm{Re}\,z) = 0,$$

and then the real part is

$$
\begin{aligned}
&|\zeta|^2_{G_0} - 2\,\mathrm{Re}\, z(\zeta, d\phi)_{G_0} - (1-|d\phi|^2_{G_0})((\mathrm{Re}\, z)^2 - (\mathrm{Im}\, z)^2)\\
&= |\zeta|^2_{G_0} + (1-|d\phi|^2_{G_0})((\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2),
\end{aligned}
\tag{16}
$$

which cannot vanish if $|d\phi|_{G_0} < 1$. But, reading off the dual metric from the principal symbol of (6),

$$
\frac{1}{4}\left| d(\log\mu - \frac{1}{2}\log(1+\mu))\right|^2_{G_0} = \left(1 - \frac{\mu}{2(1+\mu)}\right)^2 < 1
$$

for $\mu > 0$, with a strict bound as long as μ is bounded away from 0. Correspondingly, $\mu^{1/2}(1+\mu)^{-1/4}$ can be extended to a function e^{ϕ} on all of X_0 so that semiclassical ellipticity for z away from the reals is preserved, and we may even require that ϕ is constant on a fixed (but arbitrarily large) compact subset of X_0°. Then, after conjugation by $e^{-\imath\sigma\phi}$,

$$
P_{h,z} = e^{\imath z\phi/h}\mu^{-(n+1)/4-1/2}(h^2\Delta_{g_0} - z)\mu^{(n+1)/4-1/2}e^{-\imath z\phi/h}
\tag{17}
$$

is semiclassically elliptic in $\mu > 0$ (as well as in $\mu \le 0$, μ near 0, where this is already guaranteed), as desired.

Remark 0.1. We have not considered vector bundles over X_0. However, for instance for the Laplacian on the differential form bundles it is straightforward to check that slightly changing the power of μ in the conjugation the resulting operator extends smoothly across ∂X_0, has scalar principal symbol of the form (9), and the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$, which plays a role below, is also as in the scalar setting, so all the results in fact go through.

0.3. **Local dynamics near the radial set.** Let

$$
N^*S\setminus o = \Lambda_+\cup\Lambda_-,\qquad \Lambda_\pm = N^*S\cap\{\pm\xi > 0\},\qquad S=\{\mu=0\};
$$

thus $S\subset X_{-\delta_0}$ can be identified with $Y = \partial X_0 (= \partial X_{0,\mathrm{even}})$. Note that $p = 0$ at $\Lambda_\pm$ and H_p is radial there since

$$
N^*S = \{(\mu, y, \xi, \eta):\ \mu = 0,\ \eta = 0\},
$$

so

$$
\mathsf{H}_p|_{N^*S} = -4\xi^2\partial_\xi.
$$

This corresponds to $dp = 4\xi^2\,d\mu$ at N^*S, so the characteristic set $\Sigma = \{p = 0\}$ is smooth at N^*S.

Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*X_{-\delta_0}$. Next we analyze the Hamilton flow at $\Lambda_\pm$. First,

$$
\mathsf{H}_p|\eta|^2_{\mu,y} = 8(1+a_1)\mu\xi\partial_\mu|\eta|^2_{\mu,y} - 4\frac{\partial a_1}{\partial y}\mu\xi^2\cdot_h\eta
\tag{18}
$$

and

$$
\mathsf{H}_p\mu = 8(1+a_1)\xi\mu.
\tag{19}
$$

In terms of linearizing the flow at N^*S, p and μ are equivalent as $dp = 4\xi^2\,d\mu$ there, so one can simply use $\hat{p} = p/|\xi|^2$ (which is homogeneous of degree 0, like μ), in place of μ. Finally,

$$
\mathsf{H}_p|\xi| = -4\,\mathrm{sgn}(\xi) + b,
\tag{20}
$$

with b vanishing at $\Lambda_\pm$.

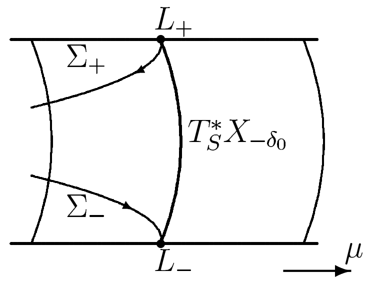

FIGURE 1. The cotangent bundle of $X_{-\delta_0}$ near $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view. The boundary of the fiber compactificaton is the cosphere bundle $S^*X_{-\delta_0}$; it is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu \leq 0$, only meeting $S^*_S X_{-\delta_0}$ at $L_\pm$. Semiclassically, i.e. in the interior of $\overline{T}^* X_{-\delta_0}$, for $z = h^{-1}\sigma > 0$, only the component of the semiclassical characteristic set containing L_+ can enter $\mu > 0$. This is reversed for $z < 0$.

It is convenient to rehomogenize (18) in terms of $\hat{\eta} = \eta/|\xi|$. This can be phrased more invariantly by working with $S^*X_{-\delta_0} = (T^*X_{-\delta_0} \setminus o)/\mathbb{R}^+$. Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*X_{-\delta_0}$. Homogeneous degree zero functions on $T^*X_{-\delta_0} \setminus o$, such as $\hat{p}$, can be regarded as functions on $S^*X_{-\delta_0}$. For semiclassical purposes, it is best to consider $S^*X_{-\delta_0}$ as the boundary at fiber infinity of the fiber-radial compactification $\overline{T}^*X_{-\delta_0}$ of $T^*X_{-\delta_0}$. Then at fiber infinity near N^*S, we can take $(|\xi|^{-1}, \hat{\eta})$ as (projective, rather than polar) coordinates on the fibers of the cotangent bundle, with $\tilde{\rho} = |\xi|^{-1}$ defining $S^*X_{-\delta_0}$ in $\overline{T}^*X_{-\delta_0}$. Then $W = |\xi|^{-1}\mathsf{H}_p$ is a $\mathcal{C}^\infty$ vector field in this region and

$$|\xi|^{-1}\mathsf{H}_p|\hat{\eta}|^2_{\mu,y} = 2|\hat{\eta}|^2_{\mu,y}\mathsf{H}_p|\xi|^{-1} + |\xi|^{-3}\mathsf{H}_p|\eta|^2_{\mu,y} = 8(\operatorname{sgn}\xi)|\hat{\eta}|^2 + \tilde{a}, \tag{21}$$

where $\tilde{a}$ vanishes cubically at N^*S. In similar notation we have

$$\mathsf{H}_p\tilde{\rho} = 4\operatorname{sgn}(\xi) + \tilde{a}', \qquad \tilde{\rho} = |\xi|^{-1}, \tag{22}$$

