MATH 256B: SCATTERING PSEUDODIFFERENTIAL OPERATORS

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We discuss basic properties of pseudodifferential and so-called scattering pseudodifferential operators, introduced in this generality by Melrose, formerly discussed by Parenti and Shubin on $\mathbb{R}^n$, where it can be also considered an example of Hörmander's Weyl calculus. These operators generalize differential operators of the form

$$A = \sum_{|\alpha|\leq m} a_\alpha D^\alpha, \text{ with } a_\alpha \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}), \tag{1}$$

as we show below in (30). Indeed, the conditions on the coefficients a_α are relaxed to be 'symbolic', so that for instance $a_0(z) = \phi(z)|z|^{-\rho}$, $\phi \equiv 0$ near the origin, $\equiv 1$ near infinity is allowed. Thus, in particular operators such as $\Delta + V$, where V is the Coulomb potential, without its singularity at the origin, fit into this framework.

More generally, we can consider Riemannian metrics g with $g_{ij} \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ such that for all $z \in \overline{\mathbb{R}^n}$, $\sum_{ij} g_{ij}(z)\zeta_i\zeta_j = 0$ implies $\zeta = 0$, i.e. g is positive definite on the compact manifold $\overline{\mathbb{R}^n}$. Then, with V as above and with $\sigma \in \mathbb{C}$, $\Delta_g + V - \sigma$ is of the form (1) with $m = 2$.

The extension of this class to scattering pseudodifferential operators allows one to construct approximate inverses (parametrices), showing Fredholm properties, for operators that are elliptic *in this class*. Ellipticity here also encodes behavior at spatial infinity, so for instance $\Delta + V - \sigma$, where V may be Coulomb type with $\rho > 0$, is elliptic for $\sigma \in \mathbb{C} \setminus [0, \infty)$, but is not elliptic for $\sigma \in [0, \infty)$. It also allows one to develop tools to study non-elliptic operators. For instance, the limiting absorption principle, i.e. the existence of the limits

$$R(\sigma \pm i0) = \lim_{\epsilon \to 0+} (\Delta + V - (\sigma \pm i\epsilon))^{-1}$$

for V real valued and $\sigma > 0$ fits very nicely into this framework.

Since there are technicalities along the way, we give an outline of this section first. First, for $m, \ell, \ell' \in \mathbb{R}$, $\delta, \delta' \in [0, 1/2)$, we define two kinds of function spaces,

$$S^{m,\ell}_{\delta,\delta'}(\mathbb{R}^n; \mathbb{R}^n) \subset S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^{2n}),$$

as well as analogues on $\mathbb{R}^{3n}$:

$$S^{m,\ell_1,\ell_2}_{\delta,\delta'}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n) \subset S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^{3n}).$$

The elements of these spaces are called *symbols*; the important point is the behavior of these symbols at infinity. Here the spaces become larger with increasing m, ℓ and ℓ_j, and $\delta = 0 = \delta'$ gives the standard classes also denoted by

$$S^{m,\ell}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) = S^{m,\ell}(\mathbb{R}^n; \mathbb{R}^n), \; S^{m,\ell}_{\infty,0}(\mathbb{R}^n; \mathbb{R}^n) = S^{m,\ell}(\mathbb{R}^n; \mathbb{R}^n),$$

and similarly for the $\mathbb{R}^{3n}$ versions. *The cases $\delta = 0 = \delta'$ are by far the most important ones.* We have projections $\pi_L, \pi_R : \mathbb{R}^{3n} \to \mathbb{R}^{2n}$, with π_L dropping the second factor of $\mathbb{R}^{3n}$ and π_R dropping the first factor:

$$\pi_L(z, z', \zeta) = (z, \zeta),\ \pi_R(z, z', \zeta) = (z', \zeta).$$

The π_L^*, π_R^* pull-back elements of the $\mathbb{R}^{2n}$ spaces to the corresponding $\mathbb{R}^{3n}$ spaces (with $\ell_1 = \ell$, $\ell_2 = 0$, resp, $\ell_2 = \ell$, $\ell_1 = 0$). We define an oscillatory integral map:

$$I : S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n) \to \mathcal{L}(\mathcal{S}, \mathcal{S}),$$

with $\mathcal{L}$ denoting continuous linear operators, and also show by duality that

$$I : S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n) \to \mathcal{L}(\mathcal{S}', \mathcal{S}'),$$

and I is closed under Fréchet space or L^2-based adjoints. The compositions

$$q_L = I \circ \pi_L^*,\ q_R = I \circ \pi_R^*,$$

are called the left and right quantization maps. Now, it turns out that I is redundant, and its range on $S^{m,\ell_1,\ell_2}_{\delta,\delta'}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n)$, resp. $S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n; \mathbb{R}^n)$, is that of q_L on $S^{m,\ell}_{\delta,\delta'}(\mathbb{R}^n; \mathbb{R}^n)$, resp. $S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n)$ with $\ell = \ell_1 + \ell_2$; the analogous statement also holds with q_L replaced by q_R. This is called left, resp. right, reduction; see Proposition 0.5. One defines pseudodifferential operators, $\Psi^{m,\ell}_{\delta,\delta'}$, resp. $\Psi^{m,\ell}_{\infty,\delta}$, to be the range of q_L (or equivalently q_R) on the spaces $S^{m,\ell}_{\delta,\delta'}(\mathbb{R}^n; \mathbb{R}^n)$, resp. $S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n; \mathbb{R}^n)$, and writes

$$\Psi^{m,\ell} = \Psi^{m,\ell}_{0,0},\ \Psi^{m,\ell}_{\infty} = \Psi^{m,\ell}_{\infty,0}.$$

Once this reducibility is shown it is straightforward to see (using the general I, which is why it is introduced) that $A \in \Psi^{m,\ell}_{\delta,\delta'}$, $B \in \Psi^{m',\ell'}_{\delta,\delta'}$ implies $AB \in \Psi^{m+m',\ell+\ell'}_{\delta,\delta'}$, i.e. that $\Psi^{\infty,\infty}_{\delta,\delta'} = \cup_{m,\ell} \Psi^{m,\ell}_{\delta,\delta'}$ is an order-filtered algebra, with the analogous statements holding for $\Psi^{\infty,\infty}_{\infty,\delta}$ as well. One also shows that composition is commutative to leading order, i.e.

$$A \in \Psi^{m,\ell}_{\delta,\delta'},\ B \in \Psi^{m',\ell'}_{\delta,\delta'} \Longrightarrow [A, B] = AB - BA \in \Psi^{m+m'-1+2\delta,\ell+\ell'-1+2\delta'};$$

the analogous statement here is

$$A \in \Psi^{m,\ell}_{\infty,\delta},\ B \in \Psi^{m',\ell'}_{\infty,\delta} \Longrightarrow [A, B] = AB - BA \in \Psi^{m+m'-1+2\delta,\ell+\ell'}_{\infty,\delta},$$

i.e. the gain is only in the first order. This is conveniently encoded by the *principal symbol*

$$\sigma_{m,\ell} : \Psi^{m,\ell}_{\delta,\delta'} \to S^{m,\ell}_{\delta,\delta'}/S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'},\ \sigma_{\infty,m,\ell} : \Psi^{m,\ell}_{\infty,\delta} \to S^{m,\ell}_{\infty}/S^{m-1+2\delta,\ell}_{\infty,\delta},$$

which are multiplicative (homomorphisms of filtered algebras); the leading order commutativity of pseudodifferential operators correspond to the commutativity of function spaces under multiplication. Here δ, δ' are suppressed in the principal symbol notation. An immediate consequence is the elliptic parametrix construction: for operators $A \in \Psi^{m,\ell}_{\delta,\delta'}$ with invertible principal symbol, which are called *elliptic*, one can construct an approximate inverse $B \in \Psi^{-m,-\ell}_{\delta,\delta'}$ such that $AB - \mathrm{Id}, BA - \mathrm{Id} : \mathcal{S}' \to \mathcal{S}$ are continuous, i.e. completely regularizing. In the case of $A \in \Psi^{m,\ell}_{\infty,\delta}$, we only have that $AB - \mathrm{Id}, BA - \mathrm{Id} : \mathcal{S}' \to \mathcal{C}^\infty(\mathbb{R}^n)$, i.e. are smoothing, but do not give decay at infinity. Since completely regularizing operators are compact from any weighted Sobolev space to any other weighted Sobolev space, and since we show that

$$A \in \Psi^{m,\ell}_{\infty,\delta} \Longrightarrow A \in \mathcal{L}(H^{r,s}, H^{r-m,s-\ell})$$

for all $r, s \in \mathbb{R}$ (so analogous statements hold for $\Psi^{m,\ell}_{\delta,\delta'} \subset \Psi^{m,\ell}_{\infty,\delta}$), we deduce that *elliptic* $A \in \Psi^{m,\ell}_{\delta,\delta'}$ are *Fredholm* on any weighted Sobolev space, with the nullspace of both A and A^* lying in $\mathcal{S}(\mathbb{R}^n)$, and is independent of the choice of the weighted Sobolev space. In particular, if $A \in \Psi^{m,0}_{\delta,\delta'}$, $m > 0$, elliptic, is symmetric with respect to the L^2 inner product, then one immediately concludes that $A \pm i\,\mathrm{Id}$ is invertible as a map $H^{m,0} \to L^2$, and thus A is self-adjoint with domain $H^{m,0}$.

