We discuss basic properties of so-called scattering pseudodifferential operators, introduced in this generality by Melrose, formerly discussed by Parenti and Shubin on $\mathbb{R}^n$, where it can be also considered an example of Hörmander’s Weyl calculus. These operators generalize differential operators of the form

$$A = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \text{ with } a_\alpha \in C^\infty(\mathbb{R}^n),$$

as we show below in (13). Indeed, the conditions on the coefficients $a_\alpha$ are relaxed to be ‘symbolic’, so that for instance $a_\alpha(z) = \phi(z)|z|^{-\rho}, \phi \equiv 0$ near the origin, $\equiv 1$ near infinity is allowed. Thus, in particular operators such as $\Delta + V$, where $V$ is the Coulomb potential, without its singularity at the origin, fit into this framework.

More generally, we can consider Riemannian metrics $g$ with $g_{ij} \in C^\infty(\mathbb{R}^n)$ such that for all $z \in \mathbb{R}^n$, $\sum_{ij} g_{ij}(z)\zeta_i\zeta_j = 0$ implies $\zeta = 0$, i.e. $g$ is positive definite on the compact manifold $\mathbb{R}^n$. Then, with $V$ as above and with $\sigma \in \mathbb{C}$, $\Delta_g + V - \sigma$ is of the form (1) with $m = 2$.

The extension of this class to scattering pseudodifferential operators allows one to construct approximate inverses (parametrices), showing Fredholm properties, for spatial infinity, so for instance $\Delta^+ + \sigma$.

Since there are technicalities along the way, we give an outline of this section first. First we define two kinds of function spaces,

$$S^{m,\ell}(\mathbb{R}^n;\mathbb{R}^n) \subset S^{m,\ell}_{\infty}(\mathbb{R}^n;\mathbb{R}^n) \subset C^\infty(\mathbb{R}^{2n}),$$

as well as analogues on $\mathbb{R}^{3n}$:

$$S^{\ell_1,\ell_2,m}(\mathbb{R}^n;\mathbb{R}^n) \subset S^{\ell_1,\ell_2,m}_{\infty}(\mathbb{R}^n;\mathbb{R}^n) \subset C^\infty(\mathbb{R}^{3n}).$$

The elements of these spaces are called symbols; the important point is the behavior of these symbols at infinity. Here the spaces become larger with increasing $m$, $\ell$ and $\ell_j$. We have projections $\pi_L, \pi_R : \mathbb{R}^{3n} \to \mathbb{R}^{2n}$, with $\pi_L$ dropping the second factor of $\mathbb{R}^{3n}$ and $\pi_R$ dropping the first factor:

$$\pi_L(z, z', \zeta) = (z, \zeta), \quad \pi_R(z, z', \zeta) = (z', \zeta).$$

The $\pi_L^*, \pi_R^*$ pull-back elements of the $\mathbb{R}^{2n}$ spaces to the corresponding $\mathbb{R}^{3n}$ spaces (with $\ell_1 = \ell, \ell_2 = 0$, resp, $\ell_2 = \ell, \ell_1 = 0$). We define an oscillatory integral map:

$$I : S^{\ell_1,\ell_2,m}_{\infty}(\mathbb{R}^n;\mathbb{R}^n) \to \mathcal{L}(S, S),$$

with $\mathcal{L}$ denoting continuous linear operators, and also show by duality that

$$I : S^{\ell_1,\ell_2,m}(\mathbb{R}^n;\mathbb{R}^n) \to \mathcal{L}(S', S').$$
and $I$ is closed under Fréchet space or $L^2$-based adjoints. The compositions

\[ q_L = I \circ \pi^*_L, \quad q_R = I \circ \pi^*_R, \]

are called the left and right quantization maps. Now, it turns out that $I$ is redundant, and its range on $S^{\ell_1,\ell_2,m}(\mathbb{R}^n;\mathbb{R}^n)$, resp. $S^{\ell_1,\ell_2,m}_c(\mathbb{R}^n;\mathbb{R}^n)$, is that of $q_L$ on $S^{m,\ell}(\mathbb{R}^n;\mathbb{R}^n)$, resp. $S^{m,\ell}_c(\mathbb{R}^n;\mathbb{R}^n)$ with $\ell = \ell_1 + \ell_2$; the analogous statement also holds with $q_L$ replaced by $q_R$. This is called left, resp. right, reduction; see Proposition 0.1. One defines pseudodifferential operators, $\Psi^{m,\ell}$, resp. $\Psi^{m,\ell}_\infty$, to be the range of $q_L$ (or equivalently $q_R$) on the spaces $S^{m,\ell}(\mathbb{R}^n;\mathbb{R}^n)$, resp. $S^{m,\ell}_c(\mathbb{R}^n;\mathbb{R}^n)$.

Once this is shown it is straightforward to see (using the general $I$, which is why it is introduced) that $A \in \Psi^{m,\ell}$, $B \in \Psi^{m',\ell'}$ implies $AB \in \Psi^{m+m',\ell+\ell'}$, i.e. that $\Psi^{m,\ell}_\infty = \cup_{m,\ell} \Psi^{m,\ell}$ is an order-filtered algebra, with the analogous statements holding for $\Psi^{m,\ell}_\infty$ as well. One also shows that composition is commutative to leading order, i.e.

\[ A \in \Psi^{m,\ell}, \quad B \in \Psi^{m',\ell'} \quad \Rightarrow \quad [A, B] = AB - BA \in \Psi^{m+m'-1,\ell+\ell'-1}, \]

the analogous statement here is

\[ A \in \Psi^{m,\ell}_\infty, \quad B \in \Psi^{m',\ell'}_\infty \quad \Rightarrow \quad [A, B] = AB - BA \in \Psi^{m+m'-1,\ell+\ell'}_\infty, \]

i.e. the gain is only in the first order. This is conveniently encoded by the principal symbol

\[ \sigma_{m,\ell} : \Psi^{m,\ell} \to S^{m,\ell} / S^{m-1,\ell-1}, \quad \sigma_{\infty,m,\ell} : \Psi^{m,\ell}_\infty \to S^{m,\ell}_\infty / S^{m-1,\ell}_\infty, \]

which are multiplicative (homomorphisms of filtered algebras); the leading order commutativity of pseudodifferential operators correspond to the commutativity of function spaces under multiplication. An immediate consequence is the elliptic parametrix construction: for operators $A \in \Psi^{m,\ell}$ with invertible principal symbol, which are called elliptic, one can construct an approximate inverse $B \in \Psi^{-m,-\ell}$ such that $AB - \Id, BA - \Id : S' \to S$ are continuous, i.e. completely regularizing.

In the case of $A \in \Psi^{m,\ell}_\infty$, we only have that $AB - \Id, BA - \Id : S' \to C^\infty(\mathbb{R}^n)$, i.e. are smoothing, but do not give decay at infinity. Since completely regularizing operators are compact from any weighted Sobolev space to any other weighted Sobolev space, and since we show that

\[ A \in \Psi^{m,\ell}_\infty \quad \Rightarrow \quad A \in \mathcal{L}(H^{r,s}, H^{r-m,s-\ell}) \]

for all $r, s \in \mathbb{R}$ (so analogous statements hold for $\Psi^{m,\ell}_\infty \subset \Psi^{m,\ell}$), we deduce that elliptic $A \in \Psi^{m,\ell}$ are Fredholm on any weighted Sobolev space, with the nullspace of both $A$ and $A^*$ lying in $S(\mathbb{R}^n)$, and is independent of the choice of the weighted Sobolev space. In particular, if $A \in \Psi^{m,0}$, $m > 0$, elliptic, is symmetric with respect to the $L^2$ inner product, then one immediately concludes that $A \pm i \Id$ is invertible as a map $H^{m,0} \to L^2$, and thus $A$ is self-adjoint.

Another important directions we explore is microlocalization, by introducing the notion of the operator wave front set, $WF'(A)$, or $WF'_\infty(A)$, which measures where in phase space $A$ is ‘trivial’. Thus, while $\sigma_{m,\ell}, \sigma_{\infty,m,\ell}$ capture the leading order behavior of operators, i.e. their behavior modulo one order lower operators, $WF'(A)$ and $WF'_\infty(A)$ give the locations where $A$ is not residual, i.e. in $\Psi^{-\infty,-\infty}$, resp. $\Psi^{-\infty,-\infty}_\infty$, so for instance the emptyness of $WF'(A)$ implies $A \in \Psi^{-\infty,-\infty}$. One should think of these of these as an analogue of the singular support of distributions, which measures where a distribution is not $C^\infty$, except that its location will not
be in the base space $\mathbb{R}^n$, but rather at infinity in phase space, $\mathbb{R}^n \times \mathbb{R}^n$. To make this concrete, it is useful to compactify $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^n_\infty$; then for $A \in \Psi^{m,\ell}$, $WF'(A) \subset \partial(\mathbb{R}^n \times \mathbb{R}^n_\infty)$ while for $A \in \Psi^{m,\ell}_\infty$, $WF'_\infty(A) \subset \mathbb{R}^n_\infty \times \partial\mathbb{R}^n_\infty$. Then one can perform a microlocal version of the elliptic parametrix construction, i.e. one that is localized, in the sense of $WF'$, near points at which the operator $A$ is elliptic; this is a first step towards understanding non-elliptic operators.

We now go through the details. Thus, starting with $\mathbb{R}^n$, we consider operators of the form

$$Au(z) = (I(a)u)(z) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{iz \cdot \zeta'} a(z,z',\zeta) u(z') \, dz', \quad u \in S(\mathbb{R}^n),$$

where $a$ is a product-type symbol of class $S^{m,\ell_1,\ell_2}$, i.e. differentiation in $z$, resp. $z'$, provides extra decay in the respective variables:

$$a \in S^{\ell_1,\ell_2,m}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n_\zeta) \quad \Rightarrow \quad a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n_\zeta),$$

$$\|D_\zeta^\alpha D_{\zeta'}^\beta D_\zeta^\gamma a\| \leq C_{\alpha\beta\gamma} \langle z \rangle^{\ell_1 - |\alpha|} \langle z' \rangle^{\ell_2 - |\beta|} \langle \zeta \rangle^{m - |\gamma|}.$$

One writes

$$\|a\|_{S^{\ell_1,\ell_2,m}} = \sum_{|\alpha| + |\beta| + |\gamma| \leq N} \sup_{z,z'} \langle z \rangle^{-\ell_1 + |\alpha|} \langle z' \rangle^{-\ell_2 + |\beta|} \langle \zeta \rangle^{m - |\gamma|} \|D_\zeta^\alpha D_{\zeta'}^\beta D_\zeta^\gamma a\|;$$

as $N$ runs over $\mathbb{N}$, these give a family of seminorms on $S^{\ell_1,\ell_2,m}$, giving it a Fréchet topology. In fact, in the beginning it is better to start with a larger class of symbols, without extra decay in the $z$, $z'$ variables upon differentiation:

$$a \in S^{\ell_1,\ell_2,m}_\infty(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n_\zeta) \quad \iff \quad a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n_\zeta),$$

$$\|D_\zeta^\alpha D_{\zeta'}^\beta D_\zeta^\gamma a\| \leq C_{\alpha\beta\gamma} \langle z \rangle^{\ell_1} \langle z' \rangle^{\ell_2} \langle \zeta \rangle^{m - |\gamma|}.$$

One writes

$$\|a\|_{S^{\ell_1,\ell_2,m}_\infty} = \sum_{|\alpha| + |\beta| + |\gamma| \leq N} \sup_{z,z'} \langle z \rangle^{-\ell_1} \langle z' \rangle^{-\ell_2} \langle \zeta \rangle^{m - |\gamma|} \|D_\zeta^\alpha D_{\zeta'}^\beta D_\zeta^\gamma a\|.$$

For $\ell_1 = \ell_2 = 0$, this is Hörmander’s uniform symbol class. Most importantly, $S^{\ell_1,\ell_2,m} \subset S^{\ell_1,\ell_2,m}_\infty$, and the inclusion map

$$\iota : S^{\ell_1,\ell_2,m} \rightarrow S^{\ell_1,\ell_2,m}_\infty$$

is continuous, with

$$\|a\|_{S^{\ell_1,\ell_2,m}_\infty} \leq \|a\|_{S^{\ell_1,\ell_2,m}}$$

for all $N$.

Note that $\ell_j \leq \ell_j'$, $m \leq m'$ implies

$$S^{\ell_1,\ell_2,m} \subset S^{\ell_1',\ell_2',m'},$$

and similarly with $S_\infty$. One writes

$$S^{\ell_1,\ell_2,-\infty} = \cap_{m \in \mathbb{R}} S^{\ell_1,\ell_2,m}, \quad S^{\ell_1,-\infty,-\infty} = \cap_{m \in \mathbb{R}, \ell_2 \in \mathbb{R}} S^{\ell_1,\ell_2,m},$$

and similarly again with $S_\infty$. One also writes

$$S^{\infty,\infty,-\infty} = \cup_{m,\ell_1,\ell_2} S^{\ell_1,\ell_2,m}.$$
conjugation, the (function-theoretic, i.e. pointwise) product (which is commutative) satisfies

\[ a \in S^{\ell_1, \ell_2, m}, \ b \in S^{\ell'_1, \ell'_2, m'} \Rightarrow ab \in S^{\ell_1 + \ell'_1, \ell_2 + \ell'_2, m + m'}, \]

as follows from Leibniz’ rule. Similarly \( S^{\infty, \infty} \) forms a commutative filtered algebra as well.

As examples, recall that if \( a \) is a polynomial of order \( \ell_1, \ell_2 \) and \( m \) in the three variables, then certainly \( a \in S^{\ell_1, \ell_2, m} \). More interestingly, if \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) = C^\infty(\mathbb{R}^n) \) then \( a \in S^{0,0,0} \), so

\[ a \in \langle \zeta \rangle^{\ell_1} \langle \zeta' \rangle^{\ell_2} \langle \zeta'' \rangle^{m} C^\infty(\mathbb{R}^n) \Rightarrow a \in S^{\ell_1, \ell_2, m}. \]

A particular example is \( a = |z|^{-\rho} \phi(z) \), where \( \phi \equiv 0 \) near 0, \( \phi \equiv 1 \) near \( \infty \), then \( a \in S^{-\rho,0,0} \), such an \( a \) can be thought of as a potential which may decay only slowly at infinity; \( \rho = 1 \) would give the Coulomb potential without its singularity at the origin. Thus, \( S^{\ell_1, \ell_2, m} \) is a \( C^\infty(\mathbb{R}^n) \)-module.