and

$$|\xi|^{-1}\mathsf{H}_p\mu = 8(\operatorname{sgn}\xi)\mu + \tilde{a}'', \tag{23}$$

with $\tilde{a}'$ smooth (indeed, homogeneous degree zero without the compactification) vanishing at N^*S, and $\tilde{a}''$ is also smooth, vanishing quadratically at N^*S. As the vanishing of $\hat{\eta}, |\xi|^{-1}$ and μ defines ∂N^*S, we conclude that $L_- = \partial\Lambda_-$ is a sink, while $L_+ = \partial\Lambda_+$ is a source, in the sense that all nearby bicharacteristics (in fact, including semiclassical (null)bicharacteristics, since $\mathsf{H}_p|\xi|^{-1}$ contains the additional information needed; see (32)) converge to $L_\pm$ as the parameter along the bicharacteristic goes to $\mp\infty$. In particular, the quadratic defining function of $L_\pm$ given by

$$\rho_0 = \widehat{\widetilde{p}} + \hat{p}^2, \text{ where } \hat{p} = |\xi|^{-2}p, \ \widehat{\widetilde{p}} = |\hat{\eta}|^2,$$

satisfies

$$(\operatorname{sgn}\xi)W\rho_0 \geq 8\rho_0 + \mathcal{O}(\rho_0^{3/2}). \tag{24}$$

We also need information on the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$ at the radial points. At $L_\pm$ this is given by

$$\sigma_1(\frac{1}{2\imath}(P_\sigma - P_\sigma^*))|_{N^*S} = -(4\,\mathrm{sgn}(\xi))\,\mathrm{Im}\,\sigma|\xi|; \tag{25}$$

here $(4\,\mathrm{sgn}(\xi))$ is pulled out due to (22), namely its size relative to $\mathsf{H}_p|\xi|^{-1}$ matters. This corresponds to the fact that $(\mu \pm \imath 0)^{\imath\sigma}$, which are Lagrangian distributions associated to $\Lambda_\pm$, solve the PDE (8) modulo an error that is two orders lower than what one might a priori expect, i.e. $P_\sigma(\mu \pm \imath 0)^{\imath\sigma} \in (\mu \pm \imath 0)^{\imath\sigma}\mathcal{C}^\infty(X_{-\delta_0})$. Note that P_σ is second order, so one should lose two orders a priori, i.e. get an element of $(\mu \pm \imath 0)^{\imath\sigma-2}\mathcal{C}^\infty(X_{-\delta_0})$; the characteristic nature of $\Lambda_\pm$ reduces the loss to 1, and the particular choice of exponent eliminates the loss. This has much in common with $e^{\imath\lambda/x}x^{(n-1)/2}$ being an approximate solution in asymptotically Euclidean scattering.

0.4. **Global behavior of the characteristic set.** By (11), points with $\xi = 0$ cannot lie in the characteristic set. Thus, with

$$\Sigma_\pm = \Sigma \cap \{\pm\xi > 0\},$$

$\Sigma = \Sigma_+ \cup \Sigma_-$ and $\Lambda_\pm \subset \Sigma_\pm$. Further, the characteristic set lies in $\mu \leq 0$, and intersects $\mu = 0$ only in $\Lambda_\pm$.

Moreover, as $\mathsf{H}_p\mu = 8(1 + a_1)\xi\mu$ and $\xi \neq 0$ on Σ, and μ only vanishes at $\Lambda_+ \cup \Lambda_-$ there, for $\epsilon_0 > 0$ sufficiently small the $\mathcal{C}^\infty$ function μ provides a negative global escape function on $\mu \geq -\epsilon_0$ which is decreasing on Σ_+, increasing on Σ_-. Correspondingly, bicharacteristics in Σ_- travel from $\mu = -\epsilon_0$ to L_-, while in Σ_+ they travel from L_+ to $\mu = -\epsilon_0$.

0.5. **High energy, or semiclassical, asymptotics.** We are also interested in the high energy behavior, as $|\sigma| \to \infty$. For the associated semiclassical problem one obtains a family of operators

$$P_{h,z} = h^2 P_{h^{-1}z},$$

with $h = |\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. Then the semiclassical principal symbol $p_{\hbar,z}$ of $P_{h,z}$ is a function on $T^*X_{-\delta_0}$, whose asymptotics at fiber infinity of $T^*X_{-\delta_0}$ is given by the classical principal symbol p. We are interested in $\mathrm{Im}\,\sigma \geq -C$, which in semiclassical notation corresponds to $\mathrm{Im}\,z \geq -Ch$. It is sometimes convenient to think of $p_{\hbar,z}$, and its rescaled Hamilton vector field, as objects on $\overline{T}^*X_{-\delta_0}$. Thus,

$$p_{\hbar,z} = \sigma_{2,\hbar}(P_{h,z}) = 4(1 + a_1)\mu\xi^2 - 4(1 + a_2)z\xi - (1 + a_3)z^2 + |\eta|^2_{\mu,y}, \tag{26}$$

so

$$\mathrm{Im}\,p_{\hbar,z} = -2\,\mathrm{Im}\,z(2(1 + a_2)\xi + (1 + a_3)\,\mathrm{Re}\,z). \tag{27}$$

In particular, for z non-real, $\mathrm{Im}\,p_{\hbar,z} = 0$ implies $2(1 + a_2)\xi + (1 + a_3)\,\mathrm{Re}\,z = 0$, so

$$\begin{aligned}\mathrm{Re}\,p_{\hbar,z} = ((1 + a_1)(1 + a_3)^2(1 + a_2)^{-2}\mu + (1 + 2a_2)(1 + a_3))(\mathrm{Re}\,z)^2 \\ + (1 + a_3)(\mathrm{Im}\,z)^2 + |\eta|^2_{\mu,y} > 0\end{aligned} \tag{28}$$

near $\mu = 0$, i.e. $p_{\hbar,z}$ is semiclassically elliptic on $T^*X_{-\delta_0}$, but *not* at fiber infinity, i.e. at $S^*X_{-\delta_0}$ (standard ellipticity is lost only in $\mu \leq 0$, of course). In $\mu > 0$ we have semiclassical ellipticity (and automatically classical ellipticity) by our choice of ϕ following (14). Explicitly, if we introduce for instance

$$(\mu, y, \nu, \hat{\eta}), \qquad \nu = |\xi|^{-1},\ \hat{\eta} = \eta/|\xi|, \tag{29}$$

as valid projective coordinates in a (large!) neighborhood of $L_\pm$ in $\overline{T}^* X_{-\delta_0}$, then