Another important directions we explore is *microlocalization*, by introducing the notion of the operator wave front set, $\mathrm{WF}'(A)$, or $\mathrm{WF}'_\infty(A)$, which measures where in phase space A is 'trivial'. Thus, while $\sigma_{m,\ell}$, $\sigma_{\infty,m,\ell}$ capture the leading order behavior of operators, i.e. their behavior modulo one order lower operators, $\mathrm{WF}'(A)$ and $\mathrm{WF}'_{\infty,\ell}(A)$ give the locations where A is not residual, i.e. in $\Psi^{-\infty,-\infty}$, resp. $\Psi^{-\infty,\ell}_\infty$, so for instance the emptyness of $\mathrm{WF}'(A)$ implies $A \in \Psi^{-\infty,-\infty}$. One should think of these of these as an analogue of the singular support of distributions, which measures where a distribution is not $\mathcal{C}^\infty$, except that its location will not be in the base space $\mathbb{R}^n$, but rather at infinity in *phase space*, $\mathbb{R}^n \times \mathbb{R}^n$. To make this concrete, it is useful to compactify $\mathbb{R}^n \times \mathbb{R}^n$ to $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$; then for $A \in \Psi^{m,\ell}$, $\mathrm{WF}'(A) \subset \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ while for $A \in \Psi^{m,\ell}_\infty$, $\mathrm{WF}'_{\infty,\ell}(A) \subset \overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$. Then one can perform a microlocal version of the elliptic parametrix construction, i.e. one that is localized, in the sense of WF', near points at which the operator A is elliptic; this is a first step towards understanding non-elliptic operators.

It turns out that it is convenient to generalize the class of operators considered here to allow their orders m and ℓ vary, namely $m = \mathsf{m}$ is a function on $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ and $\ell = \mathsf{l}$ a function on $\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$, so at different points microlocally one has an operator of different order. This is the reason we consider $\delta, \delta' > 0$ here; we naturally end up with the classes $S^{m,\ell}_{\delta,\delta'}$ and $S^{m,\ell}_{\infty,\delta}$ where δ, δ' can be taken to be arbitrarily small but positive.

We now go through the details. Thus, starting with $\mathbb{R}^n$, we consider operators of the form

$$(2)\quad Au(z) = (I(a)u)(z) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\zeta\cdot(z-z')} a(z, z', \zeta)\, u(z')\, dz',\ \ u \in \mathcal{S}(\mathbb{R}^n),$$

where a is a *product-type* symbol of class $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$, $m, \ell_1, \ell_2 \in \mathbb{R}$, $\delta, \delta' \in [0, 1/2)$, i.e. differentiation in z, resp. z', resp. ζ, provides extra decay in the respective variables:

$$\begin{aligned} a \in\, & S^{m,\ell_1,\ell_2}_{\delta,\delta'}(\mathbb{R}^n_z; \mathbb{R}^n_{z'}; \mathbb{R}^n_\zeta) \\ & \Longleftrightarrow a \in \mathcal{C}^\infty(\mathbb{R}^n_z \times \mathbb{R}^n_{z'} \times \mathbb{R}^n_\zeta), \\ & |D^\alpha_z D^\beta_{z'} D^\gamma_\zeta a| \leq C_{\alpha\beta\gamma} \langle z\rangle^{\ell_1-|\alpha|} \langle z'\rangle^{\ell_2-|\beta|} \langle\zeta\rangle^{m-|\gamma|} (\langle z\rangle + \langle z'\rangle)^{\delta'|(\alpha,\beta,\gamma)|} \langle\zeta\rangle^{\delta|(\alpha,\beta,\gamma)|} \end{aligned}$$

with

$$|(\alpha, \beta, \gamma)| = |\alpha| + |\beta| + |\gamma|$$

and

$$\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}.$$

One writes

$$\|a\|_{S^{m,\ell_1,\ell_2}_{\delta,\delta'},N} = \sum_{|\alpha|+|\beta|+|\gamma|\le N} \sup \langle z\rangle^{-\ell_1+|\alpha|}\langle z'\rangle^{-\ell_2+|\beta|}(\langle z\rangle+\langle z'\rangle)^{-\delta'|(\alpha,\beta,\gamma)|} \times \langle\zeta\rangle^{-m+|\gamma|-\delta|(\alpha,\beta,\gamma)|}|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a|;$$

as N runs over $\mathbb{N}$, these give a family of seminorms on S^{m,ℓ_1,ℓ_2}, giving it a Fréchet topology.

Note that the orders on S are reversed compared to the order of the factors, i.e. z, z', ζ; this is done in part to conform with the usual notation. Moreover, $(\langle z\rangle+\langle z'\rangle)^{\delta'|(\alpha,\beta,\gamma)|}$ can be replaced by $\langle (z,z')\rangle^{\delta'|(\alpha,\beta,\gamma)|}$. Also, z and z' play an equivalent role since as mentioned before, and as we show below, one can even eliminate, say, the z' dependence. In fact, it turns out that the behavior of a is essentially irrelevant in the region where $\frac{\langle z\rangle}{\langle z'\rangle}$ is *not* bounded between M^{-1} and M, $M>1$ is any fixed number in that if one cuts a off to be supported outside such a set, one obtains an element of $\Psi^{-\infty,-\infty}_{\delta,\delta'}$, see (24), but since this is due to the oscillatory nature of the integral in ζ, this is not obvious at this point. However, we already point out that fixing some $\chi\in\mathcal{C}^\infty_c(\mathbb{R})$, $\chi\equiv 1$ on $[\frac12,2]$, supported in $[\frac14,4]$, for $a\in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ we have the decomposition as

$$a = a_1 + a_2,\ a_1 = \chi\Big(\frac{\langle z\rangle}{\langle z'\rangle}\Big)a,\ a_2 = \Big(1-\chi\Big(\frac{\langle z\rangle}{\langle z'\rangle}\Big)\Big)a, \tag{3}$$

with a_j depending continuously on a in the $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ topology; below (24) shows that the contribution of a_2 is essentially irrelevant in the sense stated above.

In fact, in the beginning it is better to start with a larger (at least if $\delta'=0$) class of symbols, without extra decay in the z,z' variables upon differentiation: for $\delta\in[0,1/2)$,

$$a\in S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta) \Longleftrightarrow a\in\mathcal{C}^\infty(\mathbb{R}^n_z\times\mathbb{R}^n_{z'}\times\mathbb{R}^n_\zeta),\quad |D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a| \le C_{\alpha\beta\gamma}\langle z\rangle^{\ell_1}\langle z'\rangle^{\ell_2}\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|}.$$

One writes

$$\|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},N} = \sum_{|\alpha|+|\beta|+|\gamma|\le N}\sup\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}\langle\zeta\rangle^{-m+|\gamma|-\delta|(\alpha,\beta,\gamma)|}|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a|.$$

For $\ell_1=\ell_2=0$, this is Hörmander's uniform symbol class of type $1-\delta,\delta$ (i.e. ρ,δ with $\rho=1-\delta$). Note that

$$S^{m,\ell_1,\ell_2}_{\delta,0}\subset S^{m,\ell_1,\ell_2}_{\infty,\delta},$$

and the inclusion map

$$\iota : S^{m,\ell_1,\ell_2}_{\delta,0}\hookrightarrow S^{m,\ell_1,\ell_2}_{\infty,\delta}$$

is continuous, with

$$\|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},N}\le\|a\|_{S^{m,\ell_1,\ell_2}_{\delta,0},N}$$

for all N.