On the flipside, we can rewrite the estimates for \( S^{\ell_1, \ell_2, m} \):

\[ |a'| \leq |a|, \ |\beta'| \leq |\beta|, \ |\gamma'| \leq |\gamma| \Rightarrow |z^{\alpha'} D_z^{\alpha} (z')^{\beta} D_z^{\beta} \zeta' D_{\zeta}^{\gamma} a| \leq C_{\alpha, \beta, \gamma} \langle \zeta \rangle^{\ell_1} \langle \zeta' \rangle^{\ell_2} \langle \zeta'' \rangle^{m}. \]

Since \( z_j \partial_j \) and \( \partial_j \) generate all \( C^\infty \) vector fields over \( C^\infty(\mathbb{R}^n) \) which are tangent to \( \partial \mathbb{R}^n \), where the set is denoted by \( \mathcal{V}_b(\mathbb{R}^n) \), we can rewrite this equivalently as follows: let \( V_{j,k} \in \mathcal{V}_b(\mathbb{R}^n) \), \( j = 1, 2, 3 \), \( N_j \in \mathbb{N} \) (possibly 0) and 1 \( \leq k \leq N_j \) acting in the \( j \)th factor, then

\[ \langle \zeta \rangle^{\ell_1} \langle \zeta' \rangle^{\ell_2} \langle \zeta'' \rangle^{m} \prod_{j=1}^{3} \prod_{k=1}^{N_j} V_{j,k} a \in L^\infty. \]

This could be further rephrased, in terms of vector fields on \( \mathbb{R}^n \), tangent to all boundary faces: if \( V_j \) are such, 1 \( \leq j \leq N \) (possibly \( N = 0 \)), then

\[ \langle \zeta \rangle^{\ell_1} \langle \zeta' \rangle^{\ell_2} \langle \zeta'' \rangle^{m} V a \in L^\infty. \]

Since one can use any vector fields tangent to the various boundary faces, in any product decomposition \( [0,1]_{j-1} \times S^{n-1} \) near the boundary of each factor \( \mathbb{R}^n \), one automatically has smoothness in the various angular variables; in the radial variables one has iterated regularity with respect to \( r \partial_r \).

Note that unless \( m < -n \), the integral (2) is not absolutely convergent; if \( m < -n \), it is, with the result \( Au \in C(\mathbb{R}^n) \), and for \( M > \ell_2 + n \),

\[ \sup \langle \zeta \rangle^{\ell_1} Au(z) \leq C ||a||_{S^{\ell_1, \ell_2, m, 0}} ||u||_{S^{0,0,M}}, \]

where \( C \) is a universal constant (independent of \( a \) and \( u \)) and

\[ ||u||_{S,k,M} = \sum_{|\alpha| \leq k} \sum_{|\beta| \leq M} \sup |z^{\beta} D_z^{\alpha} u| \]

are the Schwartz seminorms. However, if \( m < -n \), one can also integrate by parts as usual in \( \zeta' \), noting that \((1 + \Delta_z) e^{i\xi(z - z')} = \langle \xi \rangle \langle \zeta \rangle e^{i\xi(z - z)} \), so

\[ Au(z) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle \zeta \rangle^{-2N} (1 + \Delta_z)^N e^{i\xi(z - z')} a(z, z', \zeta) u(z') dz', \]

(3)

\[ = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(z - z')} \langle \zeta \rangle^{-2N} (1 + \Delta_z)^N (a(z, z', \zeta) u(z')) dz'. \]
Expanding \((1 + \Delta_\infty)^N(a(z, z', \zeta) u(z'))\), one deduces that
\[
(4) \quad |(1 + \Delta_\infty)^N(a(z, z', \zeta) u(z'))| \leq (z')^{\ell_1} (z')^{2 - M} (\zeta)^m \|a\|_{S_{\nu,2}^m,2N} \|u\|\mathcal{S}_{2N,M},
\]
so for just \(m < -n + 2N\), the right hand side of (3) is integrable, and defining \(Au \in C(\mathbb{R}^n)\) to be the result,
\[
(5) \quad \sup |\langle z \rangle^{-\ell_1} Au(z) | \leq C\|a\|_{S_{\nu,2}^m,2N} \|u\|\mathcal{S}_{2N,M}.
\]
This gives an extension of \(A = I(a)\) to \(S_{\nu,2}^m\). Since \(S_{\nu,2}^m,\infty\) is dense in \(S_{\nu,2}^m\) in the topology of \(S_{\nu,2}^m,\nu'\) for \(\nu' > m\), and since for \(m < -n\), the expressions (3) for various \(N\) are all equal, the continuity property (5) shows that \(A\) is independent of the choice of \(N\) provided \(m < -n + 2N\) (since one can then make \(m' \in (m, -n + 2N)\), and use the \(m'\)-continuity and density statements).

Now at least \(Au \in C(\mathbb{R}^n)\), with a suitable bound, is defined, but in fact it is in \(\mathcal{S}(\mathbb{R}^n)\). To see this, first note that \(D_\infty^\alpha e^{i\zeta (z - z')} = \zeta^\alpha\), so for \(N\) sufficiently large, so that \(m + |\alpha| < -n + 2N\), differentiating under the integral sign and using the Leibniz rule,
\[
(D_\infty^\alpha A)u(z) = \sum_{\gamma + \delta \leq \alpha} C_{\gamma \delta} (2\pi)^{-\alpha} \int_{\mathbb{R}^n} D_\infty^\alpha (e^{i\zeta (z - z')}) (\zeta)^{-2N} (1 + \Delta_\infty)^N(D_\infty^\alpha a(z, z', \zeta) u(z')) dz',
\]
with \(C_{\gamma \delta}\) combinatorial constants, so by (4) with \(a\) replaced by \(D_\infty^\alpha a\), with \(M > -n + \ell_2\) still,
\[
\sup |\langle z \rangle^{-\ell_1} D_\infty^\alpha Au(z) | \leq C\|a\|_{S_{\nu,2}^m,2N+|\alpha|} \|u\|\mathcal{S}_{2N,M}.
\]
Further, \(z_\delta e^{i\zeta (z - z')} = z_\delta' e^{i\zeta (z - z')} + D_\zeta^\delta e^{i\zeta (z - z')}\), so
\[
\langle z \rangle^\beta e^{i\zeta (z - z')} = \langle z' \rangle (z + D_\zeta^\delta)^\beta e^{i\zeta (z - z')} = \sum_{\mu + \nu \leq \beta} C_{\mu \nu} \langle z' \rangle^\mu D_\zeta^\nu e^{i\zeta (z - z')}.
\]
so integration by parts in \(\zeta\) gives
\[
\langle z \rangle^\beta (D_\infty^\alpha A)u(z) = \sum_{\gamma + \delta \leq \alpha} \sum_{\mu + \nu \leq \beta} C_{\gamma \delta} C_{\mu \nu} (2\pi)^{-\alpha} \int_{\mathbb{R}^n} e^{i\zeta (z - z')} D_\zeta^\nu (\zeta^\gamma (\zeta)^{-2N} (\langle z' \rangle^\mu (1 + \Delta_\infty)^N(D_\infty^\alpha a(z, z', \zeta) u(z')) dz'
\]
\[
= \sum_{\gamma + \delta \leq \alpha} \sum_{\mu + \nu \leq \beta} \sum_{\alpha + \nu' \leq \nu} C_{\gamma \delta} C_{\mu \nu} C_{\nu' \nu} (2\pi)^{-\alpha} \int_{\mathbb{R}^n} e^{i\zeta (z - z')} D_\zeta^{\nu'} (\zeta^\gamma (\zeta)^{-2N} (\langle z' \rangle^\mu (1 + \Delta_\infty)^N(D_\infty^{\nu'} D_\zeta^\nu a(z, z', \zeta) u(z')) dz',
\]
and thus with \(M > -n + \ell_2 + |\beta|\) now, but \(m + |\alpha| < -n + 2N\) still,
\[
\sup |\langle z \rangle^{-\ell_1} \langle z \rangle^\beta D_\infty^\alpha Au(z) | \leq C\|a\|_{S_{\nu,2}^m,2N+|\alpha|} \|u\|\mathcal{S}_{2N,M},
\]
with \(C\) independent of \(a, u\). Now for \(\ell_1 \leq 0\), \(\langle z \rangle^{-\ell_1}\) can simply be dropped, while for \(\ell_1 > 0\) the \(\langle z \rangle^{-\ell_1}\) factor can be absorbed into a sum \(z^{\beta'}\) terms with \(|\beta'| \leq M'\).
where \( M' \geq \ell_1 \), so we obtain that for \( M' \geq \max(0, \ell_1) \), \( M > -n + \ell_2 + |\beta| + M' \),
\[ m + |\alpha| < -n + 2N \]
so \( Au \in S(\mathbb{R}^n) \), and the map \( A : S \to S \) is continuous, and in fact the stronger continuity property, namely that
\[ m \] is continuous, holds. In particular, one can restrict to \( S \) is continuous.

By the density of \( S \) in \( \ell_2 \), so both sides being continuous trilinear maps \( S_{\ell_1, \ell_2, m} \times S \to S \) for all \( m' \), \( \ell_1, \ell_2, m \), and we have at first for \( m < -n \),
\[ \int (I(a)u)\phi = \int u(I(\rho^* j^* a)\phi), \]

so both sides being continuous trilinear maps \( S_{\ell_1, \ell_2, m} \times S \to S \) for all \( m' \), \( \ell_1, \ell_2, m \),
the identity extends to all \( m \). Thus, the Fréchet space adjoint, \( I(a)^\dagger : S' \to S' \), defined by
\[ (I(a)^\dagger \phi)(u) = \phi(I(a)u), \ \phi \in S', \ u \in S, \]
satisfies
\[ I(a)^\dagger \phi = I(\rho^* j^* a)\phi, \ \phi \in S, \]
i.e. by the weak-* density of \( S \) in \( S' \), \( I(a)^\dagger \) is the unique continuous extension of
\( I(\rho^* j^* a) \) from \( S \) to \( S' \); one simply writes \( I(\rho^* j^* a) = I(a)^\dagger \) even as maps \( S' \to S' \).
Since \( \rho^* j^* \rho^* j^* a = a \), we deduce that for any \( a \), \( I(a) = I(\rho^* j^* a)^\dagger : S' \to S' \) is continuous.

Here we used the bilinear distributional pairing; if one uses the sesquilinear \( L^2 \)-pairing, one has
\[ \int Au(z)\overline{\phi(z)} \, dz = \int u(z') \left| \int \overline{c(z', z') a(z, z', \zeta)} \phi(z) \, dz \right| \, d\zeta \, dz' \]
\[ = \int u(z') \overline{(I(b)\phi)(z')} \, dz', \]
\[ \hat{b}(z', \zeta) = a(z', z, \zeta), \] so using * to denote the corresponding (Hilbert-space-type) adjoint

\[ (I(a))^* = I(cj^* a), \]
where $c$ is the complex conjugation map.

Note that if $a \in S^{\ell_1,\ell_2;m}$ then $c^*a \in S^{\ell_2,\ell_1;m}$, thus the adjoint of operators given by our scattering symbols is still in the same class, with $\ell_2$ and $\ell_1$ reversed.

While we have two indices $\ell_1$ and $\ell_2$ for growth in the spatial variables, this is actually redundant, $\ell_1 + \ell_2$ is the relevant quantity. To see this, we note that

$$e^{i(z-z')\cdot \zeta} = (z-z')^2 e^{i(z-z')\cdot \zeta},$$

so at first for $m<-n$, as usual,

$$(1 + \Delta_\zeta) e^{i(z-z')\cdot \zeta} = (z-z')^2 e^{i(z-z')\cdot \zeta},$$

and the analogous inequality also holds with $z$ and $z'$ interchanged, and

$$D_\zeta^a D_{\zeta'}^b (z-z')^{-2N} \leq C_{\alpha\beta} (z-z')^{-2N},$$

so for any $m, \ell_1, \ell_2, a \in S^{\ell_1,\ell_2;m}$, with $b$ defined by (8) satisfies $b \in S^{\ell_1+s,\ell_2-s,m}$ for $-2N \leq s \leq 2N$, and the map

$$S^{\ell_1,\ell_2,m} \ni a \rightarrow b \in S^{\ell_1+s,\ell_2-s,m}$$

is continuous, hence $I(a) = I(b)$ holds for all $m, \ell_1, \ell_2$ (as it holds for $m<-n$). Given any $s$, choosing sufficiently large $N$, shows that the range of $I$ on $S^{\ell_1,\ell_2,m}$ only depends on $\ell_1 + \ell_2$. We now define

$$\Psi_m(\mathbb{R}^n) = \{ I(a) : a \in S^{0,0,m}_\infty \};$$

we could have used, say, $S^{0,0,m}$ instead. Note that the orders on $\Psi_\infty$ are reversed compared to the symbol notation; this is done in part to conform with the usual notation. Note also that if $a \in S^{\ell_1,\ell_2;m}$ then $b$ defined by (8) is usually not in $S^{\ell_1+s,\ell_2-s,m}$, as derivatives in $z$ and $z'$ do not typically give extra decay when hitting $(z-z')^{-2N}$. However, we boldly make the analogous definition

$$\Psi_m(\mathbb{R}^n) = \{ I(a) : a \in S^{0,0,m} \};$$

we show below that this is the same as $\{ I(a) : a \in S^{\ell_1,\ell_2,m} \}$ with $\ell_1 + \ell_2 = \ell$.