$$\nu^2 p_{\hbar,z} = 4(1+a_1)\mu - 4(1+a_2)(\operatorname{sgn}\xi) z\nu - (1+a_3)z^2\nu^2 + |\hat{\eta}|^2_{y,\mu}$$

so

$$\nu^2 \operatorname{Im} p_{\hbar,z} = -4(1+a_2)(\operatorname{sgn}\xi)\nu \operatorname{Im} z - 2(1+a_3)\nu^2 \operatorname{Re} z \operatorname{Im} z$$

which automatically vanishes at $\nu = 0$, i.e. at $S^*X_{-\delta_0}$. Thus, for σ large and pure imaginary, the semiclassical problem adds no complexity to the 'classical' quantum problem, but of course it does not simplify it. In fact, we need somewhat more information at the characteristic set, which is thus at $\nu = 0$ when $\operatorname{Im} z$ is bounded away from 0:

$$\begin{aligned} &\nu \text{ small},\ \operatorname{Im} z \geq 0 \Rightarrow (\operatorname{sgn}\xi) \operatorname{Im} p_{\hbar,z} \leq 0 \Rightarrow \pm \operatorname{Im} p_{\hbar,z} \leq 0 \text{ near } \Sigma_{\hbar,\pm}, \\ &\nu \text{ small},\ \operatorname{Im} z \leq 0 \Rightarrow (\operatorname{sgn}\xi) \operatorname{Im} p_{\hbar,z} \geq 0 \Rightarrow \pm \operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm}, \end{aligned} \tag{30}$$

which means that for $P_{h,z}$ with $\operatorname{Im} z > 0$ one can propagate estimates forwards along the bicharacteristics where $\xi > 0$ (in particular, away from L_+, as the latter is a source) and backwards where $\xi < 0$ (in particular, away from L_-, as the latter is a sink), while for $P^*_{\hbar,z}$ the directions are reversed since its semiclassical symbol is $\overline{p_{\hbar,z}}$. The directions are also reversed if $\operatorname{Im} z$ switches sign. This is important because it gives invertibility for $z = \imath$ (corresponding to $\operatorname{Im}\sigma$ large positive, i.e. the physical halfplane), but does not give invertibility for $z = -\imath$ negative.

We now return to the claim that even semiclassically, for z almost real (i.e. when z is not bounded away from the reals; we are not fixing z as we let h vary!), when the operator is not semiclassically elliptic on $T^*X_{-\delta_0}$ as mentioned above, the characteristic set can be divided into two components $\Sigma_{\hbar,\pm}$, with $L_\pm$ in different components. The vanishing of the factor following $\operatorname{Im} z$ in (27) gives a hypersurface that separates $\Sigma_\hbar$ into two parts. Indeed, this is the hypersurface given by

$$2(1+a_2)\xi + (1+a_3)\operatorname{Re} z = 0,$$

on which, by (28), $\operatorname{Re} p_{\hbar,z}$ cannot vanish, so

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-}, \qquad \Sigma_{\hbar,\pm} = \Sigma_\hbar \cap \{\pm(2(1+a_2)\xi + (1+a_3)\operatorname{Re} z) > 0\}.$$

Farther in $\mu > 0$, the hypersurface is given, due to (15), by

$$(\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\operatorname{Re} z = 0,$$

and on it, by (16), the real part is $|\zeta|^2_{G_0} + (1 - |d\phi|^2_{G_0})((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2) > 0$; correspondingly

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-}, \qquad \Sigma_{\hbar,\pm} = \Sigma_\hbar \cap \{\pm((\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\operatorname{Re} z) > 0\}.$$

In fact, more generally, the real part is

$$\begin{aligned} &|\zeta|^2_{G_0} - 2\operatorname{Re} z(\zeta, d\phi)_{G_0} - (1 - |d\phi|^2_{G_0})((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2) \\ &= |\zeta|^2_{G_0} - 2\operatorname{Re} z((\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\operatorname{Re} z) + (1 - |d\phi|^2_{G_0})((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2), \end{aligned}$$

so for $\pm\operatorname{Re} z > 0$, $\mp((\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\operatorname{Re} z) > 0$ implies that $p_{\hbar,z}$ does not vanish. Correspondingly, only one of the two components of $\Sigma_{\hbar,\pm}$ enter $\mu > 0$, namely for $\operatorname{Re} z > 0$, only $\Sigma_{\hbar,+}$ enters, while for $\operatorname{Re} z < 0$, only $\Sigma_{\hbar,-}$ enters.

We finally need more information about the global semiclassical dynamics.

Lemma 0.2. *There exists $\epsilon_0 > 0$ such that the following holds. All semiclassical null-bicharacteristics in $(\Sigma_{\hbar,+} \setminus L_+) \cap \{-\epsilon_0 \le \mu \le \epsilon_0\}$ go to either L_+ or to $\mu = \epsilon_0$ in the backward direction and to $\mu = \epsilon_0$ or $\mu = -\epsilon_0$ in the forward direction, while all semiclassical null-bicharacteristics in $(\Sigma_{\hbar,-} \setminus L_-) \cap \{-\epsilon_0 \le \mu \le \epsilon_0\}$ go to L_- or $\mu = \epsilon_0$ in the forward direction and to $\mu = \epsilon_0$ or $\mu = -\epsilon_0$ in the backward direction.*

For $\operatorname{Re} z > 0$, only $\Sigma_{\hbar,+}$ enters $\mu > 0$, so the $\mu = \epsilon_0$ possibility only applies to $\Sigma_{\hbar,+}$ then, while for $\operatorname{Re} z < 0$, the analogous remark applies to $\Sigma_{\hbar,-}$.

