We now understand elliptic operators in \( \Psi^{m,\ell} \); the next challenge is to deal with non-elliptic operators. Let’s start with classical operators, and indeed let’s take \( m = \ell = 0 \). Thus, \( A = q_L(a) \), \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), so \( \sigma_{0,0}(A) \) is just the restriction of \( a \) to \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \). Ellipticity is just the statement that \( a_0 = a|_{\partial(\mathbb{R}^n \times \mathbb{R}^n)} \) does not vanish. Thus, the simplest (or least degenerate/complicated) way an operator can be non-elliptic is if \( a_0 \) is real-valued, and has a non-degenerate \( 0 \). As \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \) is not a smooth manifold at the corner, \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \), one has to be a bit careful. Away from the corner non-degeneracy is the statement that \( a_0(\alpha) = 0 \) implies \( \partial a_0(\alpha) \neq 0 \); in this case the characteristic set, \( \text{Char}(A) = a_0^{-1}(\{0\}) \) is a \( C^\infty \) codimension one embedded submanifold. At the corner, for \( \alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n) \), one can consider the two smooth manifolds with boundary \( \mathbb{R}^n \times \partial \mathbb{R}^n \) and \( \partial \mathbb{R}^n \times \mathbb{R}^n \), and denoting by \( a_{0, \text{fiber}} \) and \( a_{0, \text{base}} \) the corresponding restrictions, ask that \( a_0(\alpha) = 0 \) implies that \( \partial a_{0, \text{fiber}} \) and \( \partial a_{0, \text{base}} \) are not in the conormal bundle of the corner; in this case in both boundary hypersurfaces \( \text{Char}(A) \) is a smooth manifold transversal to the boundary. In many cases, such as stationary (elliptic) Schrödinger operators, the characteristic set is disjoint from the corner, \( \text{Char}(A) \) is disjoint from the corner, but this is not the case for wave propagation.

More generally, if \( A \) is classical, i.e. \( a = \langle z \rangle^\ell \langle \zeta \rangle^m \tilde{a}, \tilde{a} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), we impose the analogous condition on \( \tilde{a} \), i.e. that \( \tilde{a}_0 \) has a non-degenerate zero set. Note that if \( b \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) is elliptic, and \( \tilde{a}_0 \) has a non-degenerate zero set, then, with \( b_0 \) denoting the restriction of \( b \) to the boundary, the same holds for \( b_0 \tilde{a}_0 \) as \( d(b_0 \tilde{a}_0) = b_0 d\tilde{a}_0 + \tilde{a}_0 d b_0 \), so at when \( b_0 \tilde{a}_0 = 0 \), i.e. when \( \tilde{a}_0 = 0 \), \( d(b_0 \tilde{a}_0) \) is a non-vanishing multiple of \( d\tilde{a}_0 \).

Let’s re-interpret this from the conic point of view, for instance corresponding to \( \mathbb{R}^n \times \partial \mathbb{R}^n \), i.e. working on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \). (Note that working with \( \partial \mathbb{R}^n \times \mathbb{R}^n \), i.e. \( (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \) is completely analogous.) Away from the corner, we may drop the compactification from the first factor, and thus we may assume \( \ell = 0 \). For \( A \) classical, then, \( \sigma_{\text{fiber},0,0}(A) \) is homogeneous of degree \( m \), given by \( a_m = |\zeta|^m \tilde{a}_0 \), where \( \tilde{a}_0 \) is considered as a homogeneous degree zero function. Now, \( d(|\zeta|^m \tilde{a}_0) = |\zeta|^m d\tilde{a}_0 + \tilde{a}_0 d|\zeta|^m \), and \( \text{Char}(A) \) is given by \( \tilde{a}_0 = 0 \), so it is now a conic (invariant under the \( \mathbb{R}^+ \)-action) smooth codimension one embedded submanifold of \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \).

The next relevant structure arises from the Hamilton vector field of \( A \),

\[
H_{a_m} = \sum_{j=1}^n \left( (\partial_{\zeta_j} a_m) \partial_{z_j} - (\partial_{z_j} a_m) \partial_{\zeta_j} \right).
\]

Note that \( H_{a_m} a_m = 0 \), thus this vector field is tangent to \( \text{Char}(A) \), and thus defines a flow on \( \text{Char}(A) \). Note that \( H_{a_m} \) is homogeneous of degree \( m-1 \) in the sense that the push-forward of \( H_{a_m} \) under dilation in the fiber by \( t > 0 \), \( M_t(z, \zeta) = (z, t\zeta) \), is \( t^{-m+1} H_{a_m} \) since, using that \( \partial_{z_j} a_m \), resp. \( \partial_{\zeta_j} a_m \) are homogeneous of degree \( m \),
homogeneous of degree 2. Then
\[ H_R^\ast f \](z, \zeta) = \sum_{j=1}^{\infty} (t^{1-m}(\partial_\zeta a_m)(z, t\zeta)(\partial_z f)(z, t\zeta)
- t^{-m}(\partial_z a_m)(z, t\zeta)t(\partial_\zeta f)(z, t\zeta)) = t^{1-m}(H_{a_m} f)(z, t\zeta).

In particular, integral curves of \( H_{a_m} \) through \((z, \zeta)\) and \((z, t\zeta)\) are the 'same' up to reparameterization: denoting the first by \( \gamma \), the second by \( \tilde{\gamma} \), and denoting by \( \tilde{M}_t \) dilations on \( \mathbb{R} \): \( \tilde{M}_t(s) = ts \),
\[ \tilde{\gamma}(s) = (z(\gamma(t^{m-1}s)), t\zeta(\gamma(t^{m-1}s))) = (M_t \circ \gamma \circ \tilde{M}_{m-1})(s), \]
since \( (\tilde{M}_t)_\ast \frac{d}{ds} = t \frac{d}{ds} \), so
\[ (M_t \circ \gamma \circ \tilde{M}_{m-1})_\ast \frac{d}{ds} = t^{m-1}(M_t \circ \gamma)_\ast \frac{d}{ds} = t^{m-1}(M_t)_\ast H_{a_m} = H_{a_m}, \]
as being an integral curve of \( H_{a_m} \) means exactly that the push-forward of \( \frac{d}{ds} \) under the map is exactly \( H_{a_m} \). In particular, up to reparameterization, the integral curves can be considered curves on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+ \), i.e. on this quotient, the image of the curve is defined, but not the parameterization itself. The exception is if \( m = 1 \), in which case even the parameterization is well-defined on this quotient; in this case, \( H_{a_m} \) acts on homogeneous degree zero functions, and is thus a vector field on the quotient. In general, if \( m \) is arbitrary, one may consider instead the vector field \( |\zeta|^{-m+1} H_{a_m} \), which is homogeneous of degree 0, and thus has well-defined integral curves on the quotient: this does depend on the choice of a positive homogeneous degree 1 function, such as \( |\zeta| \), but a different choice only multiplies \( |\zeta|^{-m+1} H_{a_m} \) by a smooth non-vanishing function, and thus simply re-parameterizes the integral curves.

A different way of looking at this is that for \( m = 1 \), by the identification of \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+ \) with \( \mathbb{R}^n \times \partial \mathbb{R}^n \), one has a vector field on \( \mathbb{R}^n \times \partial \mathbb{R}^n \); if \( m \neq 1 \), then again this vector field is defined up to positive multiples.

**Definition 1.** The integral curves of \( H_{a_m} \) in \( \text{Char}(A) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \), as well as their projections to \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+ = \mathbb{R}^n \times S^{n-1} = \mathbb{R}^n \times \partial \mathbb{R}^n \), are called null-bicharacteristics, or simply bicharacteristics.

The Hamiltonian version of classical mechanics corresponding to an energy function \( p \) is just the statement that particles move on integral curves of \( H_p \). In physical terms, geometric optics for the wave equation is that at high frequencies waves follow paths given by classical mechanics. Concretely, including the time variable as part of our space, on \( \mathbb{R}^n_t = \mathbb{R}^n_y \times \mathbb{R}_t \), one has a Lorentzian metric \( g \), for instance a product-type metric, \( g = dt^2 - h(y, dy) \), and then the dual metric function \( G \) on \( \mathbb{R}^n_z \times \mathbb{R}^n_\zeta \), given by \( G(z, \zeta) = G_z(\zeta, \zeta) \), with \( G_z \) the inverse of \( g_z \). Note that \( G \) is homogeneous of degree 2. Then \( H_G \) is a vector field of homogeneity degree 1. The integral curves of \( H_G \) are the lifted geodesics; those inside \( \{ G = 0 \} \) are the lifted null-geodesics. Now, the d’Alembertian
\[ \Box_g = |\det g|^{-1/2} \sum_{i,j} D_i |\det g|^{1/2} G_{ij} D_j \]
has principal symbol \( G \), while waves are solutions of \( \Box_g u = 0 \). Mathematically, the high frequency statement translates into singularities of solutions \( u \). Concretely, we have the following theorem:
**Theorem 0.1.** Suppose that $P \in \Psi_{cl}^{m,0}$ and its principal symbol has a real homogeneous degree in representative $p$. Then in $\mathbb{R}^n \times \partial \mathbb{R}^n$, $WF^s(u) \setminus WF^{s-m+1}(Pu)$ is a union of maximally extended (null)-bicharacteristics of $p$.

We now turn to the general case, where the characteristic set Char($A$) of $A \in \Psi_{cl}^{m,\ell}$ possibly intersects the corner. So first we define the rescaled Hamilton vector field

$$H_a = H_{a,m,\ell} = \langle z \rangle^{-\ell+1} \langle \zeta \rangle^{-m+1} H_a,$$

and notice that as $\langle z \rangle \partial_{z_j}$ and $\langle \zeta \rangle \partial_{\zeta_j}$ are in $\mathcal{V}_b(\mathbb{R}^n)$, resp. $\mathcal{V}_b(\mathbb{R}^n)$, and thus in $\mathcal{V}_b(\mathbb{R}^n \times \mathbb{R}^n)$,

$$H_{a,m,\ell} \in \mathcal{V}_b(\mathbb{R}^n \times \mathbb{R}^n).$$

As above, we note that if we replace $\langle z \rangle^{-\ell+1} \langle \zeta \rangle^{-m+1}$ by $b \langle z \rangle^{-\ell+1} \langle \zeta \rangle^{-m+1}$, where $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is positive on $\partial(\mathbb{R}^n \times \mathbb{R}^n)$, then $H_{a,m,\ell}$ is only multiplied by a positive factor, $b$, at Char($A$) $\subset \partial(\mathbb{R}^n \times \mathbb{R}^n)$. Since $H_{a,m,\ell}$ is tangent to the boundary, its integral curves are globally well-defined (i.e. they are well-defined on $\mathbb{R}$). We then make the definition:

**Definition 2.** The integral curves of $H_{a,m,\ell}$ in Char($A$) $\subset \partial(\mathbb{R}^n \times \mathbb{R}^n)$, are called null-bicharacteristics, or simply bicharacteristics, and we consider them well-defined up to a direction-preserving reparameterization.