One very useful property of $\Psi_m(\mathbb{R}^n)$ is that it is in fact exactly the range of $I$ acting on symbols of a special form, namely those independent of $z'$. Thus, again reversing the orders to conform with the standard notation, let

$$a \in S^{m,\ell}(\mathbb{R}_{z'};\mathbb{R}_{\zeta}) \iff a \in C^\infty(\mathbb{R}_{z'}^n \times \mathbb{R}_{\zeta}^n),$$

with $|D_\zeta^a| \leq C_{\alpha\beta} (z')^m |\gamma|$;

so with

$$\pi_L : \mathbb{R}_{z'}^n \times \mathbb{R}_{\zeta}^n \rightarrow \mathbb{R}_{z'}^n \times \mathbb{R}_{\zeta}^n$$

the projection map dropping $z'$, $a \in S^{m,\ell}(\mathbb{R}_{z'};\mathbb{R}_{\zeta})$ if and only if

$$\pi_L a \in S^{m,\ell}(\mathbb{R}_{z'};\mathbb{R}_{\zeta}).$$
As usual, the seminorms
\[ \|a\|_{S^m,\ell,N} = \sum_{|\alpha|+|\gamma|\leq N} \sup_{z} (z^{-\ell} \langle \zeta \rangle^{-m+\gamma}) |D_z^\alpha D_\xi^\gamma a| \]
give a Fréchet topology. With \( \pi_R \) the projecting dropping the \( z \) variables, one also has \( a \in S^m,\ell(R^n;R^n) \) if and only if \( \pi_R a \in S_\infty^m,\ell(R^n;R^n;R^n) \).

Then:

**Proposition 0.1.** For any \( \ell = \ell_1 + \ell_2 \) and \( a \in S_{\infty}^{\ell_1,\ell_2,m}(R^n;R^n;R^n) \) there exists a unique \( a_L \in S_\infty^{m,\ell}(R^n;R^n;R^n) \) such that \( I(a) = I(\pi_L^m a_L) \); one writes \( q_L = I \circ \pi_L^m : S_\infty^{m,\ell} \to \Psi_\infty^{m,\ell} \). Here \( a_L \) is called the left reduced symbol of \( I(a) \), and \( q_L \) is the left quantization map.

Similarly, for any \( \ell = \ell_1 + \ell_2 \) and \( a \in S_{\infty}^{\ell_1,\ell_2,m}(R^n;R^n;R^n) \) there exists a unique \( a_R \in S_\infty^{m,\ell}(R^n;R^n;R^n) \) such that \( I(a) = I(\pi_R^n a_R) \); one writes \( q_R = I \circ \pi_R^n : S_\infty^m,\ell \to \Psi_\infty^{m,\ell} \). Here \( a_R \) is called the right reduced symbol of \( I(a) \), and \( q_R \) is the right quantization map.

Moreover, the maps \( a \mapsto a_L, a \mapsto a_R \) are continuous.

Further, with \( i : R^n \times R^n \to R^n \times R^n \times R^n \) the inclusion map as the diagonal in the first two factors, i.e. \( i(z, \zeta) = (z, z, \zeta) \),

\[
(10) \quad a_L \sim \sum_{\alpha} \frac{|\alpha|}{\alpha!} i^* D_z^\alpha D_\xi^\gamma a,
\]

and

\[
(11) \quad a_R \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} i^* D_z^\alpha D_\xi^\gamma a,
\]

with the summation asymptotic in \( \zeta \), i.e. is modulo \( S_{-\infty,\ell}^m \).

If instead \( a \in S_{\ell_1,\ell_2,m} \), then the conclusions hold with \( a_L, a_R \in S_{m,\ell} \), with the asymptotic summation being asymptotic both in \( z \) and in \( \zeta \), i.e. is modulo \( S_{-\infty,-\infty}^m \).

Notice that for \( a \in S_{\infty}^{m,\ell} \),

\[
(11) \quad q_L(a)u(z) = (2\pi)^{-n} \int_{R^n} e^{iz \cdot \zeta} a(z, \zeta) (\mathcal{F}u)(\zeta) d\zeta
\]

for \( m < -n \), but now, for \( u \in S \), the right hand side extends continuously to \( S_{m,\ell} \) for all \( m \), so one could have directly defined \( q_L(a) \) directly for all \( m \). Similarly,

\[
(12) \quad q_R(a)u = \mathcal{F}^{-1}(\zeta \mapsto \int_{R^n} e^{-iz \cdot \zeta} a(z', \zeta) u(z') dz'),
\]

where now the right hand side makes sense directly as a tempered distribution for all \( m \). However, relating \( q_L \) and \( q_R \), as well as performing other important calculations, would be rather hard without having defined the map \( I \) in general, via a continuity/regularization argument! Note that for \( a \in S_{-\infty,-\infty}^m \), in either case, one deduces that directly that \( q_R(a)u \) and \( q_L(a)u \) are in \( S \).

We remark that if \( a \in S_{m,\ell} \) is a polynomial in \( \zeta \), i.e. \( a(z, \zeta) = \sum_{|\alpha| \leq m} a_\alpha(z) \zeta^\alpha \), then one can pull the factors \( a_\alpha(z) \) out of the integral (11), and thus \( \zeta^\alpha \mathcal{F} = \mathcal{F}D^\alpha \) and the Fourier inversion formula yields

\[
q_L(a)u(z) = \sum_{|\alpha| \leq m} a_\alpha(z)(D^\alpha u)(z),
\]
Similarly,
\[ q_L(a) = \sum_{|\alpha| \leq m} a_{\alpha} D^\alpha \]

Similarly,
\[ q_R(a)(z) = \sum_{|\alpha| \leq m} (D^\alpha(a_{\alpha}u))(z), \]
i.e.
\[ q_R(a) = \sum_{|\alpha| \leq m} D^\alpha a_{\alpha}. \]

So differential operators of order \( m \) on \( \mathbb{R}^n \) with coefficients in \( S^\ell(\mathbb{R}^n) \) lie in \( \Psi^{m,\ell} \). In particular, differential operators with coefficients in \( C^\infty(\mathbb{R}^n) \) lie in \( \Psi^{m,0}(\mathbb{R}^n) \).

We now prove Proposition 0.1; we only consider the left reduction, i.e. the \( L \) subscript case as the \( R \) case is completely analogous. First, we note that the uniqueness is straightforward. Any operator \( A = I(a) \), \( a \in S_{\infty}^{\ell_1,\ell_2,m} \), has a Schwartz kernel, \( K_A \in S' \) (as it is a continuous linear map \( S \to S \), thus \( S \to S' \)). When \( m < -n \), the Schwartz kernel satisfies
\[ K_A(\phi \otimes u) = \int (Au)(z)\phi(z) \, dz = (2\pi)^{-n} \int e^{i\zeta(z-z')}a(z,z',\zeta) u(z') \phi(z) \, d\zeta \, dz \]
\[ = \int (F_\zeta^{-1}a)(z,z',z-z') \phi(z) \, dz \, dz', \]
where \( F_\zeta^{-1} \) is the inverse Fourier transform in the third variable, \( \zeta \). (\( F_3^{-1} \) is a logically better, but less self-explanatory, notation.) Thus, for such \( a \), \( K_A \) is the polynomially bounded function (hence tempered distribution) given by
\[ F_a(z,z') = (F_\zeta^{-1}a)(z,z',z-z') = (F_3^{-1}a)(z,z',z-z'). \]

If \( a \in S_{\infty}^{m,\ell} \), then, with 2 denoting that the inverse Fourier transform is in the second slot, we have
\[ F_{\pi_L^*a}(z,z') = (F_2^{-1}a)(z,z-z') = (G^*F_2^{-1}a)(z,z') \]
where \( G : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) is the invertible linear map \( G(z,z') = (z,z-z') \), thus one can pull back tempered distributions by it. Thus,
\[ K_{I(\pi_L^*a)} = G^*F_2^{-1}a, \]
and correspondingly
\[ a = F_2(G^{-1})^*K_{I(\pi_L^*a)}, \]
first for \( m < -n \), but then as both sides are continuous maps \( S_{\infty}^{m,\ell} \to S' \), this identity holds in general. In particular, given \( \tilde{a} \in S_{\infty}^{\ell_1,\ell_2,m} \) there exists at most one \( a \in S_{m,\ell_1+\ell_2} \) such that \( I(\pi_L^*a) = I(\tilde{a}) \), for
\[ a = F_2(G^{-1})^*K_{I(\tilde{a})} \]
then.

Now for existence. In principle (16) solves this problem, but then one needs to show that the \( a \) it provides, i.e. \( a_L \), in the notation of the proposition, is not merely a tempered distribution, but is in an appropriate symbol class. So we proceed differently.
With the notation of the proposition, one expands \( a \) in Taylor series in \( z' \) around the diagonal, with the integral remainder term:

\[
a(z, z', \zeta) = \sum_{|\alpha| \leq N-1} \frac{(z-z')^\alpha}{\alpha!}((\partial_{z'})^\alpha a)(z, z, \zeta) + R_N(z, z' \zeta)
\]

\[
R_N(z, z' \zeta) = \sum_{|\alpha| = N} N \frac{(z-z')^\alpha}{\alpha!} \int_0^1 (1-t)^{-N-1}((\partial_{z'})^\alpha a)(z, (1-t)z + tz', \zeta) dt.
\]

Now, for \( m < -n \), as \( (z_j - z'_j) e^{i\zeta'(z-z')} = D_\zeta e^{i\zeta' (z-z')} \),

\[
(I((z_j - z'_j) a)u)(z) = (2\pi)^{-n} \int D_\zeta e^{i\zeta' (z-z')} a(z, z', \zeta) u(z') dz' d\zeta
\]

so as

\[
S^0, m \times S \ni (a, u) \mapsto I((z_j - z'_j) a)u \in S
\]

and

\[
S^0, m \times S \ni (a, u) \mapsto I(D_\zeta a)u \in S
\]

are both continuous bilinear maps, the density of \( S^{f_1, \ell_2, m} \) in the topology of \( S^{f_1, \ell_2, m'} \) for \( m' > m \) shows that

\[
I((z - z')^o a) = I(D_\zeta^o a)
\]

for all \( m \) and \( a \in S^{f_1, \ell_2, m'} \).

Thus, for \( a \) as in (17),

\[
I(a) = \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} I((D_\zeta)^\alpha t^* \partial_{z'}^\alpha a) + I(R_N),
\]

\[
R_N^o(z, z' \zeta) = \sum_{|\alpha| = N} N \frac{1}{\alpha!} \int_0^1 (1-t)^{-N-1}(D_\zeta^o (\partial_{z'})^\alpha a)(z, (1-t)z + tz', \zeta) dt.
\]

But

\[
(D_\zeta)^\alpha t^* \partial_{z'}^\alpha a \in S^{f_1, \ell_2, m - |\alpha|}, \quad R_N^o \in S^{f_1, \ell_2, m - N},
\]

with the map

\[
S^{f_1, \ell_2, m} \ni a \mapsto (D_\zeta)^\alpha t^* \partial_{z'}^\alpha a \in S^{m - |\alpha|, \ell_1 + \ell_2}
\]

continuous, and similarly with \( R_N \). Since \( (D_\zeta)^{\alpha} t^* \partial_{z'}^{\alpha} a \) is independent of \( z' \), this proves the following weaker version of Proposition 0.1: for all \( a \in S^{f_1, \ell_2, m} \) and for all \( \ell_1, \ell_2, m, N \) there exist \( a_N \in S^{m - |\alpha|, \ell_1 + \ell_2} \) such that

\[
I(a) - I(a_N) = I(R_N'), \quad R_N' \in S^{f_1, \ell_2, m - N}.
\]

Notice that if \( a \in S^{f_1, \ell_2, m} \) then the analogous conclusions hold but with

\[
(D_\zeta)^\alpha t^* \partial_{z'}^\alpha a \in S^{m - |\alpha|, \ell_1 + \ell_2 - |\alpha|}, \quad R_N' \in S^{f_1, \ell_2 - N, m - N}.
\]

An asymptotic summation argument allows one to improve this. This means the following: suppose \( a_j \in S^{f_1, \ell_2, m - j} \) for \( j \in \mathbb{N} \). Then there exists \( a \in S^{f_1, \ell_2, m} \) such
that

\[(18) \quad a - \sum_{j=0}^{N-1} a_j \in S^{\ell_1,\ell_2,m-N}.\]

To see this, we take \( \chi \in C^\infty(\mathbb{R}^n) \) with \( \chi(\zeta) = 1 \) for \( |\zeta| \geq 2 \), \( \chi(\zeta) = 0 \) for \( |\zeta| \leq 1 \). For \( 0 < \epsilon_j < 1 \) to be determined, but with \( \epsilon_j \to 0 \), consider

\[a(z,\zeta) = \sum_{j=0}^{\infty} \chi(\epsilon_j \zeta) a_j(z,\zeta);\]

the sum is finite for \( (z,\zeta) \) with \( |\zeta| \leq R \), with \( \leq R + 1 \) terms. Thus, \( a \in C^\infty \); the question is convergence in \( S^{\ell_1,\ell_2,m-N} \), and the property (18). But by Leibniz’ rule,

\[(D_\zeta^\alpha D_z^\beta D_z^\gamma a)(z,\zeta') = \sum_{j=0}^{\infty} \sum_{\gamma \leq \alpha} C_{\alpha,\gamma} \epsilon_j^\gamma (D_\gamma^\gamma \chi)(\epsilon_j \zeta)(D_\zeta^{\alpha-\gamma} D_z^\beta D_z^\gamma a_j)(z,\zeta').\]

To get convergence of the tail in \( S^{\ell_1,\ell_2,m-N} \), we need to estimate the sup norm of

\[
(\zeta)^{-m+N+|\alpha|}(z)^{-\ell_1}(\zeta')^{-\ell_1}(D_\zeta^{\alpha} D_z^\beta D_z^\gamma)(z,\zeta')
\]

\[= \sum_{j=N+1}^{\infty} C_{\alpha,\gamma} \epsilon_j^{-N} (\zeta)^{|\gamma+N-j|} \epsilon_j^{-N-j+|\gamma|} (D_\gamma^\gamma \chi)(\epsilon_j \zeta);\]

we use the above expansion. For \( \gamma = 0 \), we use \( |\zeta| \geq \epsilon_j^{-1} \) on supp \( \chi(\epsilon_j \cdot) \), so for \( j \geq N \),

\[\epsilon_j^{N-j} (\zeta)^{N-j} = (\epsilon_j^2 + \epsilon_j^2 |\zeta|^2)^{(N-j)/2} \leq 1,\]

while for \( \gamma 
eq 0 \) we use \( \epsilon_j^{-1} \leq |\zeta| \leq 2\epsilon_j^{-1} \) on supp \( (D_\gamma^\gamma \chi)(\epsilon_j \cdot) \), so

\[1 \leq (\zeta) \epsilon_j = (\epsilon_j^2 + \epsilon_j^2 |\zeta|^2)^{1/2} \leq 5^{1/2}\]

on supp \( (D_\gamma^\gamma \chi)(\epsilon_j \cdot) \) for all \( \gamma 
eq 0 \), and thus for \( j \geq N \),

\[(\zeta)^{|\gamma+N-j|} \epsilon_j^{-N-j+|\gamma|} \leq 5^{1/2}\]

there. Thus, adding up the terms with \( |\alpha| + |\beta| = M \), there are constants \( C_M > 0 \) such that the series is absolutely summable, and hence convergent, if for all \( M \)

\[\sum_{j=N+1}^{\infty} C_M \epsilon_j \|a_j\|_{S^{\ell_1,\ell_2,m-j,M}}\]

converges. Now, if \( \|a_j\|_{S^{\ell_1,\ell_2,m-j,M}} \leq R_{j,M} \), where \( R_{j,M} \) are specified constants, then one can arrange the convergence by for instance requiring that for \( j > M \), the summand is \( \leq 2^{-j} \), i.e. that for \( j > M \),

\[\epsilon_j \leq 2^{-j} C_{M}^{-1} R_{j,M}^{-1}.\]

Note that for each \( j \) this is finitely many constraints (as only the values of \( M \) with \( M < j \) matter), which can thus be satisfied. Correspondingly, the tail of the series converges for each \( N \) in \( S^{\ell_1,\ell_2,m-N} \), and thus \( a \in S^{\ell_1,\ell_2,m} \) and also (18) holds.