Proof. We assume that $\operatorname{Re} z > 0$ for the sake of definiteness. Observe that the semiclassical Hamilton vector field is

$$\begin{aligned} \mathsf{H}_{p_{\hbar,z}} &= 4(2(1+a_1)\mu\xi - (1+a_2)z)\partial_\mu + \tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} \\ &\quad - \Big(4(1+a_1+\mu\frac{\partial a_1}{\partial \mu})\xi^2 - 4\frac{\partial a_2}{\partial \mu} z\xi + \frac{\partial a_3}{\partial \mu} z^2 + \frac{\partial |\eta|^2_{\mu,y}}{\partial \mu}\Big)\partial_\xi \\ &\quad - \Big(4\frac{\partial a_1}{\partial y}\mu\xi^2 - 4\frac{\partial a_2}{\partial y} z\xi - \frac{\partial a_3}{\partial y} z^2\Big)\partial_\eta; \end{aligned} \tag{31}$$

here we are concerned about z real. Near $S^*X_{-\delta_0} = \partial \overline{T}^* X_{-\delta_0}$, using the coordinates (29) (which are valid near the characteristic set)

$$\begin{aligned} W_\hbar = \nu \mathsf{H}_{p_{\hbar,z}} &= 4(2(1+a_1)\mu(\operatorname{sgn}\xi) - (1+a_2)z\nu)\partial_\mu + \nu\tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} \\ &\quad + (\operatorname{sgn}\xi)\Big(4(1+a_1+\mu\frac{\partial a_1}{\partial \mu}) - 4\frac{\partial a_2}{\partial \mu} z(\operatorname{sgn}\xi)\nu + \frac{\partial a_3}{\partial \mu} z^2\nu^2 \\ &\qquad + \frac{\partial |\hat\eta|^2_{\mu,y}}{\partial \mu}\Big)(\nu\partial_\nu + \hat\eta\partial_{\hat\eta}) \\ &\quad - \Big(4\frac{\partial a_1}{\partial y}\mu - 4(\operatorname{sgn}\xi)\frac{\partial a_2}{\partial y} z\nu - \frac{\partial a_3}{\partial y} z^2\nu^2\Big)\partial_{\hat\eta}, \end{aligned} \tag{32}$$

with $\nu\tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} = \sum_{ij} H_{ij}\hat\eta_i\partial_{y_j} - \sum_{ijk}\frac{\partial H_{ij}}{\partial y_k}\hat\eta_i\hat\eta_j\partial_{\hat\eta_k}$ smooth. Thus, $W_\hbar$ is a smooth vector field on the compactified cotangent bundle, $\overline{T}^*X_{-\delta_0}$ which is tangent to its boundary, $S^*X_{-\delta_0}$, and $W_\hbar - W = \nu W^\sharp$ (with W considered as a homogeneous degree zero vector field) with $W^\sharp$ smooth and tangent to $S^*X_{-\delta_0}$. In particular, by (22) and (24), using that $\tilde\rho^2 + \rho_0$ is a quadratic defining function of $L_\pm$,

$$(\operatorname{sgn}\xi)W_\hbar(\tilde\rho^2+\rho_0) \ge 8(\tilde\rho^2+\rho_0) - \mathcal{O}((\tilde\rho^2+\rho_0)^{3/2})$$

shows that there is $\epsilon_1 > 0$ such that in $\tilde\rho^2 + \rho_0 \le \epsilon_1$, $\xi > 0$, $\tilde\rho^2 + \rho_0$ is strictly increasing along the Hamilton flow except at L_+, while in $\tilde\rho^2 + \rho_0 \le \epsilon_1$, $\xi < 0$, $\tilde\rho^2 + \rho_0$ is strictly decreasing along the Hamilton flow except at L_-. Indeed, all null-bicharacteristics in this neighborhood of $L_\pm$ except the constant ones at $L_\pm$ tend to $L_\pm$ in one direction and to $\tilde\rho^2 + \rho_0 = \epsilon_1$ in the other direction.

Choosing $\epsilon_0' > 0$ sufficiently small, the characteristic set in $\overline{T}^*X_{-\delta_0} \cap \{-\epsilon_0' \le \mu \le \epsilon_0'\}$ is disjoint from $S^*X_{-\delta_0} \setminus \{\tilde\rho^2 + \rho_0 \le \epsilon_1\}$, and indeed only contains points in $\Sigma_{\hbar,+}$ as $\operatorname{Re} z > 0$. Since $\mathsf{H}_{p_{\hbar,z}}\mu = 4(2(1+a_1)\mu\xi - (1+a_2)z)$, it is negative on $\overline{T}^*_{\{\mu=0\}}X_{-\delta_0} \setminus S^*X_{-\delta_0}$. In particular, there is a neighborhood U of $\mu = 0$ in $\Sigma_{\hbar,+} \setminus S^*X_{-\delta_0}$ on which the same sign is preserved; since the characteristic set in $\overline{T}^*X_{-\delta_0} \setminus \{\tilde\rho^2+\rho_0 < \epsilon_1\}$ is compact, and is indeed a subset of $T^*X_{-\delta_0} \setminus \{\tilde\rho^2+\rho_0 < \epsilon_1\}$, we deduce that $|\mu|$ is bounded below on $\Sigma \setminus (U \cup \{\tilde\rho^2 + \rho_0 < \epsilon_1\})$, say $|\mu| \ge \epsilon_0'' > 0$ there, so with $\epsilon_0 = \min(\epsilon_0', \epsilon_0'')$, $\mathsf{H}_{p_{\hbar,z}}\mu < 0$ on $\Sigma_{\hbar,+} \cap \{-\epsilon_0 \le \mu \le \epsilon_0\} \setminus \{\tilde\rho^2+\rho_0^2 < \epsilon_1\}$.

As $\mathsf{H}_{p_{\hbar,z}}\mu < 0$ at $\mu = 0$, bicharacteristics can only cross $\mu = 0$ in the outward direction.

Thus, if γ is a bicharacteristic in $\Sigma_{\hbar,+}$, there are two possibilities. If γ is disjoint from $\{\tilde\rho^2+\rho_0 < \epsilon_1\}$, it has to go to $\mu = \epsilon_0$ in the backward direction and to $\mu = -\epsilon_0$ in the forward direction. If γ has a point in $\{\tilde\rho^2+\rho_0 < \epsilon_1\}$, then it has to go to L_+ in the backward direction and to $\tilde\rho^2+\rho_0 = \epsilon_1$ in the forward direction; if $|\mu| \geq \epsilon_0$ by the time $\tilde\rho^2+\rho_0 = \epsilon_1$ is reached, the result is proved, and otherwise $\mathsf{H}_{p_{\hbar,z}}\mu < 0$ in $\tilde\rho^2+\rho_0 \geq \epsilon_1$, $|\mu| \leq \epsilon_0$, shows that the bicharacteristic goes to $\mu = -\epsilon_0$ in the forward direction.