In general, we have the following theorem, which contains Theorem 0.1 as a special case:

**Theorem 0.2.** Suppose that $P \in \Psi_{cl}^{m,\ell}$, with real principal symbol $p$. Then in $\partial(\mathbb{R}^n \times \mathbb{R}^n)$, $WF^{s-r}(u) \setminus WF^{s-m+1,r-\ell+1}(Pu)$ is a union of maximally (null)-bicharacteristics of $p$.

A variant of this theorem in the variable order setting is the following:

**Theorem 0.3.** Suppose that $P \in \Psi_{cl}^{m,\ell}$, with real principal symbol $p$, and suppose $s \in \mathcal{C}^\infty(\mathbb{R}^n \times \partial \mathbb{R}^n)$, $r \in \mathcal{C}^\infty(\partial \mathbb{R}^n \times \mathbb{R}^n)$ are non-increasing along the rescaled Hamilton flow, i.e. $H_{p,m,\ell}s \leq 0$, $H_{p,m,\ell}r \leq 0$. Then in $\partial(\mathbb{R}^n \times \mathbb{R}^n)$, $WF^{s-r}(u) \setminus WF^{s-m+1,r-\ell+1}(Pu)$ is a union of maximally forward extended (null)-bicharacteristics of $p$.

The analogous conclusion holds if $s$, $r$ are non-decreasing and ‘forward’ is replaced by ‘backward’.

Note that all these theorems have empty statements at points at which $H_{p,m,\ell}$ vanishes (or is radial in the conic setting, i.e. is a multiple of the generator of dilations in the conic variable), so there is nothing to prove at such points. Thus, the key point is to understand what happens near points at which $H_{p,m,\ell}$ is non-vanishing.

The proof of these theorems relies on positive commutators, i.e. constructing a pseudodifferential operator $A$ such that $i[P,A]$ is of the form $B^*B$, modulo terms that we can control by our assumptions. Such an estimate actually gives bounds for the microlocal $H^{s-r}$ norm of $u$. However, the bound can also be recovered from the regularity statement of the theorem via the closed graph theorem.

**Theorem 0.4.** Suppose that $P \in \Psi_{cl}^{m,\ell}$ with real principal symbol $p$. Suppose that $B,G,Q \in \Psi^{0,0}$, $WF'(B) \subset Ell(G)$ and for every $\alpha \in WF'(B) \cap Char(P)$, there
is a point $\alpha' = \gamma(\sigma')$ on the bicharacteristic $\gamma$ through $\alpha$ with $\gamma(0) = \alpha$ such that $\alpha' \in \text{Ell}(Q)$ and such that for $\sigma \in [0, \sigma']$ (or $\sigma \in [\sigma', 0]$ if $\sigma' < 0$), $\gamma(\sigma) \in \text{Ell}(G)$.

Then for any $M, N$, there is $C > 0$ such that if $Qu \in H^{s,r}$, $GPu \in H^{s-m+1, r-\ell+1}$ then $Bu \in H^{s,r}$ and

$$\|Bu\|_{H^{s,r}} \leq C(\|Qu\|_{H^{s,r}} + \|GPu\|_{H^{s-m+1, r-\ell+1}} + \|u\|_{H^{M,N}}).$$

The analogous conclusion also holds in the variable order setting if either $s, r$ are non-increasing along the Hamilton flow and $\sigma' < 0$ or $s, r$ are non-decreasing along the Hamilton flow and $\sigma' > 0$.

**Proof.** Under the assumptions, by Theorem 0.2, $Bu \in H^{s,r}$ since $WF^{s,r}(u) = \emptyset$. Indeed, if $\alpha \notin WF'(B)$, then $\alpha \notin WF^{s,r}(Bu)$ automatically. If $\alpha \in WF'(B)$, then if $\alpha \notin \text{Char}(P)$ then $GPu \in H^{s-m+1, r-\ell+1}$ implies that $Ell(G)$, thus $WF'(B)$, are disjoint from $WF^{s-m+1, r-\ell+1}(Pu)$, and thus by elliptic regularity, $\alpha \notin WF^{s+1, r+1}(u)$.

If $\alpha \in WF'(B) \cap \text{Char}(P)$, then Theorem 0.2 states $\alpha \notin WF^{s,r}(u)$, finishing the proof of the claim.

Now,

$$X = \{u \in H^{M,N} : Qu \in H^{s,r}, GPu \in H^{s-m+1, r-\ell+1}\}$$

is complete by the lemma below, as is

$$Y = \{u \in H^{M,N} : Bu \in H^{s,r}\}.$$  

By Theorem 0.2, the identity map on $H^{M,N}$ restricts to a map $\iota : X \to Y$. Further, if $u_k \to u$ in $X$ and $u_k = \iota(u_k) \to v$ in $Y$ then in particular $u_k \to u$ in $H^{M,N}$ and $u_k \to v$ in $H^{M,N}$, so $\iota(u) = u = v$, i.e. the graph of $\iota$ is closed. The closed graph theorem thus implies that $\iota$ is continuous, which is exactly the estimate in the statement of the theorem.

**Lemma 0.5.** Suppose $A_j \in \Psi^{m_j, \ell_j}$, $j = 1, \ldots, N$. Let

$$X = \{u \in H^{r,s} : A_j u \in H^{r, s_j}, j = 1, \ldots, N\} \subset H^{r,s},$$

equipped with the norm

$$\|u\|_{X}^2 = \|u\|_{H^{r,s}}^2 + \sum_{j=1}^{N} \|A_j u\|_{H^{r, s_j}}^2.$$  

Then $X$ is complete.

Here all spaces and operators may have variable orders.

**Proof.** Suppose that $\{u_k\}_{k=1}^{\infty}$ is $X$-Cauchy. Then $u_k$, resp. $A_j u_k$, are $H^{r,s}$, resp. $H^{r, s_j}$-Cauchy, and thus converge to some $v \in H^{r,s}$, resp. $v_j \in H^{r, s_j}$. But $A_j : H^{r,s} \to H^{r-m_j, s-\ell_j}$ is continuous, so $A_j u_k \to A_j v$ in $H^{r-m_j, s-\ell_j}$. Thus, $A_j u_k \to A_j v$ and $A_j u_k \to v_j$ in $S^j$, so $v_j = A_j v$. Thus, $A_j v \in H^{r, s_j}$, so $v \in X$, and $u_k \to v$ in $X$.

To motivate the proof of Theorem 0.2, we compute (for $A \in \Psi^{m, \ell'}$ with $A = A^*$, with non-variable order for now)

$$\langle Pu, Au \rangle - \langle Au, Pu \rangle = \langle (AP - P^*A)u, u \rangle = \langle ([A, P] + (P - P^*)A)u, u \rangle.$$  

Note that if, for instance, $Pu = 0$, then the left hand side vanishes, and so if $[A, P] + (P - P^*)A$ has some definiteness properties, then we get an interesting
conclusion. Note that $P - P^* \in \Psi^{m-1,\ell-1}$ since its principal symbol in $\Psi^{m,\ell}$ as $p - \bar{p} = 0$. The principal symbol of $i([A, P] + (P - P^*)A) \in \Psi^{m' + m - 1, \ell + \ell'}$ is

$$
\sigma_{m+m'-1,\ell+\ell'-1}(i([A, P] + (P - P^*)A)) = -H_p a + \bar{p}a,
$$

where $\bar{p}$ is the principal symbol of $P - P^* \in \Psi^{m-1,\ell-1}$. Suppose we can arrange that

(1) $$H_p a - \bar{p}a = -b^2 + e, \ b \in S^{(m+m'-1)/2, (\ell+\ell'-1)/2}, \ e \in S^{m+m'-1, \ell+\ell'-1},$$

with $e$ supported in the region where we have a priori estimates on $u$. Then letting $B \in \Psi^{(m+m'-1)/2, (\ell+\ell'-1)/2}$ be such that its principal symbol is $b$, $E \in \Psi^{m+m'-1, \ell+\ell'-1}$ such that its principal symbol is $e$,

$$i([A, P] + (P - P^*)A) = B^* B - E + F,$$

with $F \in \Psi^{m+m'-2, \ell+\ell'-2}$. Then we obtain

(2) $$\langle Pu, Au \rangle - \langle Au, Pu \rangle = \|Bu\|^2 - \langle Eu, u \rangle + \langle Fu, u \rangle,$$

i.e. we can control $Bu$ in $L^2$ by controlling $Pu$, $Eu$ and $Fu$. Since $F$ is lower order, it is dealt with inductively, gradually increasing $m', \ell'$, namely one assumes that $u$ already has some a priori regularity, so that $\langle Fu, u \rangle$ is controlled (which is automatic for sufficiently low $m', \ell'$), and then one obtains a bound for $Bu$ in a stronger space than the a priori bound. Then one can increase $m', \ell'$ by 1 each to obtain new $A', B', E', F'$; the resulting $F'$ will give $\langle Fu, u \rangle$ controlled provided $WF'(F') \subset Ell(B)$ (which follows if $WF'(A') \subset Ell(B)$) by the microlocal elliptic estimate. Thus, $B'u$ is controlled in $L^2$, which is 1/2-order gain in terms of both orders over the control provided by $Bu$, etc. Here one really needs to regularize the argument to make sense of its steps.

We remark that the sign (1) is arbitrary:

(3) $$H_p a - \bar{p}a = -b^2 + e, \ b \in S^{(m+m'-1)/2, (\ell+\ell'-1)/2}, \ e \in S^{m+m'-1, \ell+\ell'-1},$$

would also work, still giving rise to the control of $\|Bu\|^2$ in terms of the other quantities mentioned above.

Before going through this in more detail, let’s see whether we can arrange (1). Indeed, let’s work with $m' = 0, \ell' = 0$ to start with, for the additional weights will not be an issue, and ignore $\bar{p}$ as well. Thus, we essentially want to find $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ whose Hamilton derivative $H_{p,m,\ell} a$ is negative (in the sense of non-positive, with some definiteness in the region of interest), apart from a region where $e$ is supported: $a$ is thus decreasing along the $H_{p,m,\ell}$ orbits. But this is very simple to achieve locally when $H_{p,m,\ell}$ is non-zero at $a$. Since the statement is slightly different (in terms of numerology) depending on whether $a$ is at the corner or not, we consider these separately:

- If $\alpha \notin \partial \mathbb{R}^n \times \partial \mathbb{R}^n$, we can choose local coordinates $q_1, q_2, \ldots, q_{2n-1}, q_{2n}$ near $a$ on $\mathbb{R}^n \times \mathbb{R}^n$, with the chart $O$ centered at $a$, such that

  $$H_{p,m,\ell} = \partial_{q_1},$$

  and $q_{2n}$ is a boundary defining function (i.e. vanishes non-degenerately at the unique boundary hypersurface, either $\partial \mathbb{R}^n \times \mathbb{R}^n$ or $\mathbb{R}^n \times \partial \mathbb{R}^n$, containing $a$). We write $q' = (q_2, \ldots, q_{2n-1})$.  