This gives a continuous asymptotic summation map on arbitrary bounded subsets of the product of the symbol spaces. (One can make the map globally defined and
continuous by letting \( \epsilon_j \) to be the minimum of, say, \( 2^{-j} C_M^{-1} (1 + \| a_j \|_{s_{\epsilon_1, \epsilon_2, \cdots, \epsilon_j, M}})^{-1} \), over \( M = 0, 1, \ldots, j - 1 \), but this is actually not important below.)

Now, let

\[
\tilde{a} \sim \sum_{\alpha} \frac{1}{\alpha!} (D_{\zeta})^\alpha \pi^\alpha a \in S^{m, \ell_1 + \ell_2}_\infty;
\]
asymptotic summation can be done so that the map \( a \mapsto \tilde{a} \) is continuous. Then \( \tilde{a} - a \in S^{\ell_1, \ell_2, m - N} \) for all \( N \), and thus

\[
I(a) - I(\tilde{a}) \in \cap_N I(S^{\ell_1, \ell_2, m - N}).
\]

If \( a \in S^{\ell_1, \ell_2, m} \) then with

\[
\tilde{a} \sim \sum_{\alpha} \frac{1}{\alpha!} (D_{\zeta})^\alpha \pi^\alpha a \in S^{m, \ell_1 + \ell_2},
\]
asymptotic sum both in the \( z \) and in the \( \zeta \) variables,

\[
I(a) - I(\tilde{a}) \in \cap_N I(S^{\ell_1, \ell_2 - N, m - N}).
\]

The following lemma then finishes the proof of Proposition 0.1:

**Lemma 0.2.** Suppose \( b \in S^{\ell_1, \ell_2, m} \) satisfies \( I(b) \in \cap_N I(S^{\ell_1, \ell_2, m - N}) \), i.e. for all \( N \in \mathbb{N} \) there is \( b_N \in S^{\ell_1, \ell_2, m - N} \) such that \( I(b) = I(b_N) \). Then there exists \( c \in S^{-\infty, \ell_1 + \ell_2} \) such that \( I(c) = I(b) \). Moreover, if there are continuous maps \( j_N : b \to b_N \), then the map \( b \to c \) is continuous.

If instead \( b \in S^{\ell_1, \ell_2, m} \) satisfies \( I(b) \in \cap_N I(S^{\ell_1, \ell_2 - N, m - N}) \), i.e. for all \( N \in \mathbb{N} \) there is \( b_N \in S^{\ell_1, \ell_2 - N, m - N} \) such that \( I(b) = I(b_N) \). Then there exists \( c \in S^{-\infty, -\infty} \) such that \( I(c) = I(b) \). Moreover, if there are continuous maps \( j_N : b \to b_N \), then the map \( b \to c \) is continuous.

The idea of the proof is to use (16), as in the present setting the Schwartz kernel can be shown to be well-behaved, so (16) immediately gives the appropriate symbolic properties of \( c \). Thus, we note that for all \( N \) there is \( b_N \in S^{\ell_1, \ell_2, m - N} \) such that \( I(b) = I(b_N) \), so taking \( N \) such that \( m - N < -n \), (14)-(15) give that the Schwartz kernel (which is independent of \( N \)) is the continuous polynomially bounded function

\[
K_{1(b_N)}(z, z') = (\mathcal{F}_\zeta^{-1} b_N)(z, z' - z);
\]
taking \( m - N < -n - k \), this is in fact \( C^k \) with polynomial bounds up to the \( k \)th derivatives. Correspondingly, it satisfies, for \( |\alpha| + |\beta| \leq k \), and writing \( D_\alpha \) for the \( \alpha \)th derivative in the \( j \)th slot, \( M_j^\alpha \) for the multiplication by the \( \alpha \)th coordinate in the \( j \)th slot,

\[
\langle z \rangle^{-\ell_1} (z')^{-\ell_2} (z - z')^\gamma D_\alpha^\gamma D_\beta^\beta K_{1(b_N)}(z, z')
\]

\[
= \langle z \rangle^{-\ell_1} (z')^{-\ell_2} M_3^\alpha (D_1 + D_3)^\alpha (D_2 - D_3)^\beta (\mathcal{F}_3^{-1} b_N)(z, z', z - z')
\]

\[
= (\mathcal{F}_3^{-1} \langle z \rangle^{-\ell_1} (z')^{-\ell_2} D_3^\alpha (D_1 + M_3)^\alpha (D_2 - M_3)^\beta b_N)(z, z', z - z').
\]

As

\[
\langle z \rangle^{-\ell_1} (z')^{-\ell_2} D_3^\alpha (D_1 + M_3)^\alpha (D_2 - M_3)^\beta b_N
\]
is bounded in \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n; L^1(\mathbb{R}_\zeta^m)) \) by a seminorm of \( b_N \) as \( |\alpha| + |\beta| \leq k \), \( m - N < -n - k \), where \( C^\infty \) stands for bounded continuous functions,

\[
\mathcal{F}_3^{-1} \langle z \rangle^{-\ell_1} (z')^{-\ell_2} D_3^\alpha (D_1 + M_3)^\alpha (D_2 - M_3)^\beta b_N
\]
is bounded in $C_\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ by a seminorm of $b_N$, hence the same holds for the pullback by the map $(z, z') \mapsto (z, z', z - z')$. Since $N$ is arbitrary, we can take arbitrary $\alpha, \beta, \gamma$ and deduce that
\[
\sup |\langle z \rangle^{-\ell_1} \langle z' \rangle^{-\ell_2} (z - z')^\gamma (D_2^\alpha D_3^\beta K_I(b))(z, z')| < \infty.
\]
Using (9) and that $\gamma$ is arbitrary, we deduce that
\[
(19) \quad \sup |\langle z \rangle^{-\ell_1 - \ell_2} (z - z')^\gamma D_2^\alpha D_3^\beta K_I(b)| < \infty.
\]
Since we want $K_{I(c)} = K_{I(b)}$, we need
\[
(F_c^{-1} - 1)(z, z') = K_{I(b)}(z, z'),
\]
i.e. with $w = z - z'$,
\[
(F_c^{-1} - 1)(z, w) = K_{I(b)}(z, z - w).
\]
Now, a linear change of variables for $K_{I(b)}$ gives that
\[
\sup |\langle z \rangle^{-\ell_1 - \ell_2} (w^\gamma D_2^\alpha D_3^\beta F_c^{-1})(z, w)| < \infty,
\]
so $\langle z \rangle^{-\ell_1 - \ell_2} D_2^\alpha D_3^\beta F_c^{-1}$ is Schwartz in $w$, uniformly in $z$, and thus $\langle z \rangle^{-\ell_1 - \ell_2} D_2^\alpha D_3^\beta$ is Schwartz in the second variable, $\zeta$, uniformly in $z$, i.e. $c \in S_{-\infty, \ell_1 + \ell_2}^{\infty}$. This also shows that any seminorm of $c$ depends only on the seminorms of $b_N$ for some $N$, and does so continuously, and thus depends on $b$ continuously.

The argument in the case of $S_{\ell_1, \ell_2, m}^{\infty}$ is completely analogous, but now even
\[
\langle z \rangle^{-\ell_1} \langle z' \rangle^{-\ell_2} (z - z')^\delta (z - z')^\gamma (D_2^\alpha D_3^\beta K_{I(b_N)})(z, z')
\]
\[
= \langle z \rangle^{-\ell_1} \langle z' \rangle^{-\ell_2} M_2^\alpha (D_1 + D_2)^\alpha (D_2 - D_3)^\beta (F_3^{-1} b_N)(z, z', z - z')
\]
\[
= \langle F_3^{-1} \rangle^{-\ell_1} \langle z' \rangle^{-\ell_2} M_2^\alpha D_3^\beta (D_1 + M_2)^\alpha (D_2 - M_3)\beta b_N)(z, z', z - z'),
\]
with the result that
\[
\sup |\langle z \rangle^{-\ell_1} \langle z' \rangle^{-\ell_2} (z - z')^\delta (z - z')^\gamma (D_2^\alpha D_3^\beta K_{I(b)})(z, z')| < \infty.
\]
Using (9) and that $\gamma, \delta$ are arbitrary, we deduce that
\[
\sup |\langle z' \rangle^\delta (z - z')^\gamma D_2^\alpha D_3^\beta K_{I(b)}| < \infty.
\]
This gives $K_{I(b)} \in \mathcal{S}(\mathbb{R}^{2n})$, and the argument is finished as before. This completes the proof of Lemma 0.2.

As a corollary of the lemma, we note that elements of $\Psi_{-\infty, \ell}$ have a $C_\infty$ Schwartz kernel, of the form $C_\infty(\mathbb{R}^n \times \mathcal{S}(\mathbb{R}^n))$, and thus give continuous linear maps $S' \rightarrow C_\infty(\mathbb{R}^n)$, i.e. are smoothing. Note that this does not mean decay at infinity. On the other hand, elements of $\Psi_{-\infty, -\infty}$ are completely regularizing, as their Schwartz kernel is in $\mathcal{S}(\mathbb{R}^{2n})$, and thus they give maps $S' \rightarrow \mathcal{S}$. Note that maps $S' \rightarrow \mathcal{S}$ are actually compact on all polynomially weighted Sobolev spaces $H_{r,s}^s$.

The isomorphism $q_L : S_m^m \rightarrow \Psi_m^m$ can be used to topologize $\Psi_m^m$. Since $q_R^{-1} \circ q_L : S_m^m \rightarrow \Psi_m^m \rightarrow \Psi_m^m$ are continuous, this is the same topology as that induced by $q_L$.

Note that if $a \in S_{\ell_1, \ell_2, m}^m$ then $\gamma^\ast a - a_L - a_R \in S_{\ell_1 - \gamma, \ell_1 + \ell_2}^{m-1}$, and if $a \in S_{\ell_1, \ell_2, m}^m$ then $\gamma^\ast a - a_L - a_R \in S_{\ell_1, \ell_1 + \ell_2, m}^{m-1}$. We thus make the following definition:

**Definition 1.** The principal symbol $\sigma_{\infty, m, \ell}(q_L(a))$ in $\Psi_m^m$ of $q_L(a)$, $a \in S_m^m$, is the equivalence class $[a]_m$ of $a$ in $S_m^m/S_m^{m-1, \ell}$.

The joint principal symbol $\sigma_{m, \ell}(q_L(a))$ in $\Psi_m^m$ of $q_L(a)$, $a \in S_m^m$, is the equivalence class $[a]$ of $a$ in $S_m^{m, \ell}/S_m^{m-1, \ell}$.
Thus, the principal symbol also satisfies
\[ \sigma_{\infty,m,\ell}(q_R(a)) = [a]_\infty, \quad \sigma_{m,\ell}(q_R(a)) = [a]. \]

For \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \subset S^{0,0} \), there is a natural identification of the equivalence class, namely the restriction of \( a \) to \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \) can be identified with its equivalence class, namely changing \( a \) by any element of \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) which vanishes in the boundary, and thus is in \( S^{-1,-1} \) does not affect the equivalence class, so the map \( a \mapsto [a] \) descends to \( a|_{\partial(\mathbb{R}^n \times \mathbb{R}^n)} \mapsto [a] \), and is injective. Note that \( \mathbb{R}^n \times \mathbb{R}^n \) is a manifold with corners with two boundary hypersurfaces, \( \partial_{\mathbb{R}^n} \times \mathbb{R}^n \) and \( \mathbb{R}^n \times \partial_{\mathbb{R}^n} \), so equivalently one can restrict to each of these separately, and keep in mind that the restrictions must agree at the corner, \( \partial_{\mathbb{R}^n} \times \partial_{\mathbb{R}^n} \).

In the case of \( \sigma_{\infty} \), a common way of understanding it is in terms of the \( \mathbb{R}^+ \)-action by dilations on the second factor of \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \):

\[ \mathbb{R}^+ \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \ni (t, z, \zeta) \mapsto (z, t\zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}). \]

The quotient of \( \mathbb{R}^n \setminus \{0\} \) by the \( \mathbb{R}^+ \) action can be identified with the unit sphere \( S^{n-1} \): every orbit of the \( \mathbb{R}^+ \)-action intersects the sphere in exactly one point. A different identification of this quotient (which is actually more relevant from the perspective of where our analysis actually takes place) is the sphere at infinity, \( \partial \mathbb{R}^n \).