If γ is a bicharacteristic in $\Sigma_{\hbar,-}$, only the second possibility exists, and the bicharacteristic cannot leave $\{\tilde\rho^2+\rho_0 < \epsilon_1\}$ in $|\mu| \leq \epsilon_0$, so it reaches $\mu = -\epsilon_0$ in the backward direction (as the characteristic set is in $\mu \leq 0$). □

If we assume that g_0 is a non-trapping metric, i.e. bicharacteristics of g_0 in $T^*X_0^\circ \setminus o$ tend to ∂X_0 in both the forward and the backward directions, then $\mu = \epsilon_0$ can be excluded from the statement of the lemma, and the above argument gives the following stronger conclusion: for sufficiently small $\epsilon_0 > 0$, and for $\operatorname{Re} z > 0$, any bicharacteristic in $\Sigma_{\hbar,+}$ in $-\epsilon_0 \leq \mu$ has to go to L_+ in the backward direction, and to $\mu = -\epsilon_0$ in the forward direction (with the exception of the constant bicharacteristics at L_+), while in $\Sigma_{\hbar,-}$, all bicharacteristics in $-\epsilon_0 \leq \mu$ lie in $-\epsilon_0 \leq \mu \leq 0$, and go to L_- in the forward direction and to $\mu = -\epsilon_0$ in the backward direction (with the exception of the constant bicharacteristics at L_-).

In fact, for applications, it is also useful to remark that for sufficiently small $\epsilon_0 > 0$, and for $\alpha \in T^*X_0$,

$$0 < \mu(\alpha) < \epsilon_0,\ p_{\hbar,z}(\alpha) = 0 \text{ and } (\mathsf{H}_{p_{\hbar,z}}\mu)(\alpha) = 0 \Rightarrow (\mathsf{H}^2_{p_{\hbar,z}}\mu)(\alpha) < 0. \tag{33}$$

Indeed, as $\mathsf{H}_{p_{\hbar,z}}\mu = 4(2(1+a_1)\mu\xi - (1+a_2)z)$, the hypotheses imply $z = 2(1+a_1)(1+a_2)^{-1}\mu\xi$ and

$$\begin{aligned}0 &= p_{\hbar,z}\\ &= 4(1+a_1)\mu\xi^2 - 8(1+a_1)\mu\xi^2 - 4(1+a_1)^2(1+a_2)^{-2}(1+a_3)\mu^2\xi^2 + |\eta|^2_{\mu,y}\\ &= -4(1+a_1)\mu\xi^2 - 4(1+a_1)^2(1+a_2)^{-2}(1+a_3)\mu^2\xi^2 + |\eta|^2_{\mu,y},\end{aligned}$$

so $|\eta|^2_{\mu,y} = 4(1+b)\mu\xi^2$, with b vanishing at $\mu = 0$. Thus, at points where $\mathsf{H}_{p_{\hbar,z}}\mu$ vanishes, writing $a_j = \mu\tilde a_j$,

$$\mathsf{H}^2_{p_{\hbar,z}}\mu = 8(1+a_1)\mu\mathsf{H}_{p_{\hbar,z}}\xi + 8\mu^2\xi\mathsf{H}_{p_{\hbar,z}}\tilde a_1 - 4z\mu\mathsf{H}_{p_{\hbar,z}}\tilde a_2 = 8(1+a_1)\mu\mathsf{H}_{p_{\hbar,z}}\xi + \mathcal{O}(\mu^2\xi^2). \tag{34}$$

Now

$$\mathsf{H}_{p_{\hbar,z}}\xi = -(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 - 4\frac{\partial a_2}{\partial\mu}z\xi + \frac{\partial a_3}{\partial\mu}z^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}).$$

Since $z\xi$ is $\mathcal{O}(\mu\xi^2)$ due to $\mathsf{H}_{p_{\hbar,z}}\mu = 0$, z^2 is $\mathcal{O}(\mu^2\xi^2)$ for the same reason, and $|\eta|^2$ and $\partial_\mu|\eta|^2$ are $\mathcal{O}(\mu\xi^2)$ due to $p_{\hbar,z} = 0$, we deduce that $\mathsf{H}_{p_{\hbar,z}}\xi < 0$ for sufficiently small $|\mu|$, so (34) implies (33). Thus, μ can be used for gluing constructions.

0.6. **Complex absorption.** The final step of fitting P_σ into our general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. We thus consider a compact manifold without boundary X for which $X_{\mu_0} = \{\mu > \mu_0\}$, $\mu_0 = -\epsilon_0 < 0$, with $\epsilon_0 > 0$ as above, is identified as an open subset with smooth boundary; it is convenient to take X to be the double of X_{μ_0}, so there are two copies of $X_{0,\mathrm{even}}$ in X.

In the case of hyperbolic space, this doubling process can be realized from the perspective of $(n+1)$-dimensional Minkowski space. Then, as mentioned in the introduction, the Poincaré model shows up in two copies, namely in the interior of the future and past light cone inside the sphere at infinity, while de Sitter space as the 'equatorial belt', i.e. the exterior of the light cone at the sphere at infinity. One can take the Minkowski equatorial plane, $t = 0$, as $\mu = \mu_0$, and place the complex absorption there, thereby decoupling the future and past hemispheres.

It is convenient to separate the 'classical' (i.e. quantum!) and 'semiclassical' problems, for in the former setting trapping for g_0 does not matter, while in the latter it does.

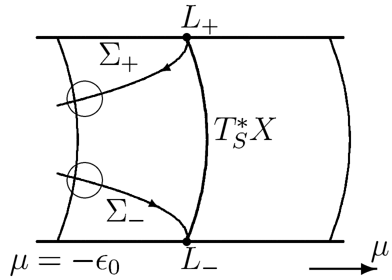

FIGURE 2. The cotangent bundle near $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view, as in Figure 1. The circles on the left show the support of q; it has opposite signs on the two disks corresponding to the opposite directions of propagation relative to the Hamilton vector field.

We then introduce a 'complex absorption' operator $Q_\sigma \in \Psi^2_{\mathrm{cl}}(X)$ with real principal symbol q supported in, say, $\mu < -\epsilon_1$, with the Schwartz kernel also supported in the corresponding region (i.e. in both factors on the product space this condition holds on the support) such that $p \pm \imath q$ is elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that $\pm q \geq 0$ near $\Sigma_\pm$. This can easily be done since $\Sigma_\pm$ are disjoint, and away from these p is elliptic, hence so is $p \pm \imath q$ regardless of the choice of q; we simply need to make q to have support sufficiently close to $\Sigma_\pm$, elliptic on $\Sigma_\pm$ at $\mu = -\epsilon_0$, with the appropriate sign near $\Sigma_\pm$. Having done this, we extend p and q to X in such a way that $p \pm \imath q$ are elliptic near ∂X_{μ_0}; the region we added is thus irrelevant at the level of bicharacteristic dynamics (of p) in so far as it is decoupled from the dynamics in X_0, and indeed also for analysis as we see shortly (in so far as we have two essentially decoupled copies of the same problem). This is accomplished, for instance, by using the doubling construction to define p on $X \setminus X_{\mu_0}$ (in a smooth fashion at ∂X_{μ_0}, as can be easily arranged; the holomorphic dependence of P_σ on σ is still easily preserved), and then, noting that the characteristic set of p still has two connected components, making q elliptic on the characteristic set of p near ∂X_{μ_0}, with the same sign in each component as near ∂X_{μ_0}. (An alternative would be to make q elliptic on the characteristic set of p near $X \setminus X_{\mu_0}$; it is just slightly more complicated to write down such a q when the high energy behavior is taken into account. With the present choice, due to the doubling, there are essentially two copies of the problem on X_0: the original, and the one from the doubling.) Finally we take Q_σ be any operator with principal symbol q with Schwartz kernel satisfying the desired support conditions and which depends on σ holomorphically. We may choose Q_σ to be independent of σ so Q_σ is indeed holomorphic; in this case we may further replace it by $\frac{1}{2}(Q_\sigma + Q_\sigma^*)$ if self-adjointness is desired.