If $α \in \partial \mathbb{R}^n \times \partial \mathbb{R}^n$, we can choose local coordinates $q_1, q_2, \ldots, q_{2n-2}, q_{2n-1}, q_{2n}$ near $α$ on $\mathbb{R}^n \times \mathbb{R}^n$, with the chart $O$ centered at $α$, such that

$$H_{p,m,ℓ} = 0,$$

and $q_{2n-1}, q_{2n}$ are boundary defining functions (i.e. vanish non-degenerately at $\partial \mathbb{R}^n \times \mathbb{R}^n$, resp. $\mathbb{R}^n \times \partial \mathbb{R}^n$). We write $q' = (q_2, \ldots, q_{2n-2})$.

Recall that in either case this is achieved by choosing a local hypersurface $S$ transversal to $H_{p,m,ℓ}(α)$ through $α$, and using that the $H_{p,m,ℓ}$-exponential map from $S \times (-ε, ε)$ ($ε > 0$ small) to a neighborhood of $α$ is a diffeomorphism. Since the $H_{p,m,ℓ}$-flow is tangent to the boundary hypersurfaces, if we choose coordinates on $S$ of which the last, resp. last two, are boundary defining functions, the pull-back under this diffeomorphism gives coordinates near $α$ which are boundary defining functions, with $q_{2n-1}$, resp. $q_{2n-2}$ being the flow parameter in $(-ε, ε)$.

In fact, it does not really matter that $H_{p,m,ℓ}q_{2n} = 0$ (and $H_{p,m,ℓ}q_{2n-1} = 0$ in the second case); this vanishes at $q_{2n} = 0$ in any case if $q_{2n}$ is any boundary defining function by the tangency of $H_{p,m,ℓ}$ to the boundary, so it is of the form $q_{2n} h$, with $h$ smooth, and we can control such terms below just as we control the terms coming from $p$.

In the first case, with $q' = (q_2, \ldots, q_{2n-1})$, for

$$a = \chi_0(q_1) \chi_1(q')^2 \chi_2(q_{2n})^2, \quad H_{p,m,ℓ} a = \chi_0'(q_1) \chi_1(q')^2 \chi_2(q_{2n})^2,$$

so we simply need to arrange that $\chi_0(q_1) \chi_1(q')^2 \chi_2(q_{2n})^2$ is supported in the region of validity of the coordinate chart, which is arranged by making $\chi_0, \chi_1$ and $\chi_2$ supported near 0, and such that $\chi_0' \leq 0$, or more precisely it is the negative of the square of a smooth function, away from the region where our error $ε$ is supported. Here $\chi_2$ is not particularly important; being away from the corner, this can be taken constant 1 near the boundary, so any contribution to symbolic calculations is trivial.

Concretely, with $γ$ the integral curve of $H_{p,m,ℓ}$ through $α$ with $γ(0) = α$, and $γ(σ) = β$ for some $σ < 0$ such that $γ([σ,0]) \subset O$, and further if we are given a neighborhood $U_2$ of $β$ and $U_1$ of $γ([σ,0])$, which we may assume is a subset of $O$ (otherwise replace these by their intersections with $O$), one can construct $χ_0$ and $χ_1$ as above with $\text{supp} \chi_0 \chi_1^2 \chi_2^2 \subset U_1$, and with $\chi_0 \chi_1^2 \chi_2^2$ the negative of the square of a smooth function outside a compact subset of $U_2$. Indeed, one simply chooses $ε > 0$ such that in the coordinates $q_j$,

$$\{q : |q'| < ε, q_{2n} < ε, q_1 \in (σ-ε, ε) \} \subset U_1, \{q : |q'| < ε, q_{2n} < ε, q_1 \in (σ-ε, σ+ε) \} \subset U_2,$$

then one considers $χ_1$ supported in $\{q' : |q'| < ε\}$, identically 1 near 0, $χ_2$ in $\{q_{2n} < ε\}$, identically 1 near 0, and lets with $F > 0$ to be determined later as convenient,

$$\bar{χ}_0(t) = e^{-t^2/(t-ε/2)}, \quad t < ε/2, \quad \bar{χ}_0(t) = 0, \quad t \geq ε/2,$$

and

$$ψ_0(t) \in C^∞(\mathbb{R}), \quad \text{supp} ψ_0 \subset (σ - ε, ∞), \quad \text{supp}(1 - ψ_0) \subset (−∞, σ + ε),$$

and

$$χ_0(t) = \bar{χ}_0(t) ψ_0(t)^2.$$
Thus, \( \psi_0(q_1) \) is constant outside \( \{ q : \ |q'| < \epsilon, \ q_1 \in (\sigma - \epsilon, \sigma + \epsilon) \} \), and \( \tilde{\chi}_0 \leq 0 \), indeed \( \sqrt{-\tilde{\chi}_0} \) is \( C^\infty \). Writing

\[
 b = \sqrt{-\tilde{\chi}_0(q_1)\psi_0(q_1)\chi_2(q_2n)}, \quad e = 2\tilde{\chi}_0(q_1)\psi_0(q_1)\psi'_0(q_1)\chi_1(q')^2\chi_2(q_2n)^2,
\]
we have

\[
 H_{p,m,\ell}a = -b^2 + e
\]
as desired. Now, choosing \( F \) large allows one to deal with additional factors giving weights or regularization, which we discuss below, if we do not choose our boundary defining functions as carefully as we have, as well as \( \tilde{\rho} \). The key point is that one can make \( -\tilde{\chi}' \) dominate \( \tilde{\chi} \) by choosing \( F > 0 \) large. Indeed,

\[
 \tilde{\chi}(t) = -F^{-1}(t - \epsilon/2)^2\tilde{\chi}'(t),
\]
so as long as \( t \) is in a fixed compact set, \( \tilde{\chi} \) is bounded by a small multiple of \( -\tilde{\chi}' \), provided \( F \) is large. Since a strictly positive \( C^\infty \) function has a \( C^\infty \) square root, given \( r \in C^\infty \), one can arrange that

\[
 H_{p,m,\ell}a + ra = -b^2 + e
\]
by taking \( F \) large so that \( 1 - F^{-1}(q_1 - \epsilon/2)^2r > 1/2 \) on \( \text{supp} \, a \). Since \( H_{p,m,\ell} \) is a derivation, so

\[
 H_{p,m,\ell}(pa) = \rho H_{p,m,\ell}a + a H_{p,m,\ell}\rho,
\]
one can add weights \( \langle \zeta \rangle^m \langle z \rangle^\ell \) easily:

\[
 \langle \zeta \rangle^{-m} \langle z \rangle^{-\ell} H_{p,m,\ell}(\langle \zeta \rangle^m \langle z \rangle^\ell a) = H_{p,m,\ell}a + ra
\]
for an appropriate \( r \), which can be handled as above. (If we use the reciprocals of \( q_{2n} \), or \( q_{2n-1} \) and \( q_{2n-2} \) at the corner, for the weights instead of \( \langle \zeta \rangle \) and \( \langle z \rangle \), we do not even get the term \( r \) from these.) Indeed, one only needs \( r \in S^{0,0} \), for then it is bounded, and for sufficiently large \( F > 0 \), \( 1 - F^{-1}(q_1 - \epsilon/2)^2r > 1/2 \) on \( \text{supp} \, a \), and thus

(4) \[
 b = (1 - F^{-1}(q_1 - \epsilon/2)^2r)^{1/2} \sqrt{-\tilde{\chi}_0(q_1)\psi_0(q_1)\chi_2(q_2n)} \in S^{0,0}.
\]

The construction in the second case, with \( \alpha \) in the corner, is completely similar with \( q = (q_2, \ldots, q_{2n-2}) \) now, taking

\[
 a = \chi_0(q_1)\chi_1(q')^2\chi_2(q_{2n-1})^2\chi_2(q_{2n})^2, \quad H_{p,m,\ell}a = \chi'_0(q_1)\chi_1(q')^2\chi_2(q_{2n-1})^2\chi_2(q_{2n})^2.
\]
Then with

\[
 b = \sqrt{-\tilde{\chi}_0(q_1)\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})},
\]
and

\[
 e = 2\tilde{\chi}_0(q_1)\psi_0(q_1)\psi'_0(q_1)\chi_1(q')^2\chi_2(q_{2n-1})^2\chi_2(q_{2n})^2,
\]
we have

\[
 H_{p,m,\ell}a = -b^2 + e.
\]
More generally, with \( r \in S^{0,0} \),

(5) \[
 b = (1 - F^{-1}(q_1 - \epsilon/2)^2r)^{1/2} \sqrt{-\tilde{\chi}_0(q_1)\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})} \in S^{0,0},
\]
gives

\[
 H_{p,m,\ell}a + ra = -b^2 + e.
\]
Note that if \( \sigma > 0 \), a similar construction works, but reversing the signs corresponding to (3). Thus, with \( \gamma \) the integral curve of \( H_{p,m,\ell} \) through \( \alpha \) with \( \gamma(0) = \alpha \), and \( \gamma(\sigma) = \beta \) for some \( \sigma > 0 \) such that \( \gamma([0, \sigma]) \subset O \), and further if we are given a
neighborhood \(U_2\) of \(\beta\) and \(U_1\) of \(\gamma([0, \sigma])\), which we may again assume is a subset of \(O\), one can construct \(\chi_0\) and \(\chi_1\) as above with \(\text{supp} \chi_0 \chi_1 \subset U_1\), and with \(\chi_0 \chi_1\) the square of a smooth function outside a compact subset of \(U_2\). Namely, \(\chi_1\) is unchanged,
\[
\tilde{\chi}_0(t) = e^{-t/(t+\epsilon/2)}, \quad t > -\epsilon/2, \quad \tilde{\chi}_0(t) = 0, \quad t \leq -\epsilon/2,
\]
and
\[
\psi_0(t) \in C^\infty(\mathbb{R}), \quad \text{supp} \psi_0 \subset (-\infty, \sigma + \epsilon), \quad \text{supp}(1 - \psi_0) \subset (\sigma - \epsilon, \infty),
\]
and
\[
\chi_0(t) = \tilde{\chi}_0(t)\psi_0(t)^2.
\]
Thus, as above but with a sign switch, given a \(C^\infty\) function \(r\) on \(\mathbb{R}^n \times \mathbb{R}^n\), one can arrange that
\[
H_{p,m,a} + ra = b^2 - e
\]
Now, returning to \(\sigma < 0\) for the sake of definiteness to deal with \(u\) that is not a priori nice, we regularize the argument. Thus, we replace \(A\) by a bounded family \(A_t, t \in [0, 1], [0, 1] \mapsto \Psi^{m', e'}, \quad \text{continuous as a map } [0, 1] \mapsto \Psi^{m'+\delta, e'+\delta} \text{ for } \delta > 0, \quad \text{such that for } t > 0, \quad A_t \in \Psi^{-m'-K, e-K} \quad \text{(for sufficiently large } K), \quad \text{and such that}
\]
\[
A_0 \in \Psi^{m', e'} \quad \text{is the operator denoted by } A \text{ above. It will be convenient to shift orders, and thus we take } A_{s,r} \in \Psi^{s,r} \text{ to be elliptic, invertible, with principal symbol}
\]
\[
\rho_{s,r} = \langle \zeta \rangle^s (\zeta)^r.
\]
Further, we arrange that as operators in \(\Psi^{m', e'}\), the principal symbols \(a_t\) satisfy for some \(M > 0\)
\[
H_p a_t - \tilde{p} a_t = -b_t^2 - M^2 p_{m-1, \ell-1} a_t + e_t,
\]
\[
(6) \quad b \in L^\infty([0, 1]; S^{n'}), \quad e \in L^\infty([0, 1]; S^{2s, 2r}), \quad \sqrt{a_t} \in L^\infty([0, 1]; S^{m'/2, e'/2});
\]
where, to simplify the notation, we let
\[
s = (m + m' - 1)/2, \quad r = (\ell + \ell' - 1)/2.
\]
In applications we also need some microsupport and ellipticity conditions for the families, so we make the following definition:

**Definition 3.** For a bounded family \(a = \{a_t : t \in [0, 1]\} \in S^{m, \ell}\) and for \(\alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n)\) we say that \(\alpha \notin \text{esssupp}_{L^\infty}(a)\) if \(a\) has a neighborhood \(U \subset \mathbb{R}^n \times \mathbb{R}^n\) such that \(a|_{U \cap (\mathbb{R}^n \times \mathbb{R}^n)}\) is bounded in \(S^{-\infty, -\infty}\) (i.e. each seminorm is bounded).