Thus, homogeneous degree zero \( C^\infty \) functions on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \) are identified with either \( C^\infty(\mathbb{R}^n \times S^{n-1}) \) or \( C^\infty(\mathbb{R}^n \times \partial \mathbb{R}^n) \). So one can correspondingly identify the principal symbol of \( A = q_L(a_L) \), \( a_L \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), as a function on \( \mathbb{R}^n \times S^{n-1} \), or instead as a homogeneous degree zero function on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \).

For
\[ a = \langle z \rangle^\ell \langle \zeta \rangle^m \tilde{a}, \quad \tilde{a} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n), \]

one cannot simply restrict \( a \) to the boundary, though as (given \( \ell \) and \( m \)) \( a \) and \( \tilde{a} \) are in a bijective correspondence, one could restrict \( \tilde{a} \) and call it the principal symbol, i.e. the actual principal symbol, as we defined it, is given by any \( C^\infty \) extension of this restriction times \( \langle z \rangle^\ell \langle \zeta \rangle^m \). In a more geometric context this is not quite natural (depends on the differentials of choices of boundary defining functions, here \( \langle z \rangle^{-1} \) and \( \langle \zeta \rangle^{-1} \), at the boundary). Taking \( \ell = 0 \) as it is the most common case, in terms of the \( \mathbb{R}^+ \) action on the second factor, it is more common then to view the part of the principal symbol corresponding to \( \partial \mathbb{R}^n \times \mathbb{R}^n \) as a homogeneous degree \( m \) function on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \). In terms of \( \tilde{a} \) and its identification with a homogeneous degree zero function on \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \), the part of the principal symbol corresponding to \( \partial \mathbb{R}^n \times \mathbb{R}^n \) is

\[ \sigma_{\text{fiber},m,0}(A) = \langle \zeta \rangle^m \tilde{a}. \]

On the other hand, the part of the principal symbol corresponding to \( \partial \mathbb{R}^n \times \mathbb{R}^n \) can be described by simply restricting to \( \partial \mathbb{R}^n \times \mathbb{R}^n \), with the result being symbolic in the second variable:

\[ \sigma_{\text{base},m,0}(A) = \langle \zeta \rangle^m \tilde{a}|_{\partial \mathbb{R}^n \times \mathbb{R}^n}. \]

Concretely, if \( A \) is a differential operator, \( A = \sum a_\alpha D^\alpha, \quad a_\alpha \in C^\infty(\mathbb{R}^n) \), then the two parts of the principal symbol under this identification are

\[ \sigma_{\text{fiber},m,0}(A)(z, \zeta) = \sum_{|\alpha| = m} a_\alpha(z)\zeta^\alpha, \quad (z, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \quad (20) \]

\[ \sigma_{\text{base},m,0}(A)(z, \zeta) = \sum_{|\alpha| = m} a_\alpha(z)\zeta^\alpha, \quad (z, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}). \]
satisfies the corresponding symbol estimates in $\sigma(21)$ and their left amplitudes $S$. This is the reason for this being a less useful point of view in this case than in that of $\sigma$ views would be needed for describing $\sigma$ on $(\mathbb{R}^n \times \mathbb{R}^n)$; the last version would be rarely considered. Thus, two different point of views would be needed for describing $\sigma$ in terms of homogeneous functions, which is the reason for this being a less useful point of view in this case than in that of $\sigma_\infty$.

**Proposition 0.3.** The sequences

\[ 0 \to \Psi^{m-1,\ell} \to \Psi^{m,\ell} \to S^{m,\ell}/S^{m-1,\ell} \to 0, \]

resp.

\[ 0 \to \Psi^{m-1,\ell-1} \to \Psi^{m,\ell} \to S^{m,\ell}/S^{m-1,\ell-1} \to 0, \]

are short exact sequences of topological vector spaces.

Here $\iota : \Psi^{m-1,\ell-1} \to \Psi^{m,\ell}$ is the inclusion map and

\[ \sigma_{m,\ell} : \Psi^{m,\ell} \to S^{m,\ell}/S^{m-1,\ell-1} \]

is the principal symbol map, with analogous definitions in the case of $\Psi_\infty$.

This is essentially tautological, given the short exact sequence

\[ 0 \to S^{m-1,\ell-1} \to S^{m,\ell} \to S^{m,\ell}/S^{m-1,\ell-1} \to 0, \]

and the isomorphisms $q_{L,m,\ell} : S^{m',\ell'} \to \Psi^{m',\ell'}$ with $m' = m, m - 1, \ell' = \ell, \ell - 1$, and that these are consistent with the inclusion $\iota_S : S^{m-1,\ell-1} \to S^{m,\ell}$, i.e. that one has a commutative diagram $q_{L,m,\ell} \circ \iota_S = \ell \circ q_{L,m-1,\ell-1}$.

We also define operator wave front sets. We first start with the microsupport of symbols:

**Definition 2.** Suppose $a \in S^{m,\ell}(\mathbb{R}^n, \mathbb{R}^n)$. We say that $a \in \partial(\mathbb{R}^n \times \mathbb{R}^n)$ is not in esssupp($a$) if there is a neighborhood $U$ of $a$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that $a|_{U \cap (\mathbb{R}^n \times \mathbb{R}^n)}$ is $S = S^{-\infty,-\infty}$ (i.e. satisfies Schwartz estimates in $U$).

Similarly, for $a \in S^{m,\ell}(\mathbb{R}^n, \mathbb{R}^n)$ we say that $a \in \mathbb{R}^n \times \partial(\mathbb{R}^n)$ is not in esssupp$_{\infty,\ell}(a)$ if there is a neighborhood $U$ of $a$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that $a|_{U \cap (\mathbb{R}^n \times \mathbb{R}^n)}$ is $S^{-\infty,\ell}$ (i.e. satisfies the corresponding symbol estimates in $U$).

In either case, esssupp is called the microsupport, or essential support, of $a$.

Now for operators we define the wave front set in terms of the microsupport of their left amplitudes $a_L$. 

\[ (21) \quad \sigma_{\text{base},m,0}(A)(z, \zeta) = \sum_{|\alpha| \leq m} a_\alpha(z)\zeta^\alpha, \quad (z, \zeta) \in \partial(\mathbb{R}^n \times \mathbb{R}^n). \]
Definition 3. Suppose that $A \in \Psi^{m,\ell}$, $A = q_L(a_L)$. We write

$$WF'(A) = \text{esssupp}(a),$$

i.e. we say that $\alpha \in \partial (\mathbb{R}^n \times \mathbb{R}^n)$ is not in $WF'(A)$ if there is a neighborhood $U$ of $\alpha$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that $a_L|_{U\cap(\mathbb{R}^n \times \mathbb{R}^n)}$ is $S^{-\infty,-\infty}$ (i.e. satisfies Schwartz estimates in $U$).

Similarly, for $A \in \Psi^\infty_{m,\ell}$, we write $WF^\infty_{\infty,\ell}(A) = \text{esssupp}_{\infty,\ell}(A)$.

Note that directly from the definition, the complement of esssupp, and thus the wave front set, is open, i.e. the wave front set itself is closed. Further, even for $WF^\infty_{\infty,\ell}$, $\ell$ is only relevant for $\alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n)$; one commonly simply writes $WF'$, or indeed $WF''$. While the principal symbol captures the leading order behavior of a pseudodifferential operator, the (complement of the) wave front set captures where it is ‘trivial’.

As an example, if $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $A = q_L(a)$, then $WF'(A) \subseteq \text{supp} a \cap \partial(\mathbb{R}^n \times \mathbb{R}^n)$, since certainly in the complement of supp $a$ vanishes, and is thus a symbol of order $-\infty,-\infty$. However, notice that the containment is not an equality, as e.g. $a \in S(\mathbb{R}^2)$ which never vanishes on $\mathbb{R}^2$ (e.g. a Gaussian) has support everywhere, but $WF'(q_L(a)) = \emptyset$. Thus, the more precise statement is that $\alpha \notin WF'(A)$ for such $a, A$, if $\alpha$ has a neighborhood $U$ in $\partial(\mathbb{R}^n \times \mathbb{R}^n)$ on which the full Taylor series of $a$ vanishes.

Again, as in the case of the principal symbol, one could consider $WF^\infty_{\infty,\ell}$ a subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ which is invariant under the $\mathbb{R}^+$-action (dilations in the second factor), i.e. which is conic; this is the standard point of view. The corresponding statement for $WF'$ is, as in the case of the principal symbol, more awkward, and is thus less common.

In view of Proposition 0.1, one could also use $a_R$ with $A = q_R(a_R)$ in place of $a_L$ in the definition. Also, as $\partial(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathbb{R}^n \times \partial(\mathbb{R}^n)$ are compact, so symbol estimates corresponding to an open cover imply symbol estimates everywhere, so

**Lemma 0.4.** If $A \in \Psi^m_{\infty,\ell}$ and $WF'(A) = \emptyset$, then $A \in \Psi^{-\infty,-\infty}_{m,\ell}$.

If $A \in \Psi^m_{\infty,\ell}$ and $WF^\infty_{\infty,\ell}(A) = \emptyset$, then $A \in \Psi^{-\infty,\ell}_{m,\ell}$.

We also have from (6) that

**Proposition 0.5.** If $A \in \Psi^m_{\infty,\ell}$ then $A^* \in \Psi^m_{\infty,\ell}$ and

$$\sigma_{\infty,m,\ell}(A^*) = \sigma_{\infty,m,\ell}(A), \ WF'(A^*) = WF'(A).$$

If $A \in \Psi^m_{\infty,\ell}$ then $A^* \in \Psi^m_{\infty,\ell}$ and

$$\sigma_{m,\ell}(A^*) = \sigma_{m,\ell}(A), \ WF'(A^*) = WF'(A).$$

We can also strengthen the surjectivity part of Proposition 0.3:

**Proposition 0.6.** For $a \in S^m_{\infty,\ell}$ there exists $A \in \Psi^m_{\infty,\ell}$ with $\sigma_{\infty,m,\ell}(A) = [a]$ and $WF'(A) \subseteq \text{esssupp}_{\infty,\ell} a$.

Similarly, for $a \in S^m_{\infty,\ell}$ there exists $A \in \Psi^m_{\infty,\ell}$ with $\sigma_{m,\ell}(A) = [a]$ and $WF'(A) \subseteq \text{esssupp} a$.

Indeed, taking $A = q_L(a)$ or $A = q_R(a)$ will do the job.

The most important part of a treatment of pseudodifferential operators is their properties under composition and commutators:
Proposition 0.7. If $A \in \Psi_m^\ell$, $B \in \Psi_m^{m',\ell'}$, then $AB \in \Psi_m^{m+m',\ell+\ell'}$, 

$$\sigma_{\infty,m+m',\ell+\ell'}(AB) = \sigma_{\infty,m,\ell}(A)\sigma_{\infty,m',\ell'}(B),$$

and

$$\WF'_\infty(AB) \subset \WF'_\infty(A) \cap \WF'_\infty(B).$$

If $A \in \Psi_m^\ell$, $B \in \Psi_m^{m',\ell'}$, then $AB \in \Psi_m^{m+m',\ell+\ell'}$, and

$$\sigma_{m+m',\ell+\ell'}(AB) = \sigma_{m,\ell}(A)\sigma_{m',\ell'}(B),$$

and

$$\WF'(AB) \subset \WF'(A) \cap \WF'(B).$$

Thus, $\Psi_\infty$ and $\Psi$ are order-filtered $*$-algebras, and in case of $\Psi_\infty$, composition is commutative to leading order in terms of the differential order, $m$, while in the case of $\Psi$, it is commutative to leading order in both the differential and the growth orders $m$ and $\ell$.

This proposition is proved easily using Proposition 0.1, taking advantage of (11) and (12). To do so, first assume $a, b \in S_\infty^{-\infty,-\infty}$, then

$$(q_L(a)q_R(b))u(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz \cdot \zeta} a(z, \zeta) \left( \mathcal{F}\mathcal{F}^{-1}(\zeta') \mapsto \int_{\mathbb{R}^n} e^{-iz' \cdot \zeta'} b(z', \zeta') u(z') dz' \right) d\zeta$$

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(z - z') \cdot \zeta} a(z, \zeta) b(z', \zeta) u(z') dz' d\zeta = (I(c)u)(z),$$

with

$$c(z, z', \zeta) = a(z, \zeta) b(z', \zeta) \in S_\infty^{-\infty,-\infty}.$$ 

However, with $c = c(a, b)$ so defined, the map

$$S_m^{m,\ell} \times S_m^{m',\ell'} \ni (a, b) \mapsto c \in S_\infty^{\ell,\ell',m+m'}$$

is continuous, so as both trilinear maps

$$(a, b, u) \mapsto q_L(a)q_R(b)u, \quad (a, b, u) \mapsto I(c(a, b))u$$

are continuous

$$S_m^{m,\ell} \times S_m^{m',\ell'} \times \mathcal{S} \to \mathcal{S}$$

for all $m, m', \ell, \ell'$, it follows that

$$q_L(a)q_R(b) = I(c(a, b)).$$

Since $q_L$, $q_R$ are isomorphisms, the closedness of $\Psi_m^{m,\ell}$ under composition is immediate, as is the continuity of composition. As for the principal symbol, this follows as if $B \in \Psi_m^{m',\ell'}$, $B = q_R(b)$, then $\sigma_{\infty,m',\ell'}(B) = b$, and then by (10), $I(c(a, b)) = q_L(c_L)$ with $c_L - ab \in S_\infty^{m+m'-1,\ell+\ell'}$. The wave front set statement is also immediate in view of (10).

In the case of $\Psi$, the same arguments go through, but corresponding to the improvement in (10), $c_L - ab \in S_\infty^{m+m'-1,\ell+\ell'-1}$.

Going one order further in the expansions, one obtains the principal symbol of the commutators. Here we recall the Poisson bracket on $\mathbb{R}^n_z \times \mathbb{R}^n_\zeta$, identified with $T^*\mathbb{R}^n$:

$$\{a, b\} = \sum_{j=1}^n \left( (\partial_\zeta_j a)(\partial_z_j b) - (\partial_z_j a)(\partial_\zeta_j b) \right).$$
Proposition 0.8. If $A \in \Psi_{m,\ell}^m$, $B \in \Psi_{m',\ell'}^m$, then $[A, B] \in \Psi_{m+m' -1,\ell+\ell'}^m$, and
\[ \sigma_{m+m' -1,\ell+\ell'}^m(AB) = \frac{1}{i}\{\sigma_{m,\ell}(A), \sigma_{m',\ell'}(B)\}. \]

If $A \in \Psi_{m,\ell}^m$, $B \in \Psi_{m',\ell'}^m$, then $[A, B] \in \Psi_{m+m' -1,\ell+\ell'-1}^m$, and
\[ \sigma_{m+m' -1,\ell+\ell'-1}(AB) = \frac{1}{i}\{\sigma_{m,\ell}(A), \sigma_{m',\ell'}(B)\}. \]

We now turn to the simplest consequences of the machinery we built up, such as the parametrix construction for elliptic operators.