In view of Subsection 0.4 we have arranged the following. For $\alpha \in S^*X \cap \Sigma$, let $\gamma_+(\alpha)$, resp. $\gamma_-(\alpha)$ denote the image of the forward, resp. backward, half-bicharacteristic of p from α. We write $\gamma_\pm(\alpha) \to L_\pm$ (and say $\gamma_\pm(\alpha)$ tends to $L_\pm$) if given any neighborhood O of $L_\pm$, $\gamma_\pm(\alpha) \cap O \neq \emptyset$; by the source/sink property this implies that the points on the curve are in O for sufficiently large (in absolute value) parameter values. Then, with $\mathrm{Ell}(Q_\sigma)$ denoting the elliptic set of Q_σ,

$$
\begin{aligned}
(35)\quad & \alpha \in \Sigma_- \setminus L_- \Rightarrow \gamma_+(\alpha) \to L_- \text{ and } \gamma_-(\alpha) \cap \mathrm{Ell}(Q_\sigma) \neq \emptyset, \\
& \alpha \in \Sigma_+ \setminus L_+ \Rightarrow \gamma_-(\alpha) \to L_+ \text{ and } \gamma_+(\alpha) \cap \mathrm{Ell}(Q_\sigma) \neq \emptyset.
\end{aligned}
$$

That is, all forward and backward half-(null)bicharacteristics of P_σ either enter the elliptic set of Q_σ, or go to $\Lambda_\pm$, i.e. $L_\pm$ in S^*X. The point of the arrangements regarding Q_σ and the flow is that we are able to propagate estimates forward near where $q \geq 0$, backward near where $q \leq 0$, so by our hypotheses we can always propagate estimates for $P_\sigma - \imath Q_\sigma$ from $\Lambda_\pm$ towards the elliptic set of Q_σ. On the other hand, for $P_\sigma^* + \imath Q_\sigma^*$, we can propagate estimates from the elliptic set of Q_σ towards $\Lambda_\pm$. This behavior of $P_\sigma - \imath Q_\sigma$ vs. $P_\sigma^* + \imath Q_\sigma^*$ is important for duality reasons.

An alternative to the complex absorption would be simply adding a boundary at $\mu = \mu_0$; this is easy to do since this is a space-like hypersurface, but this is slightly unpleasant from the point of view of microlocal analysis as one has to work on a manifold with boundary (though as mentioned this is easily done).

For the semiclassical problem, when z is almost real (namely when $\mathrm{Im}\, z$ is bounded away from 0 we only need to make sure we do not mess up the semiclassical ellipticity in $T^*X_{-\delta_0}$) we need to increase the requirements on Q_σ, and what we need to do depends on whether g_0 is non-trapping.

If g_0 is non-trapping, we choose Q_σ such that $h^2 Q_{h^{-1}z} \in \Psi^2_{\hbar,\mathrm{cl}}(X)$ with semiclassical principal symbol $q_{\hbar,z}$, and in addition to the above requirement for the classical symbol, we need semiclassical ellipticity near $\mu = \mu_0$, i.e. that $p_{\hbar,z} - \imath q_{\hbar,z}$ and its complex conjugate are elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that for z real $\pm q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. Again, we extend P_σ and Q_σ to X in such a way that $p - \imath q$ and $p_{\hbar,z} - \imath q_{\hbar,z}$ (and thus their complex conjugates) are elliptic near ∂X_{μ_0}; the region we added is thus irrelevant. This is straightforward to arrange if one ignores that one wants Q_σ to be holomorphic: one easily constructs a function $q_{\hbar,z}$ on T^*X (taking into account the disjointness of $\Sigma_{\hbar,\pm}$), and defines $Q_{h^{-1}z}$ to be h^{-2} times the semiclassical quantization of $q_{\hbar,z}$ (or any other operator with the same semiclassical and standard principal symbols). Indeed, for our purposes this would suffice since we want high energy estimates for the analytic continuation resolvent on the original space X_0 (which we will know exists by the non-semiclassical argument), and as we shall see, the resolvent is given by the same formula in terms of $(P_\sigma - \imath Q_\sigma)^{-1}$ independently whether Q_σ is holomorphic in σ (as long as it satisfies the other properties), so there is no need to ensure the holomorphy of Q_σ. However, it is instructive to have an example of a holomorphic family Q_σ in a strip at least:

in view of (27) we can take (with $C>0$)

$$q_{h,z}=2(2(1+a_2)\xi+(1+a_3)z)(\xi^2+|\eta|^2+z^2+C^2h^2)^{1/2}\chi(\mu),$$

where $\chi\geq 0$ is supported near μ_0; the corresponding full symbol is

$$\sigma_{\mathrm{full}}(Q_\sigma)=2(2(1+a_2)\xi+(1+a_3)\sigma)(\xi^2+|\eta|^2+\sigma^2)^{1/2}\chi(\mu),$$

and Q_σ is taken as a quantization of this full symbol. Here the square root is defined on $\mathbb{C}\backslash[0,-\infty)$, with real part of the result being positive, and correspondingly $q_{h,z}$ is defined away from $h^{-1}z\in\pm\imath[C,+\infty)$. Note that $\xi^2+|\eta|^2+\sigma^2$ is an elliptic symbol in $(\xi,\eta,\operatorname{Re}\sigma,\operatorname{Im}\sigma)$ as long as $|\operatorname{Im}\sigma|<C'|\operatorname{Re}\sigma|$, so the corresponding statement also holds for its square root. While $q_{h,z}$ is only holomorphic away from $h^{-1}z\in\pm\imath[C,+\infty)$, the full (and indeed the semiclassical and standard principal) symbols are actually holomorphic in cones near infinity, and indeed e.g. via convolutions by the Fourier transform of a compactly supported function can be extended to be holomorphic in $\mathbb{C}$, but this is of no importance here.