Further, we say that a bounded family \(a = \{a_t : t \in [0, 1]\} \in S^{m, \ell}\) is elliptic at \(\alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n)\) if \(a\) has a neighborhood \(U \subset \mathbb{R}^n \times \mathbb{R}^n\) such that for some \(c > 0\),
\[
|a_t|_{U \cap (\mathbb{R}^n \times \mathbb{R}^n)} \geq c(z)^{\sigma} \langle \zeta \rangle^\ell.
\]
We then have the following lemma:

**Lemma 0.6.** For \(\alpha_0 \in \partial(\mathbb{R}^n \times \mathbb{R}^n)\) with \(H_{p,m, \ell} \text{ non-vanishing at } \alpha_0\). Then \(\alpha_0\) has a neighborhood \(O \subset \partial(\mathbb{R}^n \times \mathbb{R}^n)\) on which \(H_{p,m, \ell} \text{ is non-vanishing, and if } \alpha \in O, \quad \gamma \text{ the integral curve of } H_{p,m, \ell}\) through \(\alpha\) with \(\gamma(0) = \alpha\), and \(\gamma(\sigma) = \beta\) for some \(\sigma < 0\) such that \(\gamma([\sigma, 0]) \subset O\), and further if we are given a neighborhood \(U_2\) of \(\beta\) and \(U_1\) of \(\gamma([\sigma, 0])\) contained in \(O\), then there exists \(a_t, b_t\) and \(e_t\) as in (6) such that
\[
\text{esssupp}_{L^\infty} a_t, b_t \subset U_1, \quad \text{esssupp}_{L^\infty} e_t \subset U_2,
\]
and \(b\) is elliptic on \(\gamma([\sigma, 0])\).
Proof. We assume that $\alpha_0$ is in the corner $\partial V \times \partial W$; otherwise one of the weights is irrelevant, and the corresponding coordinate can be taken one of the $q'$ coordinates with no significant changes.

We first define $\chi_0, \chi_1, \chi_2$ as above and let
$$
\tilde{a} = \langle \zeta \rangle^{m'+\ell} \chi_0(q_1) \chi_1(q') \chi_2(q_{2n-1})^2 \chi_2(q_{2n})^2.
$$
For $t \in [0, 1]$, let
$$
\phi_t(\tau) = (1 + t\tau)^{-K/2}, \quad \tau \geq 0,
$$
and note that
$$
d\phi_t(\tau) = f_t(\tau)\phi_t(\tau), \quad f_t(\tau) = t(-K/2)(1 + t\tau)^{-1},
$$
with $|f_t(\tau)| \leq K/2$, $|\tau f_t(\tau)| \leq K/2$. We then let
$$
a_t = \phi_t(|\zeta|^2)\phi_t(|z|^2)\tilde{a}.
$$
Note that $a \in L^\infty([0, 1]; S^{m', \ell'})$ satisfies $a_t = S^{m-K, \ell'-K}$ for $t > 0$, $\operatorname{esssupp}_L(a) \subset U_1$, and $a$ is elliptic on $\gamma([\sigma, 0])$. Further, $\sqrt{a} \in L^\infty([0, 1]; S^{m'/2, \ell'/2})$. As
$$
H_{p,m,\ell}(\phi_t(|\zeta|^2)\phi_t(|z|^2))
\begin{align*}
= & \langle \zeta \rangle^{-2-m+1} H_p(|\zeta|^2) \langle \zeta \rangle^{m-1} f_t(|\zeta|^2) + \langle \zeta \rangle^{-m+1} (\zeta) - 2H_p(|\zeta|^2)(\zeta)^2 f_t(|\zeta|^2) \\
& \phi_t(|\zeta|^2)\phi_t(|z|^2),
\end{align*}
$$
and the quantity inside the large parentheses is bounded in $S^{0,0}$. We then note that
$$
\begin{align*}
& \langle z \rangle^{m'-m+1} H_p a_t \\
= & \phi_t(|\zeta|^2)\phi_t(|z|^2)(\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})^2 \chi_{2}(q_{2n})^2 \chi_{0}(q_1) \\
+ & 2\phi_t(|\zeta|^2)\phi_t(|z|^2)\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})^2 \chi_{0}(q_1) \\
+ & \tilde{r}\phi_t(|\zeta|^2)\phi_t(|z|^2)\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})^2 \chi_{0}(q_1),
\end{align*}
$$
where $\tilde{r} \in L^\infty([0, 1]; S^{0,0})$. Proceeding as above, we note that for $t > 0$ sufficiently large
$$
\begin{align*}
& \langle z \rangle^{m'-m+1} \left( H_p a_t + \tilde{p} a_t \right) - M^2 \rho_{m-1, \ell-1} a_t \\
= & \phi_t(|\zeta|^2)\phi_t(|z|^2)(\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})^2 \chi_{0}(q_1) + r\chi_0(q_1)) \\
+ & 2\phi_t(|\zeta|^2)\phi_t(|z|^2)\psi_0(q_1)\chi_1(q')\chi_2(q_{2n-1})\chi_2(q_{2n})^2 \chi_{0}(q_1) \\
= & \langle z \rangle^{m'-m} \left( - b_t^2 + e_t \right),
\end{align*}
$$
with the desired properties. \hfill \Box

Let $\tilde{a}_t = \sqrt{a_t}$, and let $\tilde{A}_t = \frac{1}{2}(g_L(\tilde{a}_t) + q_L(\tilde{a}_t)^*)$, which is thus formally self-adjoint. Let $A_t = \tilde{A}_t^2$. Then for $t > 0$
$$
\langle Pu, A_t u \rangle - \langle A_t u, Pu \rangle = \|B_t u\|^2 + M^2 \|A_{(m-1)/2, (\ell-1)/2} A_t u\|^2 - \langle E_t u, u \rangle + \langle F_t u, u \rangle,
$$
where \( F \in L^{\infty}([0, 1]; \Psi^{2s-1,2r-1}) \). We write, with \( \delta > 0 \),
\[
\| \langle Pu, A_t u \rangle \| = \| \langle (\Lambda_{(m-1)/2,(\ell-1)/2})^* \mathcal{A}_t Pu, (\Lambda_{(m-1)/2,(\ell-1)/2})^* \mathcal{A}_t u \rangle \|
\leq \| (\Lambda_{(m-1)/2,(\ell-1)/2})^* \mathcal{A}_t Pu \| \| \Lambda_{(m-1)/2,(\ell-1)/2} \mathcal{A}_t u \|
\leq \frac{1}{2\delta} \| (\Lambda_{(m-1)/2,(\ell-1)/2})^* \mathcal{A}_t Pu \|^2 + \frac{\delta}{2} \| (\Lambda_{(m-1)/2,(\ell-1)/2})^* \mathcal{A}_t u \|^2
\]
One has a similar bound for \( \langle A_t u, Pu \rangle \). For \( \delta > 0 \) sufficiently small (namely \( \delta < M^2 \)), \( \delta \| \Lambda_{(m-1)/2,(\ell-1)/2} \mathcal{A}_t u \|^2 \) can be absorbed into \( M^2 \| \Lambda_{(m-1)/2,(\ell-1)/2} \mathcal{A}_t u \|^2 \), and thus we deduce that
\[
\| B_t u \|^2 \leq \| \langle E_t u, u \rangle \| + \| \langle F_t u, u \rangle \| + \delta^{-1} \| (\Lambda_{(m-1)/2,(\ell-1)/2})^* \mathcal{A}_t Pu \|^2 + C \| u \|^2_{H^{m,N}}.
\]
Now we let \( t \to 0 \); then assuming a priori control on the terms other than \( \| B_t u \|^2 \) we conclude that \( B_t u \) is bounded in \( L^2 \). But \( B_t u \to B_0 u \) in \( S' \), using the weak compactness of the unit ball in \( L^2 \), and thus that \( B_t u \) has a subsequence \( B_{t_j} u \) converging \( L^2 \)-weakly to some \( v \in L^2 \), and thus in \( S' \), we deduce that \( B_0 u = v \in L^2 \), yielding the desired regularity information.

In order to make the a priori control assumptions, we need a uniform version of the operator wave front set:

**Definition 4.** For a bounded family \( \mathcal{A} = \{ A_t : t \in [0, 1] \} \) in \( \Psi^{m,t} \) and for \( \alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^{\nu}) \) we say that \( \alpha \notin \WF'_{L^\infty}(\mathcal{A}) \) if \( \alpha \) has a neighborhood \( U \) in \( \mathbb{R}^n \times \mathbb{R}^{\nu} \) such that \( A_t = q_L(a_t) \) and \( a_t U \cap (\mathbb{R}^n \times \mathbb{R}^{\nu}) \) is bounded in \( S^{-\infty,-\infty} \) (i.e. each seminorm is bounded).

Note that if \( A_t = A_0 \) for all \( t \), then \( \WF'_{L^\infty}(\mathcal{A}) = \WF'(\mathcal{A}) \), i.e. this family wave front set is an appropriate generalization of the standard operator wave front set.

**Lemma 0.7.** If \( \mathcal{A} \) and \( \mathcal{B} \) are bounded families, then with \( \mathcal{A}\mathcal{B} = \{ A_t B_t : t \in [0, 1] \} \),
\[
\WF'_{L^\infty}(\mathcal{A}\mathcal{B}) \subset \WF'_{L^\infty}(\mathcal{A}) \cap \WF'_{L^\infty}(\mathcal{B}).
\]
In particular, if \( Q \in \Psi^{m,t} \) then
\[
\WF'_{L^\infty}(Q,\mathcal{A}), \WF'_{L^\infty}(\mathcal{A}Q) \subset \WF'(Q) \cap \WF'_{L^\infty}(\mathcal{A}).
\]

Further, \( \alpha \notin \WF'_{L^\infty}(\mathcal{A}) \) if and only if there exists \( Q \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( QA \) is bounded in \( \Psi^{-\infty,-\infty} \).