Definition 4. We say that $A$ is elliptic in $\Psi_{m,\ell}^m$, resp. $\Psi_{m',\ell'}^m$, if $[a]_\infty$, resp. $[a]_m$, is invertible, i.e. if there exists $[b]_\infty \in S_{m,\ell}^{m,\ell}$, resp. $[b]_\infty \in S_{m,\ell}^{m,\ell}$, with $[a]_\infty [b]_\infty = [1]$ in $S^{0,0}/S^{1,1}$, resp. $[a]_\infty [b]_\infty = [1]$ in $S^{0,0}/S^{1,1}$.

These definitions are equivalent to the statements that there exist $c > 0$, $R > 0$ such that
\[ |a| \geq c(z)^\ell (\zeta)^m, \quad c > 0, |\zeta| > R, \]
resp.
\[ |a| \geq c(z)^\ell (\zeta)^m, \quad c > 0, |\zeta| + |z| > R; \]
indeed, if $a$ satisfies this, the reciprocal is easily seen to satisfy the appropriate conditions in $|\zeta| > R$, resp. $|z| + |\zeta| > R$, and the multiplying by a cutoff, identically 1 near infinity, in $\zeta$, resp. $(z, \zeta)$, gives $b$. Conversely, if $b$ exists, upper bounds for $|b|$ give the desired lower bounds for $|a|$.

Concretely, if $A = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ as in (1), then under the identification of the part of the principal symbol at $\mathbb{R}^n \times \partial \mathbb{R}^n$ with a homogeneous degree $m$ function on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, while identifying the principal symbol at $\partial \mathbb{R}^n \times \mathbb{R}^n$ as an $m$th order symbol on $\partial \mathbb{R}^n \times \mathbb{R}^n$, ellipticity means:
\[ z \in \mathbb{R}^n, \zeta \neq 0 \Rightarrow \sum_{|\alpha| = m} a_\alpha \zeta^\alpha \neq 0, \]
and
\[ z \in \partial \mathbb{R}^n, \zeta \in \mathbb{R}^n \Rightarrow \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha \neq 0. \]

For $H = \Delta_g + V - \sigma$ as in (22), ellipticity does means
\[ (z, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}), \zeta \neq 0 \Rightarrow \sum g_{ij}(z) \zeta_i \zeta_j \neq 0, \]
\[ (z, \zeta) \in \partial \mathbb{R}^n \times \mathbb{R}^n \Rightarrow \sum g_{ij}(z) \zeta_i \zeta_j - \sigma \neq 0. \]

Now the first is just the statement that $g$ is a Riemannian metric on $\mathbb{R}^n$ in the uniform sense we discussed; the second holds if and only if $|\sigma| \neq 0$. Note that if $V \in S^{-\rho}(\mathbb{R}^n)$ instead, $\rho > 0$, then $V$ does affect the principal symbol in the second sense, but it does not affect ellipticity.

If $A$ is elliptic in $\Psi_{m,\ell}^m$, say, then one can construct a parametrix $B$ with a residual, or completely regularizing, error, i.e. $B \in \Psi_{-m,-\ell}^\infty$ such that
\[ AB - \text{Id}, BA - \text{Id} \in \Psi_{-\infty,-\infty}^\infty. \]

Indeed, one takes any $B_0$ with $\sigma_{-m,-\ell}(B_0)$ being the inverse for $\sigma_{m,\ell}(A)$, so
\[ \sigma_{0,0}(AB_0 - \text{Id}) = \sigma_{m,\ell}(A)\sigma_{-m,-\ell}(B_0) - 1 = 0, \]
thus \( E_0 = AB_0 - \text{Id} \in \Psi^{-1,-1} \). Now, \( AB_0 = \text{Id} + E_0 \), so one wants to invert \( \text{Id} + E_0 \) approximately; this can be done by a finite Neumann series, \( \text{Id} + \sum_{j=1}^{N} (-1)^j E_0^j \), then \( (\text{Id} + E_0)(\text{Id} + \sum_{j=1}^{N} (-1)^j E_0^j) - \text{Id} \in \Psi^{-N^{-1},-N^{-1}} \). This can be improved by writing \( E_0^j = q_L(e_j) \), then computing the asymptotic sum

\[
\hat{e} \sim \sum_{j=1}^{\infty} (-1)^j e_j \in S^{-1,-1},
\]

taking \( \hat{E} = q_L(\hat{e}) \), \( (\text{Id} + E_0)(\text{Id} + \hat{E}) - \text{Id} \in \Psi^{-\infty, -\infty} \), so \( B = B_0(\text{Id} + \hat{E}) \) provides a right parametrix: \( B = AB - \text{Id} \in \Psi^{-\infty, -\infty} \). A left parametrix \( B' \) can be constructed similarly, and the standard identities showing the identity of left and right inverses in a semigroup, as applied to the quotient by completely regularizing operators, shows that \( B - B' \in \Psi^{-\infty, -\infty} \), so one may simply replace \( B' \) by \( B \). Indeed, if \( B'A = \text{Id} + E' \),

\[
B' = B'(AB - E) = (B'A)B - B'E = B - E'B - B'E,
\]

\[
B'E, E'B' \in \Psi^{-\infty, -\infty}.
\]

Notice that all of the constructions can be done uniformly as long (24) is satisfied for a fixed \( c \) and \( R \), i.e. one can construct the maps \( A \mapsto B, E \) such that they are continuous from the set of elliptic operators to \( \Psi^{-m, -\ell} \) resp. \( \Psi^{-\infty, -\infty} \).

If \( A \in \Psi_{\infty}^{m, \ell} \) then the same argument only gains in the first order, \( m \), so one obtains a parametrix \( B \in \Psi^{-m, -\ell} \) with errors \( E, E' \in \Psi_{\infty}^{-0,0} \).

We have thus proved:

**Proposition 0.9.** If \( A \in \Psi^{m, \ell} \) is elliptic then there exists \( B \in \Psi^{-m, -\ell} \) such that \( AB - \text{Id}, BA - \text{Id} \in \Psi^{-\infty, -\infty} \). Further, the maps \( A \mapsto B \in \Psi^{-m, -\ell} \) and \( A \mapsto AB - \text{Id}, BA - \text{Id} \in \Psi^{-\infty, -\infty} \) can be taken to be continuous from the set of elliptic operators in \( \Psi^{m, \ell} \) (an open subset of \( \Psi^{m, \ell} \)), equipped with the \( \Psi^{m, \ell} \) topology.

If \( A \in \Psi_{\infty}^{m, \ell} \) is elliptic then there exists \( B \in \Psi_{\infty}^{-m, -\ell} \) such that \( AB - \text{Id}, BA - \text{Id} \in \Psi_{\infty}^{-\infty,0} \). Again, the maps \( A \mapsto B \in \Psi_{\infty}^{-m, -\ell} \) and \( A \mapsto AB - \text{Id}, BA - \text{Id} \in \Psi_{\infty}^{-0,0} \) can be taken to be continuous from the set of elliptic operators in \( \Psi_{\infty}^{m, \ell} \).

If \( A \in \Psi^{m, \ell} \) elliptic is invertible in the weak sense that there exist \( G : S \rightarrow S' \) continuous such that \( GA = \text{Id} : S \rightarrow S \) and \( AG = \text{Id} : S \rightarrow S \) then, with \( B \) a parametrix for \( A, BA - \text{Id} = E_L, AB - \text{Id} = E_R \),

\[
G = G(AB + E_R) = B + GE_R = B + (BA + E_L)GE_R = B + BE_R + E_LGE_R,
\]

with the first two terms on the right in \( \Psi^{-m, -\ell} \), resp. \( \Psi^{-\infty, -\infty} \), and the last term is residual as well since it is a continuous linear map \( S' \rightarrow S \), and thus has Schwartz kernel in \( S(\mathbb{R}^{2n}) \), thus is in \( \Psi^{-\infty, -\infty} \). Hence \( G \in \Psi^{-m, -\ell} \), and \( G - B \in \Psi^{-\infty, -\infty} \). Thus, the inverses of actually invertible elliptic operators are pseudodifferential operators themselves.

As a corollary we have elliptic regularity:

**Proposition 0.10.** If \( A \in \Psi^{m, \ell} \) is elliptic and \( Au \in S \) for some \( u \in S' \) then \( u \in S \).

**Proof.** Let \( B \) be a parametrix for \( A \) with \( BA - \text{Id} = E \in \Psi^{-\infty, -\infty} \). Then

\[
u = \text{Id} u = (BA + E)u = B(Au) + Eu,
\]

and \( Eu \in S \) as \( E \) is completely regularizing while \( Au \in S \) by assumption, hence \( B(Au) \in S \) as well. \( \square \)
We can now discuss Hörmander’s proof of $L^2$-boundedness of elements of $\Psi^{0,0}$, or indeed $\Psi^{0,0}_\infty$, via a square root construction.

**Lemma 0.11.** Suppose that $A \in \Psi^{0,0}_\infty$ is elliptic, symmetric ($A^* = A$) with principal symbol that has a positive (bounded below) representative $a$. Then there exists $B \in \Psi^{0,0}_\infty$ such that $B$ is symmetric and $A = B^2 + E$ with $E \in \Psi^{-\infty,0}_\infty$. The maps $A \mapsto B \in \Psi^{0,0}_\infty$ and $A \mapsto E \in \Psi^{-\infty,0}_\infty$ can be taken continuous from the set of $A$ satisfying these constraints (equipped with the $\Psi^{0,0}_\infty$ topology).

The same result holds with the $\infty$ subscript dropped, but with $E \in \Psi^{-\infty,-\infty}$.

**Proof.** Let $b_0 = \sqrt{a}$; one easily checks that $b_0 \in S^{0,0}_\infty$. Let $\tilde{B}_0 \in \Psi^{0,0}_\infty$ have principal symbol $b_0$, and let $B_0 = \frac{1}{2}(\tilde{B}_0 + \tilde{B}_0)^*$, so $B_0$ still has principal symbol $b_0$ and is symmetric. Then $A - B_0^2$ has vanishing principal symbol, so $E_0 = A - B_0^2 \in \Psi^{-1,0}_\infty$, providing the first step in the construction.

In general, for $j \in \mathbb{N}$, suppose one has found $B_j \in \Psi^{0,0}_\infty$ symmetric such that $E_j = A - B_j^2 \in \Psi^{-j-1,0}_\infty$; we have shown this for $j = 0$. Let $e_j$ be the principal symbol of $E_j$, and let $b_{j+1} = -\frac{1}{2b_0}e_j \in S^{j-1,0}_\infty$; this uses $b_0$ elliptic. Let $\tilde{B}_{j+1} \in \Psi^{-j-1,0}$ have principal symbol $b_{j+1}$, $B_{j+1} = 1/2(\tilde{B}_{j+1} + \tilde{B}^*_{j+1})$, $B_{j+1} = B_j + B'_{j+1}$, so $B_{j+1}$ is symmetric. Further, the principal symbol of

$$A - B_{j+1}^2 = A - (B_j + B'_{j+1})^2 = A - B_j^2 - B_j B'_{j+1} - B'_{j+1} B_j - (B'_{j+1})^2$$

$$= E_j - B_j B'_{j+1} - B'_{j+1} B_j - (B'_{j+1})^2 \in \Psi^{-j-1,0}_\infty$$

is $e_j - 2b_0 b_{j+1} = 0$, so $E_{j+1} = A - B_{j+1}^2 \in \Psi^{-j-2,0}_\infty$, providing the inductive steps. One can finish up by asymptotically summing, as in the elliptic case. \(\square\)

**Proposition 0.12.** Elements $A \in \Psi^{0,0}_\infty$ are bounded on $L^2$.

Further, if $a$ is a representative of $\sigma_{\infty,0,0}(A)$ and $C > \inf_{r \in S^{-1,0}_\infty} \sup |a + r|$ then there exists $E \in \Psi^{-\infty,0}_\infty$ such that

$$\|Au\|_{L^2} \leq C\|u\|_{L^2} + |\langle Eu, u \rangle|.$$ 

Moreover, the map $A \mapsto E \in \Psi^{-\infty,0}_\infty$ can be taken to be continuous, and thus the inclusion $\Psi^{0,0}_\infty \rightarrow L(L^2)$ is continuous.

**Proof.** We reduce the proof to the boundedness of elements of $\Psi^{-\infty,0}_\infty$ on $L^2$, which is in easy consequence of Schur’s lemma since by (19), the Schwartz kernel of elements of this space is a bounded continuous function in $z$ with values in $S(R^n_2)$ (hence with values in $L^1(R^n_2)$), and similarly with $z$ and $z'$ interchanged.

Now, suppose that $A \in \Psi^{0,0}_\infty$, so its principal symbol has a bounded representative $a$; let $M > \sup |a|$. Then $M^2 - |a|^2 \in S^{0,0}_\infty$ is bounded below by a positive constant, and is thus elliptic. By Lemma 0.11, there exists $B \in \Psi^{0,0}_\infty$ symmetric such that $M^2 - A^* A = B^2 + E$, $E \in \Psi^{-\infty,0}_\infty$. Then, first for $u \in S$, with inner products and norms the standard $L^2$ ones,

$$\langle M^2 u, u \rangle = \|Au\|^2 + \|Bu\|^2 + \langle Eu, u \rangle,$$

i.e. with $\|E\|_{L(L^2)}$ the $L^2$ bound of $E$, which is finite as discussed above,

$$\|Au\|^2 \leq M^2 \|u\|^2 + \|E\|_{L(L^2)} \|u\|^2.$$

Since $S$ is dense in $L^2$, this implies that $A$ has a unique continuous extension to $L^2$; one still denotes it by $L^2$. Since $S$ is also dense in $S'$, and the inclusion $L^2 \rightarrow S'$ is
continuous, this extension is the restriction of \( A \) acting on \( S' \). This proves the first part of the proposition.