Note that $\ell_j\le\ell_j'$, $m\le m'$ implies

$$S^{m,\ell_1,\ell_2}_{\delta,\delta'}\subset S^{m',\ell_1',\ell_2'}_{\delta,\delta'},$$

and similarly with S_∞. Further, if $\delta\le\tilde\delta$, $\delta'\le\tilde\delta'$ then

$$S^{m,\ell_1,\ell_2}_{\delta,\delta'}\subset S^{m,\ell_1,\ell_2}_{\tilde\delta,\tilde\delta'}.$$

One writes

$$S^{-\infty,\ell_1,\ell_2}_{\delta,\delta'}=\cap_{m\in\mathbb{R}}S^{m,\ell_1,\ell_2}_{\delta,\delta'},\ S^{-\infty,\ell_1,-\infty}_{\delta,\delta'}=\cap_{m\in\mathbb{R},\ell_2\in\mathbb{R}}S^{m,\ell_1,\ell_2}_{\delta,\delta'},$$

and similarly again with S_∞. Notice that for all $\delta,\delta'\in[0,1/2)$,

$$S^{-\infty,-\infty,-\infty}_{\delta,\delta'}=\mathcal{S}(\mathbb{R}^{3n})$$

while $S^{-\infty,0,0}_{\infty,\delta}$ consists of $\mathcal{C}^\infty$ functions on $\mathbb{R}^{2n}_{z,z'}$ which are bounded with all derivatives, and take values in $\mathcal{S}(\mathbb{R}^n)$. Thus, these *residual* spaces are independent of δ,δ'. One also writes

$$S^{\infty,\infty,\infty}_{\delta,\delta'}=\cup_{m,\ell_1,\ell_2\in\mathbb{R}}S^{m,\ell_1,\ell_2}_{\delta,\delta'}.$$

Further, note that $S^{\infty,\infty,\infty}_{\delta,\delta'}$ forms a commutative filtered *-algebra in the sense that in addition to $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ being a vector space for each m,ℓ_1,ℓ_2, closed under complex conjugation, the (function-theoretic, i.e. pointwise) product (which is commutative) satisfies

$$a\in S^{m,\ell_1,\ell_2}_{\delta,\delta'},\ b\in S^{m',\ell_1',\ell_2'}_{\delta,\delta'}\Rightarrow ab\in S^{m+m',\ell_1+\ell_1',\ell_2+\ell_2'}_{\delta,\delta'},$$

as follows from Leibniz' rule. Similarly $S^{\infty,\infty,\infty}_{\infty,\delta}$ forms a commutative filtered *-algebra as well. Notice also that for $\delta'=0$,

$$a\in S^{m,\ell_1,\ell_2}_{\delta,0}\Rightarrow D^\alpha_z D^\beta_{z'}D^\gamma_\zeta a\in S^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|,\ell_1-|\alpha|,\ell_2-|\beta|}_{\delta,0},\tag{4}$$

while for general δ', the a_1 piece, as defined in (3), satisfies

$$a_1\in S^{m,\ell_1,\ell_2}_{\delta,\delta'}\Rightarrow D^\alpha_z D^\beta_{z'}D^\gamma_\zeta a_1\in S^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|,\ell_1-|\alpha|+\delta'|(\alpha,\beta,\gamma)|,\ell_2-|\beta|}_{\delta,\delta'},\tag{5}$$

where by the support property of a_1, $\delta'|(\alpha,\beta,\gamma)|$ could also be shifted to the last order (and recall that a_2 will be shown to be essentially irrelevant). The analogue of (4) also holds for $S^{\infty,\infty,\infty}_{\infty,\delta}$, in which case ℓ_1 and ℓ_2 are unaffected by derivatives.

It is also to note the following lemma:

Lemma 0.1. *For $m'>m$, the residual spaces $S^{-\infty,\ell_1,\ell_2}_{\infty,\delta}=\cap_{\tilde m\in\mathbb{R}}S^{\tilde m,\ell_1,\ell_2}_{\infty,\delta}$, resp. $S^{-\infty,\ell_1,\ell_2}_{\delta,\delta'}=\cap_{\tilde m\in\mathbb{R}}S^{\tilde m,\ell_1,\ell_2}_{\delta,\delta'}$, are dense in $S^{m,\ell_1,\ell_2}_{\infty,\delta}$, resp. $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$, in the topology of $S^{m',\ell_1,\ell_2}_{\infty,\delta}$, resp. $S^{m',\ell_1,\ell_2}_{\delta,\delta'}$.*

Proof. Let $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ be such that $0\le\chi\le1$, $\chi(\zeta)=1$ for $|\zeta|\le1$, $\chi(\zeta)=0$ for $|\zeta|\ge2$, and let $a_j(z,z',\zeta)=\chi(\zeta/j)a(z,z',\zeta)$, where $a\in S^{m,\ell_1,\ell_2}_{\infty,\delta}$. Then

$$D^\alpha_z D'^\beta_z D^\gamma_\zeta(a_j-a)=\sum_{\mu+\nu=\gamma}C_{\mu\nu}j^{-|\mu|}(D^\mu_\zeta(\chi-1))(\zeta/j)(D^\alpha_z D'^\beta_z D^\nu_\zeta a)(z,z',\zeta),$$

with $C_{\mu\nu}$ combinatorial constants. The $\mu=0$ term is supported in $|\zeta|\ge j$, the $\mu\ne0$ terms are supported in $j\le|\zeta|\le2j$. Correspondingly, for $\mu=0$, the summand is bounded by

$$C_{0\gamma}\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|}\langle z\rangle^{\ell_1}\langle z\rangle^{\ell_2},\tag{6}$$

while for $\mu\ne0$, $j\sim|\zeta|$ on the support, so the summand is bounded by a constant multiple of

$$\langle\zeta\rangle^{m-|\mu|-|\nu|+\delta|(\alpha,\beta,\nu)|}\langle z\rangle^{\ell_1}\langle z\rangle^{\ell_2}.\tag{7}$$

Multiplying by

$$\langle\zeta\rangle^{-m'+|\gamma|-\delta|(\alpha,\beta,\gamma)|}\langle z\rangle^{-\ell_1}\langle z\rangle^{-\ell_2},$$

in either case we obtain a quantity bounded by a constant multiple of $\langle\zeta\rangle^{-(m'-m)}$. Since the difference is supported in $|\zeta|\geq j$, and since $m'>m$, this goes to 0 as $j\to\infty$, proving the claim.

The proof for $a\in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ is similar, with (6) replaced by

$$C_{0\gamma}\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|}\langle z\rangle^{\ell_1}\langle z\rangle^{\ell_2}\langle(z,z')\rangle^{\delta'|(\alpha,\beta,\gamma)|}, \tag{8}$$

and (7) replaced by

$$\langle\zeta\rangle^{m-|\mu|-|\nu|+\delta|(\alpha,\beta,\nu)|}\langle z\rangle^{\ell_1}\langle z\rangle^{\ell_2}\langle(z,z')\rangle^{\delta'|(\alpha,\beta,\nu)|}, \tag{9}$$

so multiplication by

$$\langle\zeta\rangle^{-m'+|\gamma|-\delta|(\alpha,\beta,\gamma)|}\langle z\rangle^{-\ell_1}\langle z\rangle^{-\ell_2}\langle(z,z')\rangle^{-\delta'|(\alpha,\beta,\gamma)|},$$

gives the desired result. □

As examples, recall that if a is a polynomial of order ℓ_1,ℓ_2 and m in the three variables, then certainly $a\in S^{m,\ell_1,\ell_2}=S^{m,\ell_1,\ell_2}_{0,0}$. More interestingly, if $a\in\mathcal{C}^\infty(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})=\mathcal{C}^\infty(\overline{\mathbb{R}^n}^3)$ then $a\in S^{0,0,0}=S^{0,0,0}_{0,0}$, so

$$a\in\langle z\rangle^{\ell_1}\langle z\rangle^{\ell_2}\langle\zeta\rangle^m\mathcal{C}^\infty((\overline{\mathbb{R}^n})^3)\Rightarrow a\in S^{m,\ell_1,\ell_2}=S^{m,\ell_1,\ell_2}_{0,0}.$$

A particular example is $a=|z|^{-\rho}\phi(z)$, where $\phi\equiv 0$ near 0, $\phi\equiv 1$ near ∞, then $a\in S^{-\rho,0,0}$, such an a can be thought of as a potential which may decay only slowly at infinity; $\rho=1$ would give the Coulomb potential without its singularity at the origin. Thus, $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ is a $\mathcal{C}^\infty(\overline{\mathbb{R}^n}^3)$-module.