If g_0 is trapping, we need to add complex absorption inside X_0 as well, at $\mu=\epsilon_0$, so we relax the requirement that Q_σ is supported in $\mu<-\epsilon_0/2$ to support in $|\mu|>\epsilon_0/2$, but we require in addition to the other classical requirements that $p_{\hbar,z}-\imath q_{\hbar,z}$ and its complex conjugate are elliptic near $\mu=\pm\epsilon_0$, and which satisfies that $\pm q_{\hbar,z}\geq 0$ on $\Sigma_{\hbar,\pm}$. This can be achieved as above for μ near μ_0. Again, we extend P_σ and Q_σ to X in such a way that $p-\imath q$ and $p_{\hbar,z}-\imath q_{\hbar,z}$ (and thus their complex conjugates) are elliptic near ∂X_{μ_0}.

In either of these semiclassical cases we have arranged that for sufficiently small $\delta_0>0$, $p_{\hbar,z}-\imath q_{\hbar,z}$ and its complex conjugate are *semiclassically non-trapping* for $|\operatorname{Im}z|<\delta_0$, namely the bicharacteristics from any point in $\Sigma_\hbar\setminus(L_+\cup L_-)$ flow to $\operatorname{Ell}(q_{\hbar,z})\cup L_-$ (i.e. either enter $\operatorname{Ell}(q_{\hbar,z})$ at some finite time, or tend to L_-) in the forward direction, and to $\operatorname{Ell}(q_{\hbar,z})\cup L_+$ in the backward direction. Here $\delta_0>0$ arises from the particularly simple choice of $q_{\hbar,z}$ for which semiclassical ellipticity is easy to check for $\operatorname{Im}z>0$ (bounded away from 0) and small; a more careful analysis would give a specific value of δ_0, and a more careful choice of $q_{\hbar,z}$ would give a better result.

0.7. **Meromorphic continuation of the resolvent.** We now state our results in the original conformally compact setting. Without the non-trapping estimate, these are a special case of a result of Mazzeo and Melrose, with improvements by Guillarmou, with 'special' meaning that evenness is assumed. If the space is asymptotic to actual hyperbolic space, the non-trapping estimate is a stronger version of the estimate of Melrose, Sá Barreto and the author, where it is shown by a parametrix construction; here conformal infinity can have arbitrary geometry. The point is thus that first, we do not need the machinery of the zero calculus here, second, we do have non-trapping high energy estimates in general (and without a parametrix construction), and third, we add the semiclassically outgoing property which is useful for resolvent gluing, including for proving non-trapping bounds microlocally away from trapping, provided the latter is mild, as shown by Datchev and Vasy.

Theorem 0.3. *Suppose that (X_0,g_0) is an n-dimensional manifold with boundary with an even conformally compact metric and boundary defining function x. Let $X_{0,\mathrm{even}}$ denote the even version of X_0, i.e. with the boundary defining function* *replaced by its square with respect to a decomposition in which g_0 is even. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-1}{2}\right)^2 - \sigma^2,$$

written as $\mathcal{R}(\sigma) : L^2 \to L^2$, has a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$,

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{D}'(X_0),$$

with poles with finite rank residues. If in addition (X_0, g_0) is non-trapping, then, with ϕ as in Subsection 0.2, and for suitable $\delta_0 > 0$, non-trapping estimates hold in every region $-C < \operatorname{Im}\sigma < \delta_0|\operatorname{Re}\sigma|$, $|\operatorname{Re}\sigma| \gg 0$: for $s > \frac{1}{2} + C$,

(36)

$$\|x^{-(n-1)/2} e^{\imath\sigma\phi}\mathcal{R}(\sigma) f\|_{H^s_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+3)/2}e^{\imath\sigma\phi} f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}.$$

If f is supported in X_0°, the $s-1$ norm on f can be replaced by the $s-2$ norm.

Furthermore, for $\operatorname{Re} z > 0$, $\operatorname{Im} z = \mathcal{O}(h)$, the resolvent $\mathcal{R}(h^{-1}z)$ is semiclassically outgoing with a loss of h^{-1} *in the sense that if f has compact support in X_0°, $\alpha \in T^*X$ is in the semiclassical characteristic set and if $\mathrm{WF}_\hbar^{s-1,0}(f)$ is disjoint from the backward bicharacteristic from α, then $\alpha \notin \mathrm{WF}_\hbar^{s,-1}(\mathcal{R}(h^{-1}z)f)$.*

We remark that although in order to go through without changes, our methods require the evenness property, it is not hard to deduce more restricted results without this. Essentially one would have operators with coefficients that have a conormal singularity at the event horizon; as long as this is sufficiently mild relative to what is required for the analysis, it does not affect the results. The problems arise for the analytic continuation, when one needs strong function spaces (H^s with s large); these are not preserved when one multiplies by the singular coefficients.

Proof. All of our microlocal results apply.

By self-adjointness and positivity of Δ_{g_0} and as $\dot{\mathcal{C}}^\infty(X_0)$ is in its domain,

$$\left(\Delta_{g_0} - \sigma^2 - \left(\frac{n-1}{2}\right)^2\right) u = f \in \dot{\mathcal{C}}^\infty(X_0)$$

has a unique solution $u = \mathcal{R}(\sigma)f \in L^2(X_0, |dg_0|)$ when when $\operatorname{Im}\sigma \gg 0$. On the other hand, let ϕ be as in Subsection 0.2, so $e^\phi = \mu^{1/2}(1+\mu)^{-1/4}$ near $\mu = 0$ (so $e^\phi \sim x$ there), $\tilde{f}_0 = e^{\imath\sigma\phi}x^{-(n+1)/2}x^{-1}f$ in $\mu \geq 0$, and $\tilde{f}_0$ still vanishes to infinite order at $\mu = 0$. Let $\tilde{f}$ be an arbitrary smooth extension of $\tilde{f}_0$ to the compact manifold X on which $P_\sigma - \imath Q_\sigma$ is defined. Let $\tilde{u} = (P_\sigma - \imath Q_\sigma)^{-1}\tilde{f}$, with $(P_\sigma - \imath Q_\sigma)^{-1}$ given by our microlocal results; this satisfies $(P_\sigma - \imath Q_\sigma)\tilde{u} = \tilde{f}$ and $\tilde{u} \in \mathcal{C}^\infty(X)$. Thus, $u' = e^{-\imath\sigma\phi}x^{(n+1)/2}x^{-1}\tilde{u}|_{\mu>0}$ satisfies $u' \in x^{(n-1)/2}e^{-\imath\sigma\phi}\mathcal{C}^\infty(X_0)$, and

$$\left(\Delta_{g_0} - \sigma^2 - \left(\frac{n-1}{2}\right)^2\right) u' = f$$

by (8) and (17) (as Q_σ is supported in $\mu < 0$). Since $u' \in L^2(X_0, |dg_0|)$ for $\operatorname{Im}\sigma > 0$, by the aforementioned uniqueness, $u = u'$.