For the second part we simply replace \( a \) by \( a + r \), choosing \( r \in S^{1,0}_\infty \) such that \( C > \sup |a + r| \), then we can take \( M = C \) in the argument above to complete the proof. \( \square \)

While elements of \( \Psi^{0,0} \) are in \( \Psi^{0,0}_0 \) and are thus \( L^2 \)-bounded, it is useful to make the bound more explicit there as well:

**Proposition 0.13.** Elements \( A \in \Psi^{0,0} \) are bounded on \( L^2 \).

Further, if \( a \) is a representative of \( \sigma_{0,0}(A) \) and \( C > \inf_{r \in S_0} \sup |a + r| \) then there exists \( E \in \Psi^{-\infty, -\infty} \) such that

\[
(27) \quad \| Au \|_{L^2} \leq C \| u \|_{L^2} + |\langle Eu, u \rangle|.
\]

Moreover, the map \( A \mapsto E \in \Psi^{-\infty, -\infty} \) can be taken to be continuous.

Concretely, if \( A = q_L(a) \) with \( a \in C_\infty(\mathbb{R}^\infty \times \mathbb{R}^\infty) \), then for any

\[
C > \sup |a|_{\partial_{\mathbb{R}^\infty \times \mathbb{R}^\infty}},
\]

(27) holds.

**Proof.** This is the same argument as above, but constructing \( B \) in \( \Psi^{0,0} \). \( \square \)

We now recall that the weighted Sobolev spaces are

\[
H^{r,s} = \{ u \in S' : \langle z \rangle^s u \in H^r \}, \quad \| u \|_{H^{r,s}} = \| \langle z \rangle^s u \|_{H^r}
\]

Further, with

\[
\Lambda_r = F^{-1}(\zeta)^r F \in \Psi^{0,0} \subset \Psi_{\infty}^{0,0},
\]

\[
H^r = \{ u : \Lambda_r u \in L^2 \text{ with } \| u \|_{H^r} = \| \Lambda_r u \|_{L^2} \}. \text{ We note here } \cup_{M,N \in \mathbb{R}} H^{M,N} = S'.
\]

\[
\text{Thus, } \Lambda_{r,s} = \Lambda_r(\zeta)^s : H^{r,s} \rightarrow L^2 \text{ is an isometry, with inverse } \Lambda_{r,-s} = \langle z \rangle^{-s} \Lambda_{-r} : L^2 \rightarrow H^{r,s}.
\]

Hence, the boundedness of some \( A \in \Psi^{m,\ell}_\infty \) as a map \( H^{r,s} \rightarrow H^{r',s'} \) is equivalent to the boundedness on \( L^2 \) of \( \Lambda_{r,s} A \Lambda_{r,-s} = A \in \Lambda_{r,-s}(\Lambda_{r,s} A \Lambda_{r,-s}) \Lambda_{r,s} \). But \( \Lambda_{r,s} A \Lambda_{r,-s} \in \Psi_{\infty}^{m+r-r',s+s'-s} \), so we conclude that

**Proposition 0.14.** An operator \( A \in \Psi^{m,\ell}_\infty \) is bounded \( H^{r,s} \rightarrow H^{r',s'} \) if \( m \leq r - r' \) and \( \ell \leq s - s' \) (thus if \( m \leq r - r' \) and \( \ell \leq s - s' \)).

This gives a quantified version of elliptic regularity:

**Proposition 0.15.** If \( A \in \Psi^{m,\ell}_\infty \) is elliptic and \( Au \in H^{r,s} \) for some \( u \in S' \) then \( u \in H^{r+m,s+\ell} \). In fact, for any \( M,N \) there is \( C > 0 \) such that

\[
\| u \|_{H^{r+m,s+\ell}} \leq C(\| Au \|_{H^{r,s}} + \| u \|_{H^{M,N}}).
\]

If \( A \in \Psi^{m,\ell}_\infty \) is elliptic and \( Au \in H^{r,s} \) for some \( u \in H^{k,s+\ell} \), \( k \in \mathbb{R} \), then \( u \in H^{r+m,s+\ell} \). In fact, for any \( k \) there is \( C > 0 \) such that

\[
\| u \|_{H^{r+m,s+\ell}} \leq C(\| Au \|_{H^{r,s}} + \| u \|_{H^{k,s+\ell}}).
\]

The point of the quantitative estimate is to allow \( M,N \) very negative, so e.g. \( H^{r+m,s+\ell} \rightarrow H^{M,N} \) is compact. One thinks of \( \| u \|_{H^{M,N}} \) as a ‘trivial’ term correspondingly.

In the case of \( \Psi^{m,\ell}_\infty \) ellipticity is too weak of a notion to gain decay at infinity; one simply has a uniform gain of Sobolev regularity.
Proof. Suppose $A \in \Psi^{m,\ell}$. Let $B \in \Psi^{-m,\ell}$ be a parametrix for $A$ with $BA - \Id = E \in \Psi^{-\infty,\infty}$. Then
\[ u = \Id u = (BA + E)u = B(Au) + Eu, \]
and $Eu \in S$ while $Au \in H^{r,s}$ by assumption, hence $B(Au) \in H^{r+m,s+\ell}$, as claimed. The bound in the proposition follows from $E : H^{M,N} \to H^{r+m,s+\ell}$ being bounded.

If $A \in \Psi^{m,\ell}_\infty$, and $B \in \Psi^{-m,\ell}_\infty$ is a parametrix, then $BA - \Id = E \in \Psi^{-\infty,0}$ then the same argument gives, using $E : H^{k,s+\ell} \to H^{r+m,s+\ell}$ bounded, the conclusion that $u \in H^{r+m,s+\ell}$, as well as the estimate. □

An immediate corollary is:

**Proposition 0.16.** Any elliptic $A \in \Psi^{m,\ell}$ is Fredholm as a map $H^{r,s} \to H^{r-m,s-\ell}$ for all $m, \ell, r, s \in \mathbb{R}$, i.e. has closed range, finite dimensional nullspace and the range has finite codimension. Further, the nullspace is a subspace of $S$, while the annihilator of the range in $H^{r-m,s-\ell}$ in the dual space $H^{r+m,s+\ell}$ is also in $S$. Correspondingly, the nullspace of $A$ as well as the annihilator of its range is independent of $r, s$; if $A$ is invertible for one value of $r, s$, then it is invertible for all.

Proof. If $B$ is a parametrix for $A$, then $B \in \mathcal{L}(H^{r-m,s-\ell}, H^{r,s})$ and $E_L = BA - \Id, E_R = AB - \Id \in \Psi^{-\infty,\infty}$, thus map $H^{r,s}$, resp. $H^{r-m,s-\ell}$ to $S$ continuously, and are thus compact as maps in $\mathcal{L}(H^{r,s})$, resp. $\mathcal{L}(H^{r-m,s-\ell})$. Then standard arguments give the Fredholm property.

The property of the nullspace being in $S$ is elliptic regularity. If $v$ is in the annihilator as stated, i.e. $\langle v, Au \rangle = 0$ for all $u \in H^{r,s}$ then $\langle A^*v, u \rangle = 0$ for all $u \in H^{r,s}$, so $A^*v = 0$ in $H^{-r,-s}$. As $A^*$ has principal symbol $a$, elliptic regularity shows that $v \in S$. □

**Corollary 0.17.** Suppose $m, \ell > 0$, $A \in \Psi^{m,\ell}$ is symmetric on $L^2$ and is elliptic. Then $A$ is self-adjoint with domain $H^{m,\ell}$.

Proof. It suffices to show that $A - \sigma : H^{m,\ell} \to L^2$ are invertible for $\sigma \in \mathbb{C} \setminus \mathbb{R}$. As $m, \ell > 0$, these are elliptic regardless of $\sigma$, thus Fredholm as stated, with nullspace and annihilator of the cokernel in $S$. But the symmetry of $A$ shows that for $u$ in the kernel, $0 = \Im \langle (A - \sigma)u, u \rangle = -\Im \sigma \|u\|^2$, so $u = 0$, hence the kernel is trivial. Thus, the kernel of $A^* = A$ is also trivial, so $A$ is surjective, thus the desired invertibility follows. □

**Corollary 0.18.** Suppose $m \geq 0$, $\ell \geq 0$, $A \in \Psi^{m,\ell}$ is symmetric on $L^2$ and is elliptic. Then $A$ is self-adjoint with domain $H^{m,\ell}$.

Proof. We have already dealt with $m, \ell > 0$; $m, \ell = 0$ is standard, so it remains to deal with $m > 0$, $\ell = 0$ as $m = 0$, $\ell > 0$ is similar. Again, it suffices to show that $A - \sigma : H^{m,\ell} \to L^2$ are invertible for $\sigma \in \mathbb{C} \setminus \mathbb{R}$. As the principal symbol has a real representative $a$,
\[ |a - \sigma|^2 = |a - \Re \sigma|^2 + |\Im \sigma|^2 \geq c|\zeta|^{2m}, \quad c > 0, \]
since $|a| \geq c_0|\zeta|^m$, $c > 0$, so for $|a| \geq 2|\Re \sigma|$, $|a - \Re \sigma|^2 \geq (|a| - |\Re \sigma|)^2 \geq |a|^2/4$, while for $|a| \leq 2|\Re \sigma|$, $|\zeta|^m \leq 2c_0^{-1}|\Re \sigma|$, so $\zeta$ is bounded, and then the $\Im \sigma$ term gives the desired positivity. So $A - \sigma$ is elliptic when $\Im \sigma \neq 0$, thus Fredholm as stated, with nullspace and annihilator of the cokernel in $S$. Again, the symmetry of $A$ shows that for $u$ in the kernel, $0 = \Im \langle (A - \sigma)u, u \rangle = -\Im \sigma \|u\|^2$, so $u = 0$,
hence the kernel of $A - \sigma$ is trivial. Thus, the kernel of $A^* = A$ is also trivial, so $A$ is surjective, thus the desired invertibility follows. \hfill \square

We summarize our results so far for the Schrödinger operators:

**Proposition 0.19.** Let $g$ be a Riemannian metric, $g_{ij} \in \mathcal{C}^\infty(\mathbb{R}^n)$, positive definite on $\mathbb{R}^n$, $V \in S^{-\rho}(\mathbb{R}^n)$ with $\rho > 0$. Let $H = \Delta_g + V$. Then for $\sigma \in \mathbb{C} \setminus [0, \infty)$, $H - \sigma : H^{r,s} \to H^{r-2,s}$ is Fredholm for all $r, s$, with nullspace in $\mathcal{S}$. If $V$ is real-valued, then $H$ is self-adjoint.

The elliptic parametrix construction can be microlocalized, i.e. if the principal symbol of $A$ is only assumed to be elliptic on (hence near) a closed subset $K$ of $\partial(\mathbb{R}^n \times \mathbb{R}^n)$, one still can construct a microlocal parametrix $B$, i.e. one whose errors $BA - \text{Id}, AB - \text{Id}$ as a parametrix have wave front set disjoint from $K$. To make this precise, first we define microlocal ellipticity:

**Definition 5.** We say that $A \in \Psi^{m,\ell}$, $\sigma_{m,\ell}(A) = [a]$, is elliptic at $\alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n)$ if $\alpha$ has a neighborhood $U$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that $a|_{U \cap \mathbb{R}^n \times \mathbb{R}^n}$ is elliptic, i.e. satisfies (24) on $U$. We say that $A$ is elliptic on a subset $K$ of $\partial(\mathbb{R}^n \times \mathbb{R}^n)$ if it is elliptic at each point of $K$. The elliptic set $\text{Ell}(A)$ is the set of points at which $A$ is elliptic; the characteristic set $\text{Char}(A)$ is its complement.

We say that $A \in \Psi^{m,\ell}_\infty$, $\sigma_{\infty,m,\ell}(A) = [a]$, is elliptic at $\alpha \in \mathbb{R}^n \times \partial\mathbb{R}^n$ if $\alpha$ has a neighborhood $U$ in $\mathbb{R}^n \times \partial\mathbb{R}^n$ such that $a|_{U \cap \mathbb{R}^n \times \partial\mathbb{R}^n}$ is elliptic, i.e. satisfies (23) on $U$. We say that $A$ is elliptic on a subset $K$ of $\mathbb{R}^n \times \partial\mathbb{R}^n$ if it is elliptic at each point of $K$. One defines $\text{Ell}_\infty(A)$, $\text{Char}_\infty(A)$ analogously to the above definition.

If $A \in \Psi^{m,\ell}$ is elliptic on a closed (hence compact) $K$, then a covering argument shows that $a$ satisfies (24) on a neighborhood of $K$. A similar statement holds for $A \in \Psi^{m,\ell}_\infty$.

**Proposition 0.20.** If $A \in \Psi^{m,\ell}$ is elliptic on a compact set $K$ then there exists $B \in \Psi^{-m,-\ell}$ such that $E_L = BA - \text{Id}$, $E_R = AB - \text{Id}$ satisfy $WF'(E_L) \cap K = \emptyset$, $WF'(E_R) \cap K = \emptyset$.