On the flipside, we can rewrite the estimates for S^{m,ℓ_1,ℓ_2}:

$$|\alpha'|\leq|\alpha|,\ |\beta'|\leq|\beta|,\ |\gamma'|\leq|\gamma|\Rightarrow|z^{\alpha'}D_z^\alpha(z')^{\beta'}D_{z'}^\beta\zeta^{\gamma'}D_\zeta^\gamma a|\leq C_{\alpha\beta\gamma}\langle z\rangle^{\ell_1}\langle z'\rangle^{\ell_2}\langle\zeta\rangle^m.$$

Since $z_i\partial_{z_j}$ and ∂_{z_j} generate all $\mathcal{C}^\infty$ vector fields over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ which are tangent to $\partial\overline{\mathbb{R}^n}$, whose set is denoted by $\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})$, we can rewrite this equivalently as follows: let $V_{j,k}\in\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})$, $j=1,2,3$, $N_j\in\mathbb{N}$ (possibly 0) and $1\leq k\leq N_j$ acting in the jth factor, then

$$\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}\langle\zeta\rangle^{-m}\prod_{j=1}^{3}\prod_{k=1}^{N_j}V_{j,k}a\in L^\infty.$$

This could be further rephrased, in terms of vector fields on $\overline{\mathbb{R}^n}^3$, tangent to all boundary faces: if V_j are such, $1\leq j\leq N$ (possibly $N=0$), then

$$\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}\langle\zeta\rangle^{-m}Va\in L^\infty.$$

Since one can use any vector fields tangent to the various boundary faces, in any product decomposition $[0,1)_{r^{-1}}\times\mathbb{S}^{n-1}$ near the boundary of each factor $\overline{\mathbb{R}^n}$, one automatically has smoothness in the various angular variables; in the radial variables one has iterated regularity with respect to $r\partial_r$.

We are also interested in the generalization of this setting in which the orders m,ℓ_1,ℓ_2 are allowed to vary. Concretely, to set this up, suppose that $\mathsf{m},\mathsf{l}_j\in S^{0,0,0}$ are real valued symbols. We write

$$\begin{aligned}a\in&S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,\delta'}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta)\\&\iff a\in\mathcal{C}^\infty(\mathbb{R}^n_z\times\mathbb{R}^n_{z'}\times\mathbb{R}^n_\zeta),\\&|D_z^\alpha D_{z'}^\beta D_\zeta^\gamma a|\leq C_{\alpha\beta\gamma}\langle z\rangle^{\mathsf{l}_1-|\alpha|}\langle z'\rangle^{\mathsf{l}_2-|\beta|}\langle\zeta\rangle^{\mathsf{m}-|\gamma|}(\langle z\rangle+\langle z'\rangle)^{\delta'|(\alpha,\beta,\gamma)|}\langle\zeta\rangle^{\delta|(\alpha,\beta,\gamma)|}.\end{aligned}$$

Notice that replacing m by m' where $\mathsf{m}-\mathsf{m}' \in S^{-\epsilon,0,0}$ for some $\epsilon > 0$ does not change the class since $\langle\zeta\rangle^{\mathsf{m}-\mathsf{m}'} = e^{(\mathsf{m}-\mathsf{m}')\log\langle\zeta\rangle}$, and $(\mathsf{m}-\mathsf{m}')\log\langle\zeta\rangle$ is a bounded function in this case. Since we are interested only in $\mathsf{m}, \mathsf{l}_j \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$, we regard m as a function on $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n}$, and take an arbitrary (smooth) extension to $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$; we proceed similarly with the ℓ_j. Thus, with

$$m = \sup \mathsf{m},\ \ \ell_j = \sup \mathsf{l}_j,$$

where the sup may be taken over the appropriate boundary of the compactification only, we have

$$a \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,\delta'} \Rightarrow a \in S^{m,\ell_1,\ell_2}_{\delta,\delta'}.$$

One can also define

$$\begin{aligned} a \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta) \Longleftrightarrow & a \in \mathcal{C}^\infty(\mathbb{R}^n_z\times\mathbb{R}^n_{z'}\times\mathbb{R}^n_\zeta),\\ & |D^\alpha_z D^\beta_{z'} D^\gamma_\zeta a| \leq C_{\alpha\beta\gamma}\langle z\rangle^{\mathsf{l}_1}\langle z'\rangle^{\mathsf{l}_2}\langle\zeta\rangle^{\mathsf{m}-|\gamma|+\delta|(\alpha,\beta,\gamma)|}, \end{aligned}$$

so with m, ℓ_j as above

$$a \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\infty,\delta} \Rightarrow a \in S^{m,\ell_1,\ell_2}_{\infty,\delta}.$$

However, these variable order space provide more precise information than simply taking $m = \sup\mathsf{m}$, etc., much like the S^{m,ℓ_1,ℓ_2} spaces provide more precise information that $S^{m,\ell_1,\ell_2}_\infty$. Further, we note that we introduced the subscript δ and δ' (limiting the gains under differentiation) since the function $b = \langle\zeta\rangle^{\mathsf{m}} = e^{\mathsf{m}\log\langle\zeta\rangle}$ is in $S^{\mathsf{m},0,0}_{\delta,0}$ for all $\delta > 0$, but not for $\delta = 0$. Indeed, differentiating in, say, z_j, gives

$$D_{z_j} b = (D_{z_j}\mathsf{m})(\log\langle\zeta\rangle)\langle\zeta\rangle^{\mathsf{m}},$$

so there is a logarithmic loss (unless m is constant). On the other hand, we formally state the regularity result as a lemma:

Lemma 0.2. *Let $b(z,z',\zeta) = \langle\zeta\rangle^{\mathsf{m}(z,z',\zeta)}$. Then $b \in S^{\mathsf{m},0,0}_{\delta,0}$ for all $\delta > 0$.*

Proof. Observe that $f = \mathsf{m}\log\langle\zeta\rangle \in S^{\epsilon,0,0}$ for all $\epsilon > 0$ since this holds for $\log\langle\zeta\rangle$, and as $\mathsf{m} \in S^{0,0,0}$. Further, if $f \in S^{\epsilon_0,\epsilon_1,\epsilon_2}$ with $0 \leq \epsilon_0,\epsilon_1,\epsilon_2 < 1$ then

$$e^{-f}D^\alpha_z D^\beta_{z'} D^\gamma_\zeta e^f \in S^{-|\gamma|+\epsilon_0|(\alpha,\beta,\gamma)|,-|\alpha|+\epsilon_1|(\alpha,\beta,\gamma)|,-|\beta|+\epsilon_2|(\alpha,\beta,\gamma)|},$$

as follows by induction on $|\alpha|+|\beta|+|\gamma|$, since it holds when α,β,γ all vanish, and

$$e^{-f}D_{z_j}(D^\alpha_z D^\beta_{z'} D^\gamma_\zeta e^f) = D_{z_j}(e^{-f}D^\alpha_z D^\beta_{z'} D^\gamma_\zeta e^f) + (D_{z_j}f)(e^{-f}D^\alpha_z D^\beta_{z'} D^\gamma_\zeta e^f),$$

so $e^{-f}D^\alpha_z D^\beta_{z'} D^\gamma_\zeta e^f \in S^{-|\gamma|+\epsilon_0|(\alpha,\beta,\gamma)|,-|\alpha|+\epsilon_1|(\alpha,\beta,\gamma)|,-|\beta|+\epsilon_2|(\alpha,\beta,\gamma)|}$ by the inductive hypothesis, and then the first term on the right hand side improves the second order by 1 keeping all others unchanged, while $D_{z_j}f \in S^{\epsilon_0,\epsilon_1-1,\epsilon_2}$, so the second term on the right hand side adds $\epsilon_0, \epsilon_1-1, \epsilon_2$ to the orders, while $|\alpha|$ is increased by 1 in both cases. The argument is symmetric for all other derivatives, giving the conclusion. Applying this with $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_0 = \epsilon$, $\epsilon > 0$ arbitrary, we deduce that for all $\delta > 0$ (namely, we take $\epsilon = \delta$), $\langle\zeta\rangle^{\mathsf{m}} \in S^{\mathsf{m},0,0}_{\delta,0}$ indeed. □

We still have, analogously to the constant order setting, that

$$a \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,\delta'},\ b \in S^{\mathsf{m}',\mathsf{l}'_1,\mathsf{l}'_2}_{\delta,\delta'} \Rightarrow ab \in S^{\mathsf{m}+\mathsf{m}',\mathsf{l}_1+\mathsf{l}'_1,\mathsf{l}_2+\mathsf{l}'_2}_{\delta,\delta'},$$

and for $\delta' = 0$

$$a \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,0} \Rightarrow D^\alpha_z D^\beta_{z'} D^\gamma_\zeta a \in S^{\mathsf{m}-|\gamma|+\delta|(\alpha,\beta,\gamma)|,\mathsf{l}_1-|\alpha|+\delta'|(\alpha,\beta,\gamma)|,\mathsf{l}_2-|\beta|}_{\delta,0}, \tag{10}$$

while for general δ', the a_1 piece, as defined in (3), satisfies

$$a_1 \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,\delta'} \Rightarrow D^\alpha_z D^\beta_{z'} D^\gamma_\zeta a_1 \in S^{\mathsf{m}-|\gamma|+\delta|(\alpha,\beta,\gamma)|,\mathsf{l}_1-|\alpha|+\delta'|(\alpha,\beta,\gamma)|,\mathsf{l}_2-|\beta|}_{\delta,\delta'}, \tag{11}$$

where by the support property of a_1, $\delta'|(\alpha,\beta,\gamma)|$ could also be shifted to the last order). The analogue of (10) also holds for $S^{\infty,\infty,\infty}_{\infty,\delta}$, in which case l_1 and l_2 are unaffected by derivatives.