**Proof.** The composition properties are automatic from the description of the product as an asymptotic sum. In particular, if \( \alpha \notin \WF'_{L^\infty}(\mathcal{A}) \) and \( Q \) is elliptic at \( \alpha \) with \( \WF'(Q) \cap \WF'_{L^\infty}(\mathcal{A}) = \emptyset \), then \( QA \) is bounded in \( \Psi^{-\infty,-\infty} \). Now, such a \( Q \) exists since \( \WF'_{L^\infty}(\mathcal{A}) \) is closed, and \( Q \) is in its complement.

Conversely, if a \( Q \) as stated exists, let \( B \) be a microlocal parametrix for \( Q \), so \( BQ = \text{Id} + E \) with \( \alpha \notin \WF'(E) \). Then \( A_t = BQA_t - EA_t \), with \( \alpha \notin \WF'_{L^\infty}(EA_t) \) since \( \alpha \notin \WF'(E) \), while \( QA \) bounded in \( \Psi^{-\infty,-\infty} \) implies that \( BQA \) is also bounded there, thus \( \alpha \notin \WF'_{L^\infty}(\mathcal{A}) \). 

In summary, we have the following result, which is the basic microlocal propagation estimate, propagating control on \( \WF'_{L^\infty}(E) \) to \( \WF'(B_0) \):

**Lemma 0.8.** Suppose that (6) is satisfied. Let \( A_t = q_L(a_t), B_t = q_L(b_t), E_t = q_L(e_t) \), and let \( Q_1, Q_2 \in \Psi^{0,0} \) such that \( Q_1 \) is elliptic on \( \WF'_{L^\infty}(\mathcal{A}) \), \( Q_2 \) is elliptic
on \( \WF'_{L} \( E) \). If \( Q_{1}u \in H^{s-1/2,r-1/2}, \) \( Q_{1}Pu \in H^{s-m+1,r-\ell+1}, \) \( Q_{2}u \in H^{s,r} \) then \( B_{0}u \in L^{2} \) and for all \( M, N \) there is \( C > 0 \) such that
\[
\|B_{0}u\|_{L^{2}} \leq C(\|Q_{2}u\|_{H^{s,r}} + \|Q_{1}Pu\|_{H^{s-m+1,r-\ell+1}} + \|Q_{1}u\|_{H^{s-1/2,r-1/2}} + \|u\|_{H^{M,N}}).
\]

Proof. We note that as \( Q_{1} \) is elliptic on \( \WF'_{L} \( A) \), hence on \( \WF'_{L} \( F) \),
\[
\|F_{1}u\|_{H^{-s+1/2,-r+1/2}} \leq C(\|Q_{1}u\|_{H^{-s+1/2,-r+1/2}} + \|u\|_{H^{M,N}}),
\]
and thus with \( \tilde{Q}_{1} \in \Psi^{0,0} \) with \( \WF'(\Id - \tilde{Q}_{1}) \cap \WF'_{L} \( A) = \emptyset, \) \( Q_{1} \) elliptic on \( \WF'(Q_{1}) \), we have \( (\Id - \tilde{Q}_{1})F_{1} \) is bounded in \( \Psi^{-\infty,\infty} \) and also
\[
\|\tilde{Q}_{1}u\|_{H^{-s+1/2,-r+1/2}} \leq C(\|Q_{1}u\|_{H^{-s+1/2,-r+1/2}} + \|u\|_{H^{M,N}}),
\]

then
\[
\langle F_{1}u, u \rangle \leq \langle (\Id - \tilde{Q}_{1})F_{1}u, u \rangle + \langle F_{1}u, \tilde{Q}_{1}u \rangle \leq C'(\|u\|_{H^{M,N}} + \|Q_{1}u\|_{H^{-s+1/2,-r+1/2}}').
\]

Combining these, we see that the right hand side of \( 9 \) remains uniformly bounded, and thus \( B_{0}u \in L^{2} \) as claimed. \( \square \)

As a corollary:

**Proposition 0.9.** For \( \alpha_{0} \in \partial(\R^{m} \times \R^{m}) \) with \( \H_{p,m,\ell} \) non-vanishing at \( \alpha_{0} \). Then \( \alpha_{0} \) has a neighborhood \( O \) in \( \partial(\R^{m} \times \R^{m}) \) on which \( \H_{p,m,\ell} \) is non-vanishing, and if \( \alpha \in O, \) \( \gamma \) the integral curve of \( \H_{p,m,\ell} \) through \( \alpha \) with \( \gamma(0) = \alpha, \) and \( \gamma(\sigma) = \beta \) for some \( \sigma < 0 \) such that \( \gamma([\sigma,0]) \subseteq O, \) and further if we are given a neighborhood \( U_{2} \) of \( \beta \) and \( U_{1} \) of \( \gamma([\sigma,0]) \) contained in \( O, \) \( Q_{1}, Q_{2} \in \Psi^{0,0} \) such that \( Q_{1} \) is elliptic on \( U_{1}, Q_{2} \) is elliptic on \( U_{2} \), then there exists \( Q_{3} \in \Psi^{0,0} \) elliptic on \( \gamma([\sigma,0]) \) such that the following holds. If \( Q_{1}u \in H^{s-1/2,r-1/2}, Q_{1}Pu \in H^{s-m+1,r-\ell+1}, Q_{2}u \in H^{s,r} \) then \( Q_{3}u \in H^{s,r} \) and for all \( M, N \) there is \( C > 0 \) such that
\[
\|Q_{3}u\|_{H^{s,r}} \leq C(\|Q_{2}u\|_{H^{s,r}} + \|Q_{1}Pu\|_{H^{s-m+1,r-\ell+1}} + \|Q_{1}u\|_{H^{s-1/2,r-1/2}} + \|u\|_{H^{M,N}}).
\]

The analogous result also holds for \( \sigma > 0 \).

Now propagation of singularities is an immediate consequence. Indeed, one can iterate, improving half an order at a time, taking \( U_{1}' = \Ell(Q_{3}) \cap U_{1} \) and \( U_{2}' = \Ell(Q_{3}) \cap U_{2} \) for the next iteration. This directly proves Theorem 0.4 when the bicharacteristic segment is contained in a single set \( O; \) in general a compactness argument proves it in a finite number of steps from this local version.

Although we required that \( P \in \Psi^{m,\ell} \) have real principal symbol, this is not quite necessary. Suppose instead that one is given a complex valued homogeneous function, \( p - iq \) with \( p, q \) real-valued. Let \( P \in \Psi^{m,\ell}, \) resp. \( Q \in \Psi^{m,\ell} \) have (real) principal symbols \( p, \) resp. \( Q. \) Note that when \( q \neq 0, p - iq \) is elliptic, while if \( q \) vanishes near a point \( \alpha, \) then the previous microlocal estimate works. Thus, the key question is what happens at the characteristic set \{ \( p = 0, q = 0 \} \) at points in \( \text{supp } q. \) The typical application here would involve a \( q \) that acts as microlocal ‘absorption’ along the Hamilton flow; absorbing any singularity propagating along
\( H_{p,m,\ell} \) in \( \{ p = 0 \} \) away from \( \text{supp} \, u \). Thus, one should consider \( q \) a bump function along the \( H_{p,m,\ell} \)-bicharacteristics, whose sign will be very important; along a fixed \( H_{p,m,\ell} \)-bicharacteristic \( \gamma \) in \( \{ p = 0 \} \), where \( q \neq 0 \), the problem is elliptic, and thus if \( (P - iQ)u \) is regular, so is \( u \), where \( q \equiv 0 \), singularities propagate, so for a bump-function like \( q \), the singularities are absorbed at the boundary of \( \text{supp} \, q \). Since \( q \) is mostly a technical tool, and thus it is reasonable to make technically convenient assumptions on \( Q \) below.

Then for \( A \) formally self-adjoint,

\[
\langle i(P - iQ)u, Au \rangle - \langle iAu, (P - iQ)u \rangle = \langle [A, P] + (P - P^*)A, u, u \rangle + \langle (AQ + Q^*A)u, u \rangle,
\]

\( i([A, P] + (P - P^*)A) \in \Psi^{m+m' - 1, \ell + \ell' - 1}, \quad AQ + Q^*A \in \Psi^{m+m', \ell + \ell'} \).

Now, if the principal symbol \( a \) of \( A \) is \( \geq 0 \) as above, with the principal symbol of \( i([A, P] + (P - P^*)A) \) being

\[-(H_p a - \tilde{p}a) = b^2 - c,
\]

then for \( q \geq 0 \), the principal symbol of the second term on the right hand side of (10) has the same sign as that of the first. However, there is an issue with such an argument since the second term is higher order than the first one. Thus, it is convenient to assume that \( Q = T^2 \), with \( T \in \Psi^{m/2, l/2}, \quad T = T^* \). Note that at the cost of changing \( P \) without changing its principal symbol, if \( q = t^2 \) for a non-negative symbol \( t \), then this may always be arranged; simply take \( T \in \Psi^{m/2, l/2} \) with principal symbol \( t \) and with \( T = T^* \); then \( Q - T^2 \in \Psi^{m + m' - 1, \ell + \ell' - 1} \), so replacing \( P \) by \( P - i(Q - T^2) \), and \( Q \) by \( T^2 \), we have the desired form of \( Q \). From now on we assume that

\[ Q = T^2, \quad T \in \Psi^{m/2, l/2}, \quad T = T^*, \]

When \( a = \tilde{a}^2 \) with \( \tilde{a} \in S'^{m'/2, \ell'/2} \), as is arranged above, then with \( \tilde{A} \in \Psi^{m'/2, \ell'/2} \) of principal symbol \( \tilde{a} \) and \( \tilde{A}^* = \tilde{A} \), let \( A = \tilde{A}^2 \) (so \( A^* = A \)) – we are simply a bit more careful in our specification of \( A \). Then

\[ AQ + Q^*A = \tilde{A}^2 Q + Q \tilde{A}^2 = 2 \tilde{A} Q \tilde{A} + [\tilde{A}, Q] \tilde{A} = 2 \tilde{A} T^2 \tilde{A} + [\tilde{A}, [\tilde{A}, Q]], \]

and \( [\tilde{A}, [\tilde{A}, Q]] \in \Psi^{m + m' - 2, \ell + \ell' - 2} \), which is thus the same order as \( F \) above, so is controlled by the a priori assumptions. On the other hand, \( \langle AT^2 \tilde{A} u, u \rangle = \| T \tilde{A} u \|^2 \), so we obtain that

\[
\langle i(P - iQ)u, Au \rangle - \langle iAu, (P - iQ)u \rangle = \| Bu \|^2 - \langle Eu, u \rangle + \langle (F + [\tilde{A}, [\tilde{A}, Q]])u, u \rangle + 2 \| T \tilde{A} u \|^2.
\]

Regularizing \( \tilde{A} \) as \( \hat{A} \), all the previous arguments go through. Note also that if instead \( \sigma > 0 \), i.e. we have

\[-(H_p a - \tilde{p}a) = -b^2 + c,
\]

then we need to change our requirements on \( Q \), namely we need \( q \leq 0 \), and for the technically convenient setting we need \( q = -t^2, \quad t \geq 0 \).