**Proof.** If $A$ is elliptic on $K$, there is a neighborhood $U$ of $K$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that $a|_{U \cap \mathbb{R}^n \times \mathbb{R}^n}$ is elliptic, i.e. satisfies (24) on $U$. We may shrink $U$ so that $|z| + |\zeta| > R$ on $U$; thus $a|_U$ has a lower bound on all of $U$. Let $q \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be identically 1 near $K$, be supported in $U$, and let $Q \in \Psi^{0,0}$ be given by $Q = q_L(q)$. Thus, $Q$ has principal symbol $\sigma_{0,0}(Q) = q|_{\partial(\mathbb{R}^n \times \mathbb{R}^n)}$, and $WF'(Q) \subset U$, $WF'(\text{Id} - Q) \cap K = \emptyset$. Now let $[a]$ be the principal symbol of $A$, let $b_0 = qa^{-1} \in S^{-m,-\ell}$ since $a$ is elliptic on $U$. Let $B_0 = q_L(b_0)$, so $\sigma_{-m,-\ell}(B_0) = b_0$ and $WF'(B_0) \subset U$. Let $q_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be identically 1 near $K$, have disjoint support from $1 - q$, so $q_0(1 - q) = 0$, and let $Q_0 = q_L(q_0)$. Note that $WF'(\text{Id} - Q_0) \cap K = \emptyset$. Then $E_{0,L} = Q_0(B_0A - \text{Id}) \in \Psi^{0,0}$, $E_{0,R} = (AB_0 - \text{Id})Q_0 \in \Psi^{0,0}$ have vanishing principal symbols, so $E_{0,L}, E_{0,R} \in \Psi^{-1, -1}$. As in the globally elliptic case, one may asymptotically sum the amplitudes $e_{L,j}$ of $(-1)^j E_{0,L}^j$ to obtain $\hat{E}_L$ such that
\[ F_N = \tilde{E}_L - \sum_{j=1}^{N} (-1)^j E_{0,L}^j \in \Psi^{-N-1,-N-1} \] for all \( N \). Thus,

\[(Id + \tilde{E}_L)Q_0 B_0 \Delta = (Id + \tilde{E}_L)(E_{0,L} + Id) + (Id + \tilde{E}_L)(Q_0 - Id)\]

\[= (Id + \sum_{j=1}^{N} (-1)^j E_{0,L}^j + F_N)(Id + E_{0,L}) + (Id + \tilde{E}_L)(Q_0 - Id)\]

\[= Id + (-1)^{N+1} E_{0,L}^{-N+1} + F_N(Id + E_{0,L}) + (Id + \tilde{E}_L)(Q_0 - Id).\]

Now, \((-1)^{N+1} E_{0,L}^{-N+1} + F_N(Id + E_{0,L}) \in \Psi^{-N-1,-N-1}\), and is independent of \( N \)
(since it plus Id is \((Id + \sum_{j=1}^{N} (-1)^j E_{0,L}^j + F_N)(Id + E_{0,L})\)) so it is in \( \Psi^{-\infty,-\infty}\), and
\( WF'(\{Id + \tilde{E}_L\}(Q_0 - Id)) \subset WF'(Q_0 - Id) \), which is disjoint from \( K \). Thus, we may

\[ B_L = (Id + \tilde{E}_L)Q_0 B_0 \]
as our microlocal left parametrix, and similarly obtain a microlocal right parametrix \( B_R \). The parametrix identity (26) now shows that \( WF'(B_L - B_R) \cap K = \emptyset \), completing the proof. \( \square \)

One corollary is the following.

**Corollary 0.21.** Suppose \( u \in S' \), \( A \in \Psi^{m,\ell} \), and \( Au \in H^{r,s} \) then for \( Q \in \Psi^{0,0} \) with \( WF'(Q) \cap \text{Char}(A) = \emptyset \), \( Qu \in H^{r+m,s+\ell} \). Further, for all \( M, N \) there exists \( C > 0 \) such that

\[ \|Qu\|_{H^{r+m,s+\ell}} \leq C(\|Au\|_{H^{r,s}} + \|u\|_{H^{M,N}}). \]

**Proof.** Let \( B \) be a microlocal parametrix for \( A \) near \( WF'(Q) \). Then \( BA - Id = E \) with \( WF'(E) \cap WF'(Q) = \emptyset \). Thus,

\[ Qu = Q(BA - E)u = QB(Au) - (QE)u. \]

Now, \( WF'(QE) = WF'(Q) \cap WF'(E) = \emptyset \), so \( QE \in \Psi^{-\infty,-\infty} \), and thus \( QE \in S \), while \( QB \in \Psi^{-m,-\ell} \), so the proof is finished as for global elliptic regularity. \( \square \)

Here the assumption \( Au \in H^{r,s} \) is too strong; it only matters that \( Au \) is such microlocally near \( WF'(Q) \). That is:

**Corollary 0.22.** *(Microlocal elliptic regularity; operator version.)* Suppose \( u \in S' \), \( A \in \Psi^{m,\ell} \), and for some \( Q' \in \Psi^{0,0} \), \( Q'(Au) \in H^{r,s} \). Then for \( Q \in \Psi^{0,0} \) with \( WF'(Q) \subset Ell(A) \cap Ell(Q') \), \( Qu \in H^{r+m,s+\ell} \). Further, for all \( M, N \) there exists \( C > 0 \) such that

\[ \|Qu\|_{H^{r+m,s+\ell}} \leq C(\|Q'Au\|_{H^{r,s}} + \|u\|_{H^{M,N}}). \]

**Proof.** We just note that \( Q'A \) is elliptic on \( Ell(A) \cap Ell(Q') \), so the previous corollary is applicable. \( \square \)

One can restate the corollary in terms of microlocalizing the distributions instead of adding the microlocalizers explicitly as operators.

**Definition 6.** Suppose \( \alpha \in \partial(\mathbb{R}^m \times \mathbb{R}^n), u \in S' \). We say that \( \alpha \notin WF^{m,\ell}(u) \) if there exists \( A \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Au \in H^{m,\ell} \). We say that \( \alpha \notin WF(u) \) if there exists \( A \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Au \in S \).

For \( k, \ell, m \in \mathbb{R} \), \( u \in H^{k,\ell} \), \( WF^{m,\ell}(u) \) is defined similarly: if \( \alpha \in \mathbb{R}^m \times \partial \mathbb{R}^n \), we say \( \alpha \notin WF^{m,\ell}(u) \) if there exists \( A \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Au \in H^{m,\ell} \). We say that \( \alpha \notin WF(u) \) if there exists \( A \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Au \in H^{m,\ell} \).
The most important property of WF and pseudodifferential operators is microlocality:

**Proposition 0.23.** If \( A \in \Psi^{m,\ell} \) and \( u \in S' \) then

\[
WF^r,s(Au) \subset WF^r(A) \cap WF^{r+m,s+\ell}(u)
\]

and

\[
WF(Au) \subset WF^r(A) \cap WF(u).
\]

**Proof.** We need to show that \( WF^r,s(Au) \subset WF^r(A) \cap WF^{r+m,s+\ell}(u) \).

We start with the former, which is straightforward. Suppose \( \alpha \notin WF^r(A) \). Let \( Q \in \Psi^{0,0} \) be elliptic at \( \alpha \) but with \( WF^r(Q) \cap WF^r(A) = \emptyset \); one can achieve this as \( WF^r(A) \) is closed, so one simply needs to take \( q \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) equal to 1 near \( \alpha \) and with essential support disjoint from \( WF^r(A) \). Then \( WF^r(QA) \subset WF^r(Q) \cap WF^r(A) = \emptyset \), so \( QA \in \Psi^{-\infty,-\infty} \), thus \( QAu \in S \).

Now for the second inclusion. Suppose \( \alpha \notin WF^{r+m,s+\ell}(u) \). Then there exists \( B \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Bu \in H^{r+m,s+\ell} \). Let \( G \in \Psi^{0,0} \) be a microlocal parametrix for \( B \), so \( GB = \text{Id} + E \) with \( \alpha \notin WF^r(E) \). Then \( Au = AGBu - AEu \), and \( AG \in \Psi^{m,\ell} \), so \( AGBu \in H^{r,s} \). On the other hand, \( \alpha \notin WF^r(AE) \subset WF^r(E) \).

Thus, while the wave front set definition is a ‘there exists’ statement, in fact it is equivalent to a ‘for all’ statement, namely for all \( Q \in \Psi^{0,0} \) with \( WF^r(Q) \) in a sufficiently neighborhood of \( \alpha \), \( Qu \in H^{r,s} \). (The other direction is simply because these \( Q \) include those elliptic at \( \alpha \).)

Also, as immediate from the proof below, one can take \( U \) to be the elliptic set of the \( B \in \Psi^{0,0} \), elliptic at \( \alpha \), with \( Bu \in H^{r,s} \), whose existence is guaranteed by \( \alpha \notin WF^{r,s}(u) \).

**Lemma 0.24.** If \( \alpha \notin WF^{r,s}(u) \) then there is a neighborhood \( U \) of \( \alpha \) such that for all \( Q \in \Psi^{0,0} \) with \( WF^r(Q) \subset U \), \( Qu \in H^{r,s} \).

Further, with the same \( U \), for all \( Q \in \Psi^{m,\ell} \) with \( WF^r(Q) \subset U \), \( Qu \in H^{r-m,s-\ell} \).

Thus, while the wave front set definition is a ‘there exists’ statement, in fact it is equivalent to a ‘for all’ statement, namely for all \( Q \in \Psi^{0,0} \) with \( WF^r(Q) \) in a sufficiently neighborhood of \( \alpha \), \( Qu \in H^{r,s} \). (The other direction is simply because these \( Q \) include those elliptic at \( \alpha \).)

**Proof.** Suppose \( \alpha \notin WF^{r,s}(u) \). Then there exists \( B \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Bu \in H^{r,s} \); let \( G \in \Psi^{0,0} \) be a microlocal parametrix for \( B \), so \( GB = \text{Id} + E \) with \( \alpha \notin WF^r(E) \). Let \( U \) be the complement of \( WF^r(E) \); this is a neighborhood of \( \alpha \). Then for any \( Q \in \Psi^{0,0} \) with \( WF^r(Q) \subset U \), \( QE \in \Psi^{-\infty,-\infty} \), so \( Qu = QGBu - QE u \in H^{r,s} \) as \( QG \in \Psi^{0,0} \).

The second statement is proved the same way, noticing that \( QG \in \Psi^{m,\ell} \) now.

**An immediate consequence is:**

**Lemma 0.25.** If \( u \in S' \) and \( WF^{m,\ell}(u) = \emptyset \) then \( u \in H^{m,\ell} \).

If \( u \in H^{k,\ell} \) and \( WF^\infty(u) = \emptyset \) then \( u \in H^{m,\ell} \).
Proof. Suppose \( u \in S' \) and \( \text{WF}^{m,\ell}(u) = \emptyset \). Then for all \( \alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n) \) there exists \( U_\alpha \) open such that for all \( Q \in \Psi^{0,0} \) with \( \text{WF}'(Q) \subset U_\alpha \), \( Qu \in H^{m,\ell} \). Now \( \{ U_\alpha : \alpha \in \partial(\mathbb{R}^n \times \mathbb{R}^n) \} \) is an open cover of the compact set \( \partial(\mathbb{R}^n \times \mathbb{R}^n) \), so there is a finite subcover, say \( \{ U_{\alpha_j} : j = 1, \ldots, N \} \). Let \( \tilde{U}_{\alpha_j} \) be open in \( \mathbb{R}^n \times \mathbb{R}^n \) with \( \tilde{U}_{\alpha_j} \cap \partial(\mathbb{R}^n \times \mathbb{R}^n) = U_{\alpha_j} \). Then, with \( \tilde{U}_{\alpha_j} \cap \partial(\mathbb{R}^n \times \mathbb{R}^n) = U_{\alpha_j} \), there exists a finite subcover, say \( \{ U_{\alpha_j} : j = 1, \ldots, N \} \). Let \( \tilde{U}_{\alpha_j} \) be open in \( \mathbb{R}^n \times \mathbb{R}^n \) with \( \tilde{U}_{\alpha_j} \cap \partial(\mathbb{R}^n \times \mathbb{R}^n) = U_{\alpha_j} \). Then, with \( \tilde{U}_{\alpha_j} \cap \partial(\mathbb{R}^n \times \mathbb{R}^n) = U_{\alpha_j} \), there exists a finite subcover, say \( \{ U_{\alpha_j} : j = 1, \ldots, N \} \). Let \( \sum_{j=0}^{N} q_j = 1 \) be a subordinate partition of unity, and let \( Q_j = q_j(q_j) \). Then \( \{ Q_j : j = 0, 1, \ldots, N \} \) is a finite open cover of \( \mathbb{R}^n \times \mathbb{R}^n \). Let \( \sum_{j=0}^{N} Q_j = \text{Id}, Q_0 \in \Psi^{-\infty,-\infty} \) since \( q_0 \) has compact support, while for \( j = 1, \ldots, N \), \( \text{WF}(Q_j) \subset U_{\alpha_j} \) since \( \text{supp} q_j \subset \tilde{U}_{\alpha_j} \). Thus, \( Q_j u \in H^{r,s} \) for all \( j \), and thus \( u = \sum Q_j u \in H^{r,s} \) as claimed.

The argument for \( \text{WF}_\infty \) is analogous. □

The distributional version of microlocal elliptic regularity then is:

**Corollary 0.26.** *(Microlocal elliptic regularity; distributional version.)* Suppose \( u \in S', A \in \Psi^{m,\ell} \). Then
\[
\text{WF}^{r+m,s+\ell}(u) \subset \text{Char}(A) \cup \text{WF}^{r,s}(Au).
\]

Proof. Suppose \( \alpha \notin \text{Char}(A) \cup \text{WF}^{r,s}(Au) \), we need to show \( \alpha \notin \text{WF}^{r+m,s+\ell}(u) \). As \( \alpha \notin \text{WF}^{r,s}(Au) \) there exists \( Q' \in \Psi^{0,0} \) elliptic at \( \alpha \) such that \( Q' Au \in H^{r,s} \). Let \( Q \in \Psi^{0,0} \) be such that \( \text{WF}'(Q) \subset \text{Ell}(A) \cap \text{Ell}(Q') \), note that the set on the right is open and includes \( \alpha \). Then by Corollary 0.22, \(Qu \in H^{r,s} \). Taking \( Q \) which is in addition elliptic at \( \alpha \) completes the proof. □

The consequence of what we proved so far for Schrödinger operators is:

**Proposition 0.27.** Let \( g \) be a Riemannian metric, \( g_{ij} \in C^\infty(\mathbb{R}^n) \), positive definite on \( \mathbb{R}^n \), \( V \in S^{-\rho}(\mathbb{R}^n) \) with \( \rho > 0 \). Let \( H = \Delta_g + V \). Then for \( \sigma \subset [0,\infty) \), \( (H - \sigma)u \in H^{r,s} \) implies
\[
\text{WF}^{r+2,s}(u) \subset \{ (z,\zeta) \in \partial(\mathbb{R}^n \times \mathbb{R}^n) : \sum g_{ij}(z) \zeta_i \zeta_j = \sigma \}.
\]