Having discussed symbols in some detail, we now turn to operators, starting with the constant order $S_{\infty,\delta}$-type setting. Note that unless $m < -n$, the integral (2) with $a \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$ is not absolutely convergent; if $m < -n$, it is, with the result $Au \in C(\mathbb{R}^n)$, and for $M > \ell_2 + n$,

$$\sup |\langle z\rangle^{-\ell_1} Au(z)| \leq C \|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},0} \|u\|_{\mathcal{S},0,M},$$

where C is a universal constant (independent of a and u) and

$$\|u\|_{\mathcal{S},k,M} = \sum_{|\alpha|\leq k} \sum_{|\beta|\leq M} \sup |z^\beta D^\alpha_z u|$$

are the Schwartz seminorms. However, if $m < -n$, one can also integrate by parts as usual in z', noting that $(1+\Delta_{z'}) e^{i\zeta\cdot(z-z')} = \langle\zeta\rangle^2 e^{i\zeta\cdot(z-z')}$, so

$$\begin{aligned} Au(z) &= (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} \langle\zeta\rangle^{-2N} (1+\Delta_{z'})^N e^{i\zeta\cdot(z-z')} a(z,z',\zeta)\, u(z')\, d\zeta\, dz' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\zeta\cdot(z-z')} \langle\zeta\rangle^{-2N} (1+\Delta_{z'})^N \big(a(z,z',\zeta)\, u(z')\big)\, d\zeta\, dz'. \end{aligned} \tag{12}$$

Expanding $(1+\Delta_{z'})^N(a(z,z',\zeta)\,u(z'))$, one deduces that

$$|(1+\Delta_{z'})^N (a(z,z',\zeta)\, u(z'))| \leq \langle z\rangle^{\ell_1} \langle z'\rangle^{\ell_2 - M} \langle\zeta\rangle^{m+2N\delta} \|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},2N} \|u\|_{\mathcal{S},2N,M}, \tag{13}$$

so for just $m + 2N\delta < -n + 2N$, i.e.

$$2(1-\delta)N > m+n,$$

the right hand side of (12) is integrable, and defining $Au \in C(\mathbb{R}^n)$ to be the result,

$$\sup |\langle z\rangle^{-\ell_1} Au(z)| \leq C \|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},2N} \|u\|_{\mathcal{S},2N,M}. \tag{14}$$

This gives an extension of $A = I(a)$ to $S^{m,\ell_1,\ell_2}_{\infty,\delta}$. Since $S^{\ell_1,\ell_2,-\infty,\delta}_{\infty,\delta}$ is dense in $S^{m,\ell_1,\ell_2}_{\infty,\delta}$ in the topology of $S^{m,\ell_1,\ell_2'}_{\infty,\delta}$ for $m' > m$, and since for $m < -n$, the expressions (12) for various N are all equal, the continuity property (14) shows that A is independent of the choice of N provided $m < -n + 2(1-\delta)N$ (since one can then take $m' \in (m, -n+2(1-\delta)N)$, and use the m'-continuity and density statements).

Now at least $Au \in C(\mathbb{R}^n)$, with a suitable bound, is defined, but in fact it is in $\mathcal{S}(\mathbb{R}^n)$. To see this, first note that $D^\alpha_z e^{i\zeta\cdot(z-z')} = \zeta^\alpha$, so for N sufficiently large, so that $m + |\alpha| < -n + 2(1-\delta)N$, differentiating under the integral sign and using the Leibniz rule,

$$(D_z^\alpha A)u(z) = \sum_{\gamma+\lambda\le\alpha} C_{\gamma\lambda}(2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} D_z^\gamma(e^{i\zeta\cdot(z-z')})\langle\zeta\rangle^{-2N} (1+\Delta_{z'})^N(D_z^\lambda a(z,z',\zeta)\,u(z'))\,d\zeta\,dz'$$

(15)

$$= \sum_{\gamma+\lambda\le\alpha} C_{\gamma\lambda}(2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\zeta\cdot(z-z')}\zeta^\gamma\langle\zeta\rangle^{-2N} (1+\Delta_{z'})^N(D_z^\lambda a(z,z',\zeta)\,u(z'))\,d\zeta\,dz',$$

with $C_{\gamma\lambda}$ combinatorial constants, so by (13) with a replaced by $D_z^\lambda a$, with $M > n+\ell_2$ still,

$$\sup|\langle z\rangle^{-\ell_1} D_z^\alpha Au(z)| \le C\|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},2N+|\alpha|}\|u\|_{\mathcal{S},2N,M}.$$

Further, $z_j e^{i\zeta\cdot(z-z')} = z'_j e^{i\zeta\cdot(z-z')} + D_{\zeta_j} e^{i\zeta\cdot(z-z')}$, so

$$z^\beta e^{i\zeta\cdot(z-z')} = (z'+D_\zeta)^\beta e^{i\zeta\cdot(z-z')} = \sum_{\mu+\nu\le\beta} C_{\mu\nu}(z')^\mu D_\zeta^\nu e^{i\zeta\cdot(z-z')},$$

so integration by parts in ζ gives

(16)

$$(z^\beta D_z^\alpha A)u(z) = \sum_{\gamma+\lambda\le\alpha}\sum_{\mu+\nu\le\beta} C_{\gamma\lambda}C_{\mu\nu}(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\zeta\cdot(z-z')} D_\zeta^\nu\big(\zeta^\gamma\langle\zeta\rangle^{-2N}(z')^\mu(1+\Delta_{z'})^N(D_z^\lambda a(z,z',\zeta)\,u(z'))\big)\,d\zeta\,dz'$$

$$= \sum_{\gamma+\lambda\le\alpha}\sum_{\mu+\nu\le\beta}\sum_{\nu'+\nu''\le\nu} C_{\gamma\lambda}C_{\mu\nu}C_{\nu'\nu''}(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\zeta\cdot(z-z')} D_\zeta^{\nu'}(\zeta^\gamma\langle\zeta\rangle^{-2N})(z')^\mu(1+\Delta_{z'})^N(D_\zeta^{\nu''}D_z^\lambda a(z,z',\zeta)\,u(z')))\,d\zeta\,dz'.$$

Thus with

$$M > n+\ell_2+|\beta| \text{ and } m+|\gamma|-|\nu'|-2N+(2N+|\nu''|+|\lambda|)\delta < -n,$$

the latter of which is implied by

$$m+|\alpha|+|\beta|\delta < -n+2(1-\delta)N,$$

we have

$$\sup|\langle z\rangle^{-\ell_1} z^\beta D_z^\alpha Au(z)| \le C\|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},2N+|\alpha|}\|u\|_{\mathcal{S},2N,M},$$

with C indendent of a, u. Now for $\ell_1 \le 0$, $\langle z\rangle^{-\ell_1}$ can simply be dropped, while for $\ell_1 > 0$ the $\langle z\rangle^{-\ell_1}$ factor can be absorbed into a sum $z^{\beta'}$ terms with $|\beta'| \le M'$ where $M' \ge \ell_1$, so we obtain that for

$$M' \ge \max(0,\ell_1),\ M > n+\ell_2+|\beta|+M',\ m+|\alpha|+|\beta|\delta < -n+2(1-\delta)N$$

we have

$$\sup|z^\beta D_z^\alpha Au(z)| \le C\|a\|_{S^{m,\ell_1,\ell_2}_{\infty,\delta},2N+|\alpha|}\|u\|_{\mathcal{S},2N,M},$$

so $Au \in \mathcal{S}(\mathbb{R}^n)$, and the map $A : \mathcal{S} \to \mathcal{S}$ is continuous, and in fact the stronger continuity property, namely that

$$S^{m,\ell_1,\ell_2}_{\infty,\delta} \times \mathcal{S} \ni (a,u) \mapsto I(a)u \in \mathcal{S}$$

is continuous, holds. Thus, we have the first claim of the following lemma, as well as the second in case $\delta'=0$:

Lemma 0.3. *The maps*

$$S^{m,\ell_1,\ell_2}_{\infty,\delta}\times\mathcal{S}\ni(a,u)\mapsto I(a)u\in\mathcal{S},$$
$$S^{m,\ell_1,\ell_2}_{\delta,\delta'}\times\mathcal{S}\ni(a,u)\mapsto I(a)u\in\mathcal{S},$$

are continuous.