**Proposition 0.10.** For \( \alpha_0 \in \partial(\mathbb{R}^n \times \mathbb{R}^n) \) with \( H_{p,m,\ell} \) non-vanishing at \( \alpha_0 \). Suppose also that \( Q = T^2 \) as above. Then \( \alpha_0 \) has a neighborhood \( O \) in \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \) on which \( H_{p,m,\ell} \) is non-vanishing, and if \( \alpha \in O \), \( \gamma \) the integral curve of \( H_{p,m,\ell} \) through \( \alpha \) with \( \gamma(0) = \alpha \), and \( \gamma(\sigma) = \beta \) for some \( \sigma < 0 \) such that \( \gamma([\sigma, 0]) \subset O \), and
further if we are given a neighborhood $U_2$ of $\beta$ and $U_1$ of $\gamma([\sigma, 0])$ contained in $O$, $Q_1, Q_2 \in \Psi^{0,0}$ such that $Q_1$ is elliptic on $U_1$, $Q_2$ is elliptic on $U_2$, then there exists $Q_3 \in \Psi^{0,0}$ elliptic on $\gamma([\sigma, 0])$ such that the following holds. If $Q_1 u \in H^{s-1/2, r-1/2}$, $Q_1(P - iq)u \in H^{s-\frac{m+1}{2}, r-\frac{\ell+1}{2}}$, $Q_2 u \in H^{s,r}$ then $Q_3 u \in H^{s,r}$ and for all $M, N$ there is $C > 0$ such that
\[
\|Q_3 u\|_{H^{s,r}} \leq C(\|Q_2 u\|_{H^{s,r}} + \|Q_1(P - iq)u\|_{H^{s-\frac{m+1}{2}, r-\frac{\ell+1}{2}}} + \|Q_1 u\|_{H^{s-1/2, r-1/2}} + \|u\|_{H^{M,N}}).
\]

The analogous result also holds for $\sigma > 0$, provided $Q = -T^2$.

Thus, for $q \geq 0$, estimates propagate forward along the Hamilton flow; for $q \leq 0$, they propagate backwards. This means that singularities, as measured by the wave front set, propagate in the opposite direction: if $q \geq 0$, and there is WF at $\alpha$, then for $\sigma < 0$, there is also WF at $\gamma(\sigma)$, for otherwise our estimate (and the corresponding regularity statement) would give the absence of WF at $\alpha$.

While we used $Q = T^2$ here, this was for a purely technical point. If $Q = Q^*$ (which may always be arranged at the cost of changing $P$ without changing its principal symbol, namely replacing $P$ by $P - i(Q - Q^*)/2$, and replacing $Q$ by $(Q + Q^*)/2$ and $Q \geq 0$, then $(AQ\hat{A}u, \hat{A}u) \geq (Q\hat{A}u, \hat{A}u) \geq 0$ still. In general, just because $q \geq 0$ and $Q = Q^*$, we do not have $Q \geq 0$, but by the sharp Gårding inequality, this holds modulo a one order lower error term, i.e. $(Qv, v) \geq -C\|v\|^2_{H^{(m-1)/2, (\ell-1)/2}}$. Applying this with $v = \hat{A}u$, we obtain a term from the right hand side that is controlled by $C\|\Lambda_{(m-1)/2, (\ell-1)/2}\hat{A}u\|^2$, which can be controlled as above by choosing $T > 0$ large to absorb this into $\|Bu\|^2$.

This gives rise to the simplest non-elliptic Fredholm problem. So suppose that $P \in \Psi^{m, \ell}$ with real homogeneous principal symbol. Suppose also that $Q \in \Psi^{m, \ell}$ is such that its principal symbol $q$ satisfies $q = t^2$ as above, and that for all $\alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^\ell)$ in $\Sigma(p) = \{p = 0\}$ the integral curve $\gamma$ of $H_p$ with $\gamma(0) = \alpha$ reaches $\{q > 0\}$ in finite time in both the forward and backward direction, i.e. there exist $T_\pm > 0$ such that $\gamma(T_\pm)) > 0$. Let
\[
\mathcal{X}^{s,r} = \{u \in H^{s,r} : (P - iq)u \in H^{s-m+1, r-\ell+1}\}, \quad \mathcal{Y}^{s,r} = H^{s,r}.
\]

Then
\[
P - iq : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-m+1, r-\ell+1}, \quad P^* + iq^* : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-m+1, r-\ell+1}
\]
are Fredholm for all $s, r$. Note that this follows estimates of the kind
\[
\|u\|_{H^{s,r}} \leq C(\|(P - iq)u\|_{H^{s-m+1, r-\ell+1}} + \|u\|_{H^{M,N}}),
\]
\[
\|u\|_{H^{s,r}} \leq C(\|(P^* + iq^*)u\|_{H^{s-m+1, r-\ell+1}} + \|u\|_{H^{M,N}}),
\]
for sufficiently negative $M, N$, which in turn follow from the propagation of singularities result as discussed beforehand in the real principal symbol setting. Thus, we just need to check that if $u \in S'$ and $(P - iq)u \in H^{s-m+1, r-\ell+1}$ then $u \in H^{s,r}$, i.e. $WF^{s,r}(u) = \emptyset$. But this is straightforward: if either $p$ or $q$ is elliptic at $\alpha$, then $\alpha \notin WF^{s+1, r+1}(u)$ and thus $\alpha \notin WF^{s,r}(u)$ by microlocal elliptic regularity, and otherwise $p$ vanishes at $\alpha$, and thus the backward bicharacteristic $\gamma$ from $\alpha$ reaches $q > 0$, where we know there is no $WF^{s,r}(u)$, so propagation of singularities gives that $\alpha \notin WF^{s,r}(u)$. A similar argument works for the adjoint, using the forward bicharacteristic. This proves the claim.

The final ingredient for scattering theory is radial points. These are the points in the conic perspective where $H_p$ is a multiple of the generator of dilations, and
from the compactification point of view the points \( \alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n) \) where \( H_{p,m,\ell} = \langle z \rangle^{-m+1/2} H_p \), as a smooth vector field on \( \mathbb{R}^n \times \mathbb{R}^n \), vanishes. At such points the argument given above breaks down, since there are no local coordinates on the boundary in which this rescaled Hamilton vector field would be a coordinate vector field. Also, at such points, one cannot use the derivative of a function on the boundary to dominate terms when \( H_p \) falls on weights or regularizers, with the result that the weights are required to produce the correct signs for the commutator argument.

In fact, it is convenient to proceed more generally, using a generalization of this setting to a submanifold \( L \) of \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \) closed under the \( H_{p,m,\ell} \) flow, i.e. \( H_{p,m,\ell} \) is tangent to it, and which acts as a source or sink for the flow in a neighborhood of \( L \). Since we have corners here, we need to be more specific, and we require that \( L \) is a submanifold of one of the two boundary hypersurfaces which is transversal to the other boundary face (allowing of course that \( L \) does not intersect the other boundary face at all). For the sake of definiteness, we assume that \( L \) is a subset of \( \mathbb{R}^n \times \partial \mathbb{R}^n \), i.e. is at fiber infinity, and is transversal to \( \partial \mathbb{R}^n \times \partial \mathbb{R}^n \), possibly via an empty intersection. Next, suppose that \( L \) is defined by \( \{ \rho_{1,j} = 0 : \ j = 1, \ldots, k \} \) in \( \mathbb{R}^n \times \partial \mathbb{R}^n \), i.e. the \( \rho_{1,j} \) have linearly independent differentials on their joint zero set. Let

\[
\rho_1 = \sum_{j=1}^k \rho_{1,j}^2,
\]

which is thus a quadratic defining function of \( L \), i.e. it vanishes there quadratically in a non-degenerate manner. Since \( H_{p,m,\ell} \) is tangent to \( L \), \( H_{p,m,\ell} \rho_{1,j} \) vanishes at \( L \), i.e. is a linear combination of the \( \rho_{1,i} \) with \( C^\infty \) coefficients, so \( H_{p,m,\ell} \rho_1 \) vanishes quadratically at \( L \). The source(+)sink(−) assumption is that there is a function \( \beta_1 > 0 \) such that

\[
\pm H_{p,m,\ell} \rho_1 = \beta_1 \rho_1 + F_2 + F_3,
\]

where \( F_2 \geq 0 \) and \( F_3 \) vanishes cubically at \( L \). Under this assumption, \( L \) is a source or sink, since \( \rho_1 \) is an increasing (+), resp. decreasing (−), function near \( L \), as \( |F_2| \leq C \rho_1^{3/2} \leq \beta_1 \rho_1/2 \) in a neighborhood of \( L \). In particular, notice that if \( \phi \in C^\infty_c([0, \infty)) \) is such that \( \phi \equiv 1 \) near 0, \( \phi \geq 0 \) with \( \sqrt{\phi} \) smooth and \( \phi' \leq 0 \) with \( \sqrt{-\phi'} \) smooth, with support sufficiently close to 0 then

\[
\pm H_{p,m,\ell} \phi(\rho_1) = (-\phi'(\rho_1))(\beta_1 \rho_1 + F_2 + F_3) \geq 0,
\]

and further the second factor on the right hand side is \( > 0 \) on supp \( \phi' \), so indeed

\[
\phi_1 = \sqrt{\pm H_{p,m,\ell} \phi(\rho_1)}
\]

is smooth, and vanishes near \( L \).

To explain this source/sink condition note that if \( H_{p,m,\ell} \) actually vanishes at \( L \), then one can consider its linearization at each point \( \alpha \in L \), i.e. \( H_{p,m,\ell} \) then maps the ideal \( \mathcal{I} \) of functions vanishing at \( \alpha \) to itself, and thus \( \mathcal{I}^2 \) to itself, and thus acts on \( \mathcal{I}/\mathcal{I}^2 \), which can be identified with the cotangent space at \( \alpha \). Further, the conormal space of \( L \) at \( \alpha \) is preserved by \( H_{p,m,\ell} \) as \( H_{p,m,\ell} \rho_{1,j} \) vanishes at \( L \), i.e. \( H_{p,m,\ell} \) acts on this finite dimensional vector space. If the \( \rho_{1,j} \) are such that \( d\rho_{1,j} \) is an eigenvector with eigenvalue \( \beta_{1,j}(\alpha) \) at each \( \alpha \in L \), then \( H_{p,m,\ell} \rho_1 \) is given by

\[
\sum_{j=1}^k 2 \beta_{1,j} \rho_{1,j}^2 \mod \text{cubically vanishing functions},
\]

and thus if all \( \beta_{1,j} \) have the same non-zero sign, \( \pm \), then letting \( \beta_1 \) to be \( 2 \min_j |\beta_{1,j}| \), the above condition is
satisfied (with ± corresponding to the sign ± here). In fact, in applications often $H_{p,m,ℓ}$ is a multiple of the identity operator on this finite dimensional vector space, and then all the $β_{1,j}$ are the same, and any defining functions $ρ_{1,j}$ can be used to construct $ρ_1$.