To make the extension from $X_{0,\mathrm{even}}$ to X more systematic, let $E_s : H^s(X_{0,\mathrm{even}}) \to H^s(X)$ be a continuous extension operator, $R_s : H^s(X) \to H^s(X_{0,\mathrm{even}})$ the restriction map. Then, as we have just seen, for $f \in \dot{\mathcal{C}}^\infty(X_0)$,

$$\mathcal{R}(\sigma)f = e^{-\imath\sigma\phi}x^{(n+1)/2}x^{-1}R_s(P_\sigma - \imath Q_\sigma)^{-1}E_{s-1}e^{\imath\sigma\phi}x^{-(n+1)/2}x^{-1}f. \tag{37}$$

While, for the sake of simplicity, Q_σ is constructed in Subsection 0.6 in such a manner that it is not holomorphic in all of $\operatorname{Im}\sigma > -C$ due to a cut in the upper half plane, this cut can be moved outside any fixed compact subset, so taking into account that $\mathcal{R}(\sigma)$ is independent of the choice of Q_σ, the theorem follows immediately from our earlier microlocal results. $\square$

Our argument proves that every pole of $\mathcal{R}(\sigma)$ is a pole of $(P_\sigma - \imath Q_\sigma)^{-1}$ (for otherwise (37) would show $\mathcal{R}(\sigma)$ does not have a pole either), but it is possible for $(P_\sigma - \imath Q_\sigma)^{-1}$ to have poles which are not poles of $\mathcal{R}(\sigma)$. However, in the latter case, the Laurent coefficients of $(P_\sigma - \imath Q_\sigma)^{-1}$ would be annihilated by multiplication by R_s from the left, i.e. the resonant states (which are smooth) would be supported in $\mu \leq 0$, in particular vanish to infinite order at $\mu = 0$.

In fact, a stronger statement can be made: by a calculation completely analogous to what we just performed, we can easily see that in $\mu < 0$, P_σ is a conjugate (times a power of μ) of a Klein-Gordon-type operator on n-dimensional de Sitter space with $\mu = 0$ being the boundary (i.e. where time goes to infinity). Thus, if σ is not a pole of $\mathcal{R}(\sigma)$ and $(P_\sigma - \imath Q_\sigma)\tilde{u} = 0$ then one would have a solution u of this Klein-Gordon-type equation near $\mu = 0$, i.e. infinity, that rapidly vanishes at infinity. In fact, one can use a Carleman-type estimate to show that this cannot happen. Thus, if Q_σ is supported in $\mu < c$, $c < 0$, then $\tilde{u}$ is also supported in $\mu < c$. This argument can be iterated for Laurent coefficients of higher order poles; their range (which is finite dimensional) contains only functions supported in $\mu < c$.

Remark 0.4. We now return to our previous remarks regarding the fact that our solution disallows the conormal singularities $(\mu \pm i0)^{\imath\sigma}$ from the perspective of conformally compact spaces of dimension n. Recalling that $\mu = x^2$, the two indicial roots on these spaces correspond to the asymptotics $\mu^{\pm\imath\sigma/2+(n-1)/4}$ in $\mu > 0$. Thus for the operator

$$\mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2},$$

or indeed P_σ, they correspond to

$$\Big(\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2}\Big)^{-1}\mu^{\pm\imath\sigma/2+(n-1)/4} = \mu^{\imath\sigma/2\pm\imath\sigma/2}.$$

Here the indicial root $\mu^0 = 1$ corresponds to the smooth solutions we construct for P_σ, while $\mu^{\imath\sigma}$ corresponds to the conormal behavior we rule out. Back to the original Laplacian, thus, $\mu^{-\imath\sigma/2+(n-1)/4}$ is the allowed asymptotics and $\mu^{\imath\sigma/2+(n-1)/4}$ is the disallowed one. Notice that $\operatorname{Re}\imath\sigma = -\operatorname{Im}\sigma$, so the disallowed solution is growing at $\mu = 0$ relative to the allowed one, as expected in the physical half plane, and the behavior reverses when $\operatorname{Im}\sigma < 0$. Thus, in the original asymptotically hyperbolic picture one has to distinguish two different rates of growths, whose relative size changes. On the other hand, in our approach, we rule out the singular solution and allow the non-singular (smooth one), so there is no change in behavior at all for the analytic continuation.

Remark 0.5. For *even* asymptotically de Sitter metrics on an n-dimensional manifold X_0' with boundary, the methods for asymptotically hyperbolic spaces work, except $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma^*$ switch roles, which does not affect Fredholm properties. Again, evenness means that we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 - h}{x^2} \tag{38}$$

there, where h is an even family of Riemannian metrics; as above, we take x to be a globally defined boundary defining function. Then with $\tilde{\mu} = x^2$, so $\tilde{\mu} > 0$ is the Lorentzian region, $\overline{\sigma}$ in place of σ (recalling that our aim is to get to $P_\sigma^* + \imath Q_\sigma^*$) the above calculations for $\Box_{g_0} - \frac{(n-1)^2}{4} - \overline{\sigma}^2$ in place of $\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2$ leading to (7) all go through with μ replaced by $\tilde{\mu}$, σ replaced by $\overline{\sigma}$ and Δ_h replaced by $-\Delta_h$. Letting $\mu = -\tilde{\mu}$, and conjugating by $(1+\mu)^{\imath\overline{\sigma}/4}$ as above, yields

$$-4\mu D_\mu^2 + 4\overline{\sigma} D_\mu + \overline{\sigma}^2 - \Delta_h + 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \overline{\sigma}/2 - \imath(n-1)/4), \tag{39}$$

modulo terms that can be absorbed into the error terms in operators in the class (8), i.e. this is indeed of the form $P_\sigma^* + \imath Q_\sigma^*$ in the framework of Subsection 0.6, at least near $\tilde{\mu} = 0$. If now X_0' is extended to a manifold without boundary in such a way that in $\tilde{\mu} < 0$, i.e. $\mu > 0$, one has a classically elliptic, semiclassically non-trapping problem, then all of our microlocal tools are applicable.