Proof. To deal with general (not necessarily vanishing) $\delta'\in[0,1/2)$, let by using $\chi\in\mathcal{C}^\infty_c(\mathbb{R})$, $\chi\equiv 1$ on $[\frac{1}{2},2]$, supported in $[\frac{1}{4},4]$. Then we can write $a\in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ as

$$a=a_1+a_2,\ a_1=\chi\Big(\frac{\langle z\rangle}{\langle z'\rangle}\Big)a,\ a_2=\Big(1-\chi\Big(\frac{\langle z\rangle}{\langle z'\rangle}\Big)\Big)a,$$

with a_j depending continuously on a in the $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ topology. Now, since $\langle z\rangle\sim\langle z'\rangle\sim\langle(z,z')\rangle$ on $\operatorname{supp}a_1$, and since differentiation is local, a_1 satisfies estimates

$$|D^\alpha_z D^\beta_{z'} D^\gamma_\zeta a_1|\leq C_{\alpha\beta\gamma}\langle z\rangle^{\ell_1+\ell_2-|\alpha|-|\beta|+\delta'|(\alpha,\beta,\gamma)|}\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\beta,\gamma)|}.$$

Denoting the corresponding seminorms by $\|.\|_{\tilde S^{m,\ell_1+\ell_2}_{\delta,\delta'},N}$ temporarily, note that a_1 in $\tilde S^{m,\ell_1+\ell_2}_{\delta,\delta'}$ depends continuously on a. The right hand side of (13) becomes

$$\langle z\rangle^{\ell_1+\ell_2+2N\delta'}\langle z'\rangle^{-M}\langle\zeta\rangle^{m+2N\delta}\|a_1\|_{\tilde S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N}\|u\|_{\mathcal{S},2N,M},$$

so for $M>n$ and $m+2N\delta<-n+2N$ the right hand side of (12) is integrable, and (14) becomes

$$\sup|\langle z\rangle^{-\ell_1-\ell_2-2N\delta'}A_1u(z)|\leq C\|a_1\|_{\tilde S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N}\|u\|_{\mathcal{S},2N,M}. \tag{17}$$

In fact, using $\langle z\rangle\sim\langle z'\rangle$ on $\operatorname{supp}a_1$, taking $M>n+\ell_1+\ell_2+2N\delta+|\beta|$, $m+2N\delta<-n+2N$ (i.e. first choose N sufficiently large, then M sufficiently large), this even gives

$$\sup|z^\beta A_1u(z)|\leq C\|a\|_{\tilde S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N}\|u\|_{\mathcal{S},2N,M}.$$

To deal with derivatives, use (15) and note that the integrand is bounded by a constant multiple of

$$\sup_{|\gamma|+|\lambda|=|\alpha|}\langle z\rangle^{\ell_1+\ell_2+2N\delta'+|\lambda|\delta'}\langle z'\rangle^{-M}\langle\zeta\rangle^{m+2N\delta-2N+|\gamma|+|\lambda|\delta}\|a_1\|_{\tilde S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N+|\alpha|}\|u\|_{\mathcal{S},2N,M},$$

which in turn is bounded by

$$\sup_{|\gamma|+|\lambda|=|\alpha|}\langle z\rangle^{\ell_1+\ell_2+2N\delta'+|\alpha|\delta'}\langle z'\rangle^{-M}\langle\zeta\rangle^{m+2N\delta-2N+|\alpha|}\|a_1\|_{\tilde S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N+|\alpha|}\|u\|_{\mathcal{S},2N,M},$$

so in view of the support of a_1 first choosing N such that $m+2N\delta-2N+|\alpha|<-n$ and then M such that $M>n+\ell_1+\ell_2+2N\delta'+|\alpha|\delta'+|\beta|$

$$\sup|z^\beta D^\alpha_z A_1u(z)|\leq C\|a_1\|_{\tilde S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N}\|u\|_{\mathcal{S},2N,M}$$

follows, with C indendent of a_1, u. This shows that a_1 satisfies the conclusion of the lemma.

Now, to deal with a_2, integrate by parts in ζ, starting with (12) for $A_2=I(a_2)$ in place of $A=I(a)$, using

$$e^{i(z-z')\cdot\zeta}=\langle z-z'\rangle^{-2}(1+\Delta_\zeta)e^{i(z-z')\cdot\zeta},$$

so first for $m < -n$

$$
\begin{aligned}
A_2 u(z) &= (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\zeta\cdot(z-z')} \langle z - z'\rangle^{-2K} \\
&\qquad (1+\Delta_\zeta)^K \Big(\langle\zeta\rangle^{-2N} (1+\Delta_{z'})^N (a_2(z,z',\zeta)\, u(z')) \Big)\, d\zeta\, dz' \\
&= \sum_{|\mu|+|\nu|\le 2K} \tilde{C}_{\mu\nu} (2\pi)^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\zeta\cdot(z-z')} \langle z-z'\rangle^{-2K} (D_\zeta^\mu \langle\zeta\rangle^{-2N}) \\
&\qquad (1+\Delta_{z'})^N (D_\zeta^\nu a_2(z,z',\zeta)\, u(z'))\, d\zeta\, dz',
\end{aligned} \tag{18}
$$

where $\tilde{C}_{\mu\nu}$ are combinatorial constants. On the support of a_2,

$$\langle z - z'\rangle \ge C'(\langle z\rangle + \langle z'\rangle)$$

for some C', and now the integrand on the right hand hand side is bounded by a constant multiple of

$$\langle z\rangle^{\ell_1} \langle z'\rangle^{\ell_2 - M} \langle (z,z')\rangle^{-2K+(2N+2K)\delta'} \langle\zeta\rangle^{-2N+m+(2N+2K)\delta} \|a_2\|_{S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N+2K} \|u\|_{\mathcal{S},2N,M}.$$

For a given β, we can now even take $M = 0$, and take N, K so that

$$2N\delta' - (1-\delta')2K < -n - |\beta| - \ell_1 - \ell_2$$

and

$$-(1-\delta)2N + 2K\delta + m < -n;$$

to see that such a choice exists, take $K = N$, in which case sufficiently large N works as $1-2\delta, 1-2\delta' > 0$. We then deduce

$$\sup |z^\beta A_2 u(z)| \le C \|a\|_{S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N+2K} \|u\|_{\mathcal{S},2N,M}.$$

To deal with derivatives, we again use a calculation similar to (15) to obtain that

$$
\begin{aligned}
D_z^\alpha A_2 u(z) &= \sum_{\gamma+\kappa+\lambda\le\alpha} C_{\gamma\kappa\lambda} \sum_{|\mu|+|\nu|\le 2K} \tilde{C}_{\mu\nu} (2\pi)^{-n} \\
&\qquad \int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta^\gamma e^{i\zeta\cdot(z-z')} (D_z^\kappa \langle z-z'\rangle^{-2K}) (D_\zeta^\mu \langle\zeta\rangle^{-2N}) \\
&\qquad\qquad (1+\Delta_{z'})^N (D_\zeta^\nu D_z^\lambda a_2(z,z',\zeta)\, u(z'))\, d\zeta\, dz'.
\end{aligned} \tag{19}
$$

Since $D_z^\kappa \langle z-z'\rangle^{-2K} \le C\langle z-z'\rangle^{-2K}$ (indeed, one even has a bound $C\langle z - z'\rangle^{-2K-|\kappa|}$), so now the integrand on the right hand hand side is bounded by a constant multiple of

$$\langle z\rangle^{\ell_1} \langle z'\rangle^{\ell_2-M} \langle (z,z')\rangle^{-2K+(2N+2K+|\alpha|)\delta'} \langle\zeta\rangle^{-2N+m+(2N+2K+|\alpha|)\delta} \|a_2\|_{S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N+2K+|\alpha|} \|u\|_{\mathcal{S},2N,M},$$

which gives

$$\sup |z^\beta D_z^\alpha A_2 u(z)| \le C \|a_2\|_{S^{m,\ell_1,\ell_2}_{\delta,\delta'},2N+2K+|\alpha|} \|u\|_{\mathcal{S},2N,M}$$

when $M = 0$, and take N, K so that

$$2N\delta' - (1-\delta')2K + \delta'|\alpha| < -n - |\beta| - \ell_1 - \ell_2$$

and

$$-(1-\delta)2N + 2K\delta + m + |\alpha|\delta < -n,$$

which can be arranged exactly as in the $\alpha = 0$ case above. This completes the proof of the lemma. □