At the possible radial points in $L$, the only way one can have a positive commutator estimate is by taking into account the weights. Consider $H_{p,m,ℓ}$ as a vector field on $\mathbb{R}^n \times \mathbb{R}^m$ which is tangent to the boundary of this compact space, and recall that $ρ_{\text{base}} = (z)^{-1}$ and $ρ_{\text{fiber}} = (ζ)^{-1}$ are defining functions of the boundary hypersurfaces. In particular, as already mentioned, $H_{p,m,ℓ}ρ_{\text{base}} = a_{\text{base}}ρ_{\text{base}}$ and $H_{p,m,ℓ}ρ_{\text{fiber}} = a_{\text{fiber}}ρ_{\text{fiber}}$ with $a_{\text{base}}, a_{\text{fiber}}$ well defined on $∂(\mathbb{R}^n \times \mathbb{R}^m)$ (though they depend on the choice of $ρ_{\text{base}}$ and $ρ_{\text{fiber}}$). Moreover, if $H_{p,m,ℓ}$ actually vanishes at $α$, which is, say, at fiber infinity, then

$$H_{p,m,ℓ}(fρ_{\text{fiber}}) = (H_{p,m,ℓ}f)ρ_{\text{fiber}} + f(H_{p,m,ℓ}ρ_{\text{fiber}}) = (a_{\text{fiber}}f + H_{p,m,ℓ}f)ρ_{\text{fiber}},$$

with $(a_{\text{fiber}}f + H_{p,m,ℓ}f) = a_{\text{fiber}}(α)f(α)$, shows that $a_{\text{fiber}}$ is actually well-defined at such $α$, independent even of the choice of the boundary defining function. We then require that $L$ is a source or sink even taking into account the infinitesimal ‘jet dynamics’ at the boundary, i.e. $a_{\text{fiber}}$ in our case (since we are working at fiber infinity) has the same sign as $H_{p,m,ℓ}ρ_0$ of $ρ_0$ at $(z, ζ)$ such that

$$\mp H_{p,m,ℓ}ρ_0 = β_0ρ_0, \quad β_0|_L > 0.$$

By the remarks above, if $L$ consists of radial points, then one can simply take our preferred boundary defining function, $ρ_0 = ρ_{\text{fiber}}$. Note that $β_0$ is thus bounded below by a positive constant in a neighborhood of $L$; we can always restrict work to such a neighborhood below.

If $L$ intersects the corner (which the reader may ignore at first reading), we also need to take care of weights in terms of $ρ_{\text{base}}$, and we assume that we have a defining function (a positive smooth multiple of $ρ_{\text{base}}$) $ρ_2$ such that

$$H_{p,m,ℓ}ρ_2 = 2β_2β_0ρ_2, \quad β_2|_L = 0.$$

Further, as before, $P - P^* ∈ Ψ^{m-1, ℓ-1}$ also plays a role in these arguments, and at radial points it cannot be absorbed into other terms, just like the weight, i.e. powers of $ρ_0$, could not be thus absorbed. We normalize this this and write that

$$-\tilde{ρ}/2 = σ_{m-1, ℓ-1}(1/21(P - P^*) = ±β_0\tilde{β}ρ_0^{m-1}.$$

Then we compute that with

$$a = φ(ρ_1)^2ρ_0^{-m}ρ_0^{-ℓ}$$

the principal symbol $-(Hρ - ρa)$ of $i[A, P] + (P - P^*)A$ is

$$\mp ρ_0^{-m' - m + 1}ρ_2^{-ℓ' - ℓ + 1}(φ(ρ_1)^2 + β_0m'φ(ρ_1)^2 + 2β_2β_0ℓ'φ(ρ_1)^2 + 2β_0\tilde{β}φ(ρ_1)^2).$$

Of the terms in the parentheses, the last three have essential support at $L$ itself, and they have a definite sign if $m' + 2β_2ℓ' + 2\tilde{β}$ has a definite sign. With $s = (m + m' - 1)/2$, $r = (ℓ + ℓ' - 1)/2$, this means that

$$s - (m - 1)/2 + β_2(r - (ℓ - 1)/2 + \tilde{β}$$

needs to have a definite sign when we prove $H^{s,r}$ regularity. Further, the first term in the parentheses has the same sign as these when the sign of $s - (m - 1)/2 + β_2 + \tilde{β}$ is negative. Thus, when $s - (m - 1)/2 + β_2(r - (ℓ - 1)/2 + \tilde{β} > 0$, then all signs agree, and
we have a result that has a different character from the standard propagation result in so far as not having to assume a priori \( H^s \) regularity anywhere to conclude \( H^s \) regularity near \( L \). On the other hand, when \( s - (m - 1)/2 + \beta > 0 \) we must assume regularity on \( \text{supp} \phi \), i.e. in a punctured neighborhood of \( L \), in order to conclude \( H^s \) regularity at \( L \). In summary, using that we can make \( \phi \) have sufficiently small support so that the definite sign of this expression just as \( L \) implies the analogous conclusion on \( \text{supp} \phi \), modulo justifying the calculations via a regularization argument, we have

**Proposition 0.11.** Suppose that \( L \) is as above, and \( s - (m - 1)/2 + \beta > 0 \) on \( L \), and \( s' - (m - 1)/2 + \tilde{\beta} > 0 \) on \( L \), \( s' \in [s-1/2, s) \). Then there exist \( Q \in \Psi^{0,0} \) elliptic on \( L \) such that for \( \phi \), \( L \) elliptic on \( \text{WF}'(Q) \), if \( Q_\phi \in H^{s', r-1/2} \), \( Q_\phi \in H^{s-m+1, r-\ell+1}, \) then \( Q_\phi \in H^{s, r} \) and for all \( M, N \) there is \( C > 0 \) such that

\[
\|Q\phi\|_{H^{s, r}} \leq C(\|Q_\phi\|_{H^{s-m+1, r-\ell+1}} + \|Q_\phi\|_{H^{s', r-1/2}} + \|\phi\|_{H^{s, r}}).
\]

On the other hand, suppose now that \( s - (m - 1)/2 + \beta < 0 \) on \( L \). Suppose also that we are given neighborhoods \( U_1, U_2 \) of \( L \) with \( \overline{U_1} \subseteq U_2 \), and \( Q_1, Q_2 \in \Psi^{0,0} \) with \( \text{WF}'(Q_2) \subseteq U_2 \setminus \overline{U_1} \), \( \text{WF}'(Q_1) \subseteq U_2 \), and such that for every \( \alpha \in (\text{Char}(p) \cap U_2) \setminus L \), the bicharacteristic \( \gamma \) of \( p \) with \( \gamma(0) = \alpha \) enters \( \text{Ell}(Q_2) \) while remaining in \( \text{Ell}(Q_1) \). Then there exist \( Q \in \Psi^{0,0} \) elliptic on \( L \) such that if \( Q_\phi \in H^{s', r}, Q_\phi \in H^{s-1/2, r-1/2} \) and \( Q_\phi \in H^{s-m+1, r-\ell+1}, \) then \( Q_\phi \in H^{s, r} \) and for all \( M, N \) there is \( C > 0 \) such that

\[
\|Q\phi\|_{H^{s, r}} \leq C(\|Q\phi\|_{H^{s-m+1, r-\ell+1}} + \|Q\phi\|_{H^{s', r-1/2}} + \|\phi\|_{H^{s, r}}).
\]

**Proof.** Modulo justifying the pairing argument, the standard calculation, (2) (with possibly a switch of the sign of the \( \|B\phi\|\) term) takes care of the \( s - (m - 1)/2 + \beta \) \( L \) \( > 0 \) case, with no \( E \) term, and \( B^*B \) replaced by \( \sum_{j=1}^2 B_j^* B_j \), with \( B_j \) having symbol \( b_j \) satisfying

\[
b_1 = \sqrt{2\beta_0} \sqrt{s - (m - 1)/2 + \beta_0s'} + \beta \phi(\rho_1)\rho_0^s\rho_2, \\
b_2 = \sqrt{\phi(\rho_1)\phi(\rho_1)\rho_0^s\rho_2^r}.
\]

Here we use that one can make \( \phi \) supported in a specified neighborhood of \( L \); in particular in one in which \( s - (m - 1)/2 + \beta_0s' \) is bounded below by a positive constant since it is \( s - (m - 1)/2 + \beta \) bounded by \( L \) \( > 0 \). In a similar vein, but now regarding the \( \phi_1 \) term as part of the error \( e \), in the case \( s - (m - 1)/2 + \beta < 0 \), we can take

\[
b = \sqrt{2\beta_0} \sqrt{s - (m - 1)/2 + \beta_0s'} + \beta \phi(\rho_1)\rho_0^s\rho_2^r, \\
e = \phi(\rho_1)\phi(\rho_1)\rho_0^s\rho_2^r.
\]

In order to justify the argument, we use \( \phi_1 \) given in (7) with \( \tau = \rho_0^{-1} \) and \( K = 2(s - s') \). Then

\[
\mp H_{\rho, m, s} \phi_1(\rho_0^{-1}) = -\rho_0^{-1} f_\rho(\rho_0^{-1}) \phi_1(\rho_0^{-1}) \beta_0,
\]

with

\[
0 \leq -\rho_0^{-1} f_\rho(\rho_0^{-1}) \leq s - s'.
\]

Thus, with

\[
a_\rho = \phi(\rho_1)^2 \rho_0^{-m'}\rho_2^{-\ell'} \phi_1(\rho_0^{-1})^2
\]
the principal symbol \(-\langle H_\rho a_t - \tilde{\rho}_a \rangle_i \) of \(i\langle [A_t, P] + (P - P^*) A_t \rangle\) is
\[
\mp \rho_0^{-m'} - m + 1 \rho_2^{-1} \tilde{\rho}_0^{-1} \phi_{\tilde{\rho}_0^{-1}}(\phi_0^{-1})^2 + \beta_0 \rho \phi_0^{-1} \phi_1^{-1} + 2 \beta_0 \phi_1^{-1} \phi_1^{-2}
\]
Then the above calculation of the commutator is unchanged provided either the
new term \(2(\rho_0^{-1} f_\rho(\rho_0^{-1})) \beta_0\) has the same sign as the previous \(b_j\) or \(b\) terms, which
indeed happens when \(s - (m - 1)/2 + \beta|L < 0\), or if it can be absorbed in the \(b_j\)
terms. For the latter we need
\[
s - (m - 1)/2 + \beta_2(r - (\ell - 1)/2) + \beta - (\rho_0^{-1} f_\rho(\rho_0^{-1})) > 0,
\]
which is satisfied if
\[
s' - (m - 1)/2 + \beta_2(r - (\ell - 1)/2) + \beta > 0
\]
in view of (11). This proves the proposition, provided the amount of regularization
provided suffices for the proof to go through for \(t > 0\). There are no issues with
the argument provided we actually know that
(12) \[
\langle Pu, A_t u \rangle - \langle A_t u, Pu \rangle = \langle (A_t P - P^* A_t) u, u \rangle;
\]
for the rest of the argument it is straightforward to check that all steps work,
provided we take \(s' = s - 1/2\) in the second case, and have \(s'\) as stated in the first
case. (In fact, in the second case we may simply take \(s' = s - 1\), and then the whole
argument goes through directly, including the proof of (12); the same is true in the
first case if \((s'-1/2) - (m-1)/2 + \beta > 0\) on \(L\) with the notation of the statement
of the proposition.)