Note that for such an A with $m<-n$ to start, $u\in\mathcal{S}$, $\phi\in\mathcal{S}$,

$$\begin{aligned}\int Au(z)\phi(z)\,dz&=\int u(z')\Big(\int e^{i(-\zeta)\cdot(z'-z)}a(z,z',\zeta)\phi(z)\,dz\,d\zeta\Big)\,dz'\\&=\int u(z')\Big(\int e^{i\zeta\cdot(z'-z)}a(z,z',-\zeta)\phi(z)\,dz\,d\zeta\Big)\,dz'\\&=\int u(z')(I(b)\phi)(z')\,dz',\end{aligned}$$

where $b(z,z',\zeta)=a(z',z,-\zeta)$, so $b\in S^{\ell_2,\ell_1,m}_{\infty,\delta}$. Letting j to be the transposition map $j(z,z',\zeta)=(z',z,\zeta)$, ρ the reflection map $\rho(z,z',\zeta)=(z,z',-\zeta)$, so $\rho^*,j^*:S^{m,\ell_1,\ell_2}_{\infty,\delta}\to S^{\ell_2,\ell_1,m}_{\infty,\delta}$ are continuous for all m,ℓ_1,ℓ_2, and we have at first for $m<-n$,

$$\int(I(a)u)\phi=\int u(I(\rho^*j^*a)\phi),$$

so both sides being continuous trilinear maps $S^{m,\ell_1,\ell_2'}_{\infty,\delta}\times\mathcal{S}\times\mathcal{S}\to\mathbb{C}$ for all m',ℓ_1,ℓ_2, by the density of $S^{\ell_1,\ell_2,-\infty}_{\infty,\delta}$ in $S^{m,\ell_1,\ell_2}_{\infty,\delta}$ in the $S^{m,\ell_1,\ell_2'}_{\infty,\delta}$ topology for $m'>m$, the identity extends to all m. Thus, the Fréchet space adjoint, $I(a)^\dagger:\mathcal{S}'\to\mathcal{S}'$, defined by

$$(I(a)^\dagger\phi)(u)=\phi(I(a)u),\ \phi\in\mathcal{S}',\ u\in\mathcal{S},$$

satisfies

$$I(a)^\dagger\phi=I(\rho^*j^*a)\phi,\ \phi\in\mathcal{S},$$

i.e. by the weak-* density of $\mathcal{S}$ in $\mathcal{S}'$, $I(a)^\dagger$ is the unique continuous extension of $I(\rho^*j^*a)$ from $\mathcal{S}$ to $\mathcal{S}'$; one simply writes $I(\rho^*j^*a)=I(a)^\dagger$ even as maps $\mathcal{S}'\to\mathcal{S}'$. Since $\rho^*j^*\rho^*j^*a=a$, we deduce that for any a, $I(a)=I(\rho^*j^*a)^\dagger:\mathcal{S}'\to\mathcal{S}'$ is continuous.

Here we used the bilinear distributional pairing; if one uses the sesquilinear L^2-pairing, one has

$$\begin{aligned}\int Au(z)\overline{\phi(z)}\,dz&=\int u(z')\,\overline{\int e^{i\zeta\cdot(z'-z)}\overline{a(z,z',\zeta)}\phi(z)\,dz\,d\zeta}\,dz'\\&=\int u(z')\,\overline{(I(\tilde b)\phi)(z')}\,dz',\end{aligned}$$

$\tilde b(z,z',\zeta)=\overline{a(z',z,\zeta)}$, so using $*$ to denote the corresponding (Hilbert-space-type) adjoint

$$(I(a))^*=I(cj^*a),\tag{20}$$

where c is the complex conjugation map.

Note that if $a\in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ then $cj^*a\in S^{\ell_2,\ell_1,m}_{\delta,\delta'}$, thus the adjoint of operators given by our scattering symbols is still in the same class, with ℓ_2 and ℓ_1 reversed.

While we have two indices ℓ_1 and ℓ_2 for growth in the spatial variables, this is actually redundant, $\ell_1+\ell_2$ is the relevant quantity, as we have already seen signs of in the proof of Lemma 0.3 in the case of $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$: for the a_1 term the orders were interchangeable due to support properties, while the a_2 term was irrelevant.

Lemma 0.4. *Given $\ell\in\mathbb{R}$, the range of the map $a\mapsto I(a)$ is independent of the choice of ℓ_1 and ℓ_2 as long as $\ell_1+\ell_2=\ell$.*

Definition 1. We now define

$$\Psi^{m,\ell}_{\infty,\delta}(\mathbb{R}^n)=\{I(a):\ a\in S^{m,\ell,0}_{\infty,\delta}\}$$

and

$$\Psi^{m,\ell}_{\delta,\delta'}(\mathbb{R}^n)=\{I(a):\ a\in S^{m,\ell,0}_{\delta,\delta'}\};$$

we could have used $S^{m,\ell_1,\ell_2}_{\infty,\delta}$, resp. $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ instead for any ℓ_1,ℓ_2 with $\ell_1+\ell_2=\ell$.

Proof. To see this lemma for $S^{m,\ell_1,\ell_2}_{\infty,\delta}$, we note as in the proof of Lemma 0.3 that $(1+\Delta_\zeta)e^{i(z-z')\cdot\zeta}=\langle z-z'\rangle^2 e^{i(z-z')\cdot\zeta}$, so at first for $m<-n$, as usual, for $a\in S^{m,\ell_1,\ell_2}_{\infty,\delta}$,

(21)

$$\begin{aligned}(I(a)u)(z)&=(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle z-z'\rangle^{-2N}(1+\Delta_\zeta)^N(e^{i\zeta\cdot(z-z')})a(z,z',\zeta)\,u(z')\,dz',\\ &=(2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{i\zeta\cdot(z-z')}(\langle z-z'\rangle^{-2N}(1+\Delta_\zeta)^N a(z,z',\zeta))\,u(z')\,dz'=(I(b)u)(z),\end{aligned}$$

where

$$b(z,z',\zeta)=\langle z-z'\rangle^{-2N}(1+\Delta_\zeta)^N a(z,z',\zeta). \tag{22}$$

Notice that

$$\langle z\rangle^2=1+|z|^2\leq 1+(|z-z'|+|z|)^2\leq 1+2|z'|^2+2|z-z'|^2\leq 2\langle z-z'\rangle^2\langle z'\rangle^2, \tag{23}$$

and the analogous inequality also holds with z and z' interchanged, and

$$D^\alpha_z D^\beta_{z'}\langle z-z'\rangle^{-2N}\leq C_{\alpha\beta}\langle z-z'\rangle^{-2N},$$

so for any m,ℓ_1,ℓ_2, $a\in S^{m,\ell_1,\ell_2}_{\infty,\delta}$, with b defined by (22) satisfies $b\in S^{m,\ell_1+s,\ell_2-s}_{\infty,\delta}$ for $-2N\leq s\leq 2N$, and the map

$$S^{m,\ell_1,\ell_2}_{\infty,\delta}\ni a\mapsto b\in S^{m,\ell_1+s,\ell_2-s}_{\infty,\delta}$$

is continuous, hence $I(a)=I(b)$ holds for all m,ℓ_1,ℓ_2 (as it holds for $m<-n$). Given any s, choosing sufficiently large N, shows that the range of I on $S^{m,\ell_1,\ell_2}_{\infty,\delta}$ only depends on $\ell_1+\ell_2$.