So it remains to prove (12) for \(t > 0\). The point is that while both sides make
sense by the a priori assumptions, they are not necessarily equal in principle since
e.g. \(\langle P^* A_t u, u \rangle\) is not defined for \(u\) with \(WF^{s',r-1/2}(u)\) disjoint from \(WF^{s'}_L(\{A_t\})\).
This, however, is straightforward to overcome by using an additional regularizer
family \(\Lambda_\tau\), for then
\[
\langle Pu, A_t u \rangle - \langle A_t u, Pu \rangle = \lim_{\tau \to 0} \langle (\Lambda_\tau Pu, A_t u) - \langle \Lambda_\tau A_t u, Pu \rangle \rangle
\]
\[
= \lim_{\tau \to 0} \langle (\Lambda_\tau A_t Pu, u) - \langle P^* \Lambda_\tau A_t u, u \rangle \rangle
\]
so the argument is finished by showing that \(\langle A_t \Lambda_\tau P - P^* \Lambda_\tau A_t \rangle \to A_t P - P^* A_t\)
strongly. But
\[
(A_t \Lambda_\tau P - P^* \Lambda_\tau A_t) = \Lambda_\tau (A_t P - P^* A_t) + [A_t, \Lambda_\tau] P - [P^*, \Lambda_\tau] A_t,
\]
and \(A_t P - P^* A_t \in \Psi^{2s',2r-1}\), so \(\Lambda_\tau (A_t P - P^* A_t) \to (A_t P - P^* A_t)\) in \(\Psi^{2s',2r-1}\), while \([A_t, \Lambda_\tau] P, [P^*, \Lambda_\tau] A_t\) are
uniformly bounded in \(\Psi^{2s',2r-1}\) and converge to 0 in \(\Psi^{2s',2r-1}\) for \(\delta > 0\), giving
the desired strong convergence. This completes the proof of the proposition.

As usual, one can iterate the argument and obtain:

Proposition 0.12. Suppose that \(L\) is as above, and \(s - (m - 1)/2 + \beta > 0\) on \(L\),
and \(s' - (m - 1)/2 + \beta > 0\) on \(L\) and \(r, r' \in \mathbb{R}\). Then there exist \(Q \in \Psi^{0,0}\) elliptic
on $L$ such that for $Q_1$ elliptic on $WF'(Q)$, if $Q_1 u \in H^{s',r'}$, $Q_1 P u \in H^{s-m+1,r-\ell+1}$, then $Q u \in H^{s,r}$ and for all $M, N$ there is $C > 0$ such that

$$\|Q u\|_{H^{s,r}} \leq C (\|Q_1 P u\|_{H^{s-m+1,r-\ell+1}} + \|Q_1 u\|_{H^{s',r'}} + \|u\|_{H^{M,N}}).$$

On the other hand, suppose now that $s - (m-1)/2 + \tilde{\beta} < 0$ on $L$. Then we are given neighborhoods $U_1, U_2$ of $L$ with $\overline{U_1} \subset U_2$, and $Q_1, Q_2 \in \Psi^{0,0}$ with $WF'(Q_2) \subset U_2 \setminus U_1$, $WF'(Q_1) \subset U_2$, and such that for every $\alpha \in (\text{Char}(p) \cap U_2) \setminus L$, the bicharacteristic $\gamma$ of $p$ with $\gamma(0) = \alpha$ enters $\text{Ell}(Q_2)$ while remaining in $\text{Ell}(Q_1)$. Then there exist $Q \in \Psi^{0,0}$ elliptic on $L$ such that if $Q_2 u \in H^{s',r}$, and $Q_1 P u \in H^{s-m+1,r-\ell+1}$, then $Q u \in H^{s,r}$ and for all $M, N$ there is $C > 0$ such that

$$\|Q u\|_{H^{s,r}} \leq C (\|Q_2 u\|_{H^{s',r}} + \|Q_1 P u\|_{H^{s-m+1,r-\ell+1}} + \|u\|_{H^{M,N}}).$$

This gives rise to another non-elliptic Fredholm problem. So suppose that $P \in \Psi^{m,\ell}$ with real homogeneous principal symbol. Suppose also that $Q \in \Psi^{m,\ell}$ is such that its principal symbol $q$ satisfies $q = \ell^2$ as above, and that for all $\alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n)$ in $\Sigma(p) = \{p = 0\}$ the integral curve $\gamma$ of $H_p$ with $\gamma(0) = \alpha$ reaches $q < 0$ in finite time in the backward direction, and tends to $L$ in the forward direction. Again let

$$X^{s,r} = \{u \in H^{s,r}: (P - iQ) u \in H^{s-m+1,r-\ell+1}\}, \quad Y^{s,r} = H^{s,r}.$$

Then

$$P - iQ: X^{s,r} \to Y^{s-m+1,r-\ell+1}, \quad P^* + iQ^*: X^{s,\tilde{r}} \to Y^{s-m+1,\tilde{r}-\ell+1}$$

are Fredholm for all $r$ and for all $s$ with

$$s - (m-1)/2 + \tilde{\beta} > 0, \quad \tilde{s} - (m-1)/2 - \tilde{\beta} < 0.$$

Note that this is exactly the condition for ‘high regularity’ $H^{s,r}$ estimates for $P$ at the radial points, as well as for ‘low regularity’ $H^{s,\tilde{r}}$ estimates for $P^*$ at these. Further, for

$$\tilde{s} = m - 1 - s, \quad \tilde{r} = \ell - 1 - r,$$

the stated Sobolev spaces for $P^* + iQ^*$ are the duals of those stated for $P - iQ$, i.e.

$$H^{\tilde{s},\tilde{r}} = (H^{s-m+1,r-\ell+1})^*, \quad H^{\tilde{s}-m+1,\tilde{r}-\ell+1} = (H^{s,r})^*.$$

Again, this Fredholm statement follows estimates of the kind

$$\|u\|_{H^{s,r}} \leq C ((P - iQ) u \|_{H^{s-m+1,r-\ell+1}} + \|u\|_{H^{M,N}}),$$

for $M < s, N < r, \tilde{M} < \tilde{s}, \tilde{N} < \tilde{r}$, which in turn follow from the propagation of singularities and the result at the radial points. Thus, now we just need to check that if $u \in H^{s',r'}$ for some $r'$ and with $s'$ as above, and $(P - iQ) u \in H^{s-m+1,r-\ell+1}$ then $u \in H^{s,r}$, i.e. $WF^{s,r}(u) = \emptyset$. But if either $p$ or $q$ is elliptic at $\alpha$, then $\alpha \notin WF^{s+1,r+1}(u)$ and thus $\alpha \notin WF^{s,r}(u)$ by microlocal elliptic regularity. Further, there is a neighborhood $U$ of $\alpha$ such that $\alpha \in U$ implies $\alpha \notin WF^{s,r}(u)$ by the high-regularity radial set result. Now propagating the regularity backwards (which is what forces the sign of $q$), if $p$ vanishes at $\alpha$, and thus the forward bicharacteristic $\gamma$ from $\alpha$ reaches $U$, where we know there is no $WF^{s,r}(u)$, so propagation of singularities gives that $\alpha \notin WF^{s,r}(u)$. For the adjoint, again we do not need to be concerned with elliptic points. If $p$ vanishes at $\alpha$ but $\alpha \notin L$, then the backward bicharacteristic from $\alpha$ reaches $q < 0$, so the propagation of singularities result gives
\( \alpha \notin \text{WF}^{k,\tilde{\tau}}(u) \). Finally, at \( L \), we use the low-regularity radial set result to complete the proof of the Fredholm claim.

This can be generalized when the characteristic set of \( \rho \) has two components \( \Sigma_{\pm} \), and in \( \Sigma_+ \) the radial set \( L_+ \) is a source, in \( \Sigma_- \) it is a sink \( L_- \). Namely, suppose that \( P \in \Psi^{m,\ell} \) with real homogeneous principal symbol, \( Q \in \Psi^{m,\ell} \) is such that its principal symbol \( q \) satisfies \( q = t^2_+ - t^2_- \) with \( \text{supp} t_+ \cap \Sigma_- = \emptyset \), \( \text{supp} t_- \cap \Sigma_+ = \emptyset \), and that for all \( \alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n) \) in \( \Sigma_+ \), resp. \( \Sigma_- \), the integral curve \( \gamma \) of \( H_\rho \) with \( \gamma(0) = \alpha \) reaches \( \{ q > 0 \} \), resp. \( \{ q < 0 \} \) in finite time in the forward, resp. backward direction, and tends to \( L_+ \), resp. \( L_- \) in the backward, resp. forward direction. Then \( P - iQ, P^* + iQ^* \) are Fredholm as in (13) subject to (14).

This setup suffices for instance for the analysis of scattering theory on asymptotically hyperbolic spaces. On asymptotically Euclidean spaces only minor changes in the functional analytic setup are required, in which the artificial complex absorption is eliminated, and the bicharacteristics connect two components of the radial set \( L^\pm \) (so \( L^\pm \) intersect the same component of the characteristic set). For this, we note that if \( s_- - (m - 1)/2 + \tilde{\beta}_{L_-} > 0 \), i.e. we are in the high regularity regime at \( L_- \), then we can propagate \( H^{s_-,-\tilde{\tau}} \) estimates out of \( L_- \) for elements of \( H^{s_-,-\tilde{\tau}} \) with \( s' - (m - 1)/2 + \tilde{\beta}_{L_+} > 0 \). These can be propagated to a punctured neighborhood of \( L^+ \). Now at \( L^+ \) we can propagate then to \( L^+ \) itself in \( H^{s_+,-\tilde{\tau}} \) provided \( s_+ - (m - 1)/2 + \tilde{\beta}_{L_+} < 0 \), i.e. we are in the low regularity regime. Thus, we need spaces with differing regularity at \( L^\pm \). Notice that the adjoint operator, one the dual spaces corresponding to these microlocal Sobolev spaces will then proceed in the opposite direction, starting with high regularity estimates from \( L^+ \), propagating them to a punctured neighborhood at \( L^- \), and then propagating these to \( L^- \) itself in the low regularity regime.