Now, if $a\in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ then b defined by (22) is usually not in $S^{m,\ell_1+s,\ell_2-s}_{\delta,\delta'}$, as derivatives in z and z' do not typically give extra decay when hitting $\langle z-z'\rangle^{-2N}$. However, for the decomposition $a=a_1+a_2$ used in the proof of Lemma 0.3, on the support of the a_2 piece derivatives of $\langle z-z'\rangle^{-2N}$ have the required decay (indeed, one has decay in (z,z') jointly upon differentiation in either z or z'), so the corresponding b_2 satisfies $b_2\in S^{m,\ell_1-s,\ell_2-s'}_{\delta,\delta'}$ if $s+s'\leq 2N(1-\delta')$ (with δ' coming from the ζ derivatives), while the a_1 piece the weights ℓ_1 and ℓ_2 are directly equivalent as $\langle z\rangle\sim\langle z'\rangle$ on $\operatorname{supp} a_1$. □

We use this opportunity to remark that for the a_2 piece $I(a_2)$ of $I(a)$ in fact one has

$$I(a_2)\in\cap_{m',\ell'\in\mathbb{R}}\Psi^{m',\ell'}_{\delta,\delta'}=\Psi^{-\infty,-\infty}_{\delta,\delta'}. \tag{24}$$

We have already seen above that the analogue of this holds with $m'=m$ fixed, $l'\in\mathbb{R}$. In order to see that m' can be taken arbitrary as well, note that due to the support of a_2, we can use $\Delta_\zeta e^{i(z-z')\cdot\zeta}=|z-z'|^2 e^{i(z-z')\cdot\zeta}$ and integrate by parts in

ζ (noting that the diagonal singularity of $|z-z'|^{-2}$ is irrelevant due to the support of a_2) to see that

(25)
$$\begin{aligned}(I(a_2)u)(z) &= (2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} |z-z'|^{-2N}\Delta_\zeta^N(e^{i\zeta\cdot(z-z')})a_2(z,z',\zeta)\,u(z')\,dz',\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\zeta\cdot(z-z')}(|z-z'|^{-2N}\Delta_\zeta^N a_2(z,z',\zeta))\,u(z')\,dz' = (I(b_2)u)(z),\end{aligned}$$

where

$$b_2(z,z',\zeta) = |z-z'|^{-2N}\Delta_\zeta^N a_2(z,z',\zeta) \in S^{m-(1-\delta)2N,\ell_1-s,\ell_2-s'}_{\delta,\delta'} \tag{26}$$

if $s+s'\leq 2N(1-\delta')$. This shows (24). The analogue also holds on $S^{m,\ell_1,\ell_2}_{\infty,\delta}$, namely in that case the similarly defined a_2 gives rise to $I(a_2)\in\Psi^{-\infty,-\infty}_{\infty,\delta}$.

One very useful property of $\Psi^{m,\ell}_{\infty,\delta}(\mathbb{R}^n)$ is that it is in fact exactly the range of I acting on symbols of a special form, namely those independent of z'. Thus, let

$$\begin{aligned} a\in S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_\zeta) \Longleftrightarrow & a\in\mathcal{C}^\infty(\mathbb{R}^n_z\times\mathbb{R}^n_\zeta),\\ & |D_z^\alpha D_\zeta^\gamma a|\leq C_{\alpha\gamma}\langle z\rangle^\ell\langle\zeta\rangle^{m-|\gamma|+\delta|(\alpha,\gamma)|};\end{aligned}$$

so with

$$\pi_L:\mathbb{R}^n_z\times\mathbb{R}^n_{z'}\times\mathbb{R}^n_\zeta\to\mathbb{R}^n_z\times\mathbb{R}^n_\zeta$$

the projection map dropping z', $a\in S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_\zeta)$ if and only if

$$\pi_L^* a\in S^{m,\ell,0}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta).$$

As usual, the seminorms

$$\|a\|_{S^{m,\ell}_{\infty,\delta},N} = \sum_{|\alpha|+|\gamma|\leq N}\sup\langle z\rangle^{-\ell}\langle\zeta\rangle^{-m+|\gamma|-\delta|(\alpha,\gamma)|}|D_z^\alpha D_\zeta^\gamma a|$$

give a Fréchet topology. With π_R the projecting dropping the z variables, one also has $a\in S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n;\mathbb{R}^n)$ if and only if $\pi_R^* a\in S^{m,0,\ell}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta)$.

Then:

Proposition 0.5. *For any* $\ell=\ell_1+\ell_2$ *and* $a\in S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta)$ *there exists a unique* $a_L\in S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_\zeta)$ *such that* $I(a)=I(\pi_L^*a_L)$*; one writes* $q_L=I\circ\pi_L^*:S^{m,\ell}_{\infty,\delta}\to\Psi^{m,\ell}_{\infty,\delta}$*. Here* a_L *is called the* left reduced symbol *of* $I(a)$*, and* q_L *is the* left quantization *map.*

Similarly, for any $\ell=\ell_1+\ell_2$ *and* $a\in S^{m,\ell_1,\ell_2}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_{z'};\mathbb{R}^n_\zeta)$ *there exists a unique* $a_R\in S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n_z;\mathbb{R}^n_\zeta)$ *such that* $I(a)=I(\pi_R^*a_R)$*; one writes* $q_R=I\circ\pi_R^*:S^{m,\ell}_{\infty,\delta}\to\Psi^{m,\ell}_{\infty,\delta}$*. Here* a_R *is called the* right reduced symbol *of* $I(a)$*, and* q_R *is the* right quantization *map.*

Moreover, the maps $a\mapsto a_L, a\mapsto a_R$ *are continuous.*

Further, with $\iota:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ *the inclusion map as the diagonal in the first two factors, i.e.* $\iota(z,\zeta)=(z,z,\zeta)$*,*

$$a_L\sim\sum_\alpha\frac{i^{|\alpha|}}{\alpha!}\iota^* D^\alpha_{z'}D^\alpha_\zeta a, \tag{27}$$

and

$$a_R \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!}\iota^* D_z^\alpha D_\zeta^\alpha a,$$

with the summation asymptotic in ζ*, i.e. is modulo* $S^{-\infty,\ell}_{\infty,\delta}$.

If instead $a \in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$*, then the conclusions hold with* $a_L, a_R \in S^{m,\ell}_{\delta,\delta'}$*, with the asymptotic summation being asymptotic both in* z *and in* ζ*, i.e. is modulo* $S^{-\infty,-\infty}$.

In the case of variable orders, stated for $S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,\delta'}$ only:

Corollary 0.6. *If* $a \in S^{\mathsf{m},\mathsf{l}_1,\mathsf{l}_2}_{\delta,\delta'}$ *then* $a_L, a_R \in S^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}$*, where*

$$\mathsf{l}(z,\zeta) = \mathsf{l}_1(z,z,\zeta) + \mathsf{l}_2(z,z,\zeta).$$

This corollary is an immediate consequence of the asymptotic expansion in Proposition 0.5, for the αth term there is in $S^{\mathsf{m}-(1-2\delta)|\alpha|,\mathsf{l}-(1-2\delta')|\alpha|}_{\delta,\delta'}$.

Notice that for $a \in S^{m,\ell}_{\infty,\delta}$,

$$q_L(a)u(z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\zeta\cdot z} a(z,\zeta)\,(\mathcal{F}u)(\zeta)\,d\zeta \tag{28}$$

for $m < -n$, but now, for $u \in \mathcal{S}$, the right hand side extends continuously to $S^{m,\ell}_{\infty,\delta}$ for all m, so one could have directly defined $q_L(a)$ directly for all m. Similarly,

$$q_R(a)u = \mathcal{F}^{-1}(\zeta \mapsto \int_{\mathbb{R}^n} e^{-iz'\cdot\zeta} a(z',\zeta)\,u(z')\,dz'), \tag{29}$$

where now the right hand side makes sense directly as a tempered distribution for all m. However, relating q_L and q_R, as well as performing other important calculations, would be rather hard without having defined the map I in general, via a continuity/regularization argument! Note that for $a \in S^{-\infty,-\infty}_\infty$, in either case, one deduces that directly that $q_R(a)u$ and $q_L(a)u$ are in $\mathcal{S}$.

We remark that if $a \in S^{m,\ell}_\infty$ is a polynomial in ζ, i.e. $a(z,\zeta) = \sum_{|\alpha|\le m} a_\alpha(z)\zeta^\alpha$, then one can pull the factors $a_\alpha(z)$ out of the integral (28), and thus $\zeta^\alpha\mathcal{F} = \mathcal{F}D^\alpha$ and the Fourier inversion formula yields

$$q_L(a)u(z) = \sum_{|\alpha|\le m} a_\alpha(z)(D^\alpha u)(z),$$

i.e., with a_α acting as multiplication operators,

$$q_L(a) = \sum_{|\alpha|\le m} a_\alpha D^\alpha \tag{30}$$

Similarly,

$$q_R(a)u(z) = \sum_{|\alpha|\le m} (D^\alpha(a_\alpha u))(z),$$

i.e.

$$q_R(a) = \sum_{|\alpha|\le m} D^\alpha a_\alpha.$$

So differential operators of order m on $\mathbb{R}^n$ with coefficients in $S^\ell(\mathbb{R}^n)$ lie in $\Psi^{m,\ell}$. In particular, differential operators with coefficients in $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ lie in $\Psi^{m,0}(\mathbb{R}^n)$.

We now prove Proposition 0.5; we only consider the left reduction, i.e. the L subscript case as the R case is completely analogous. First, we note that the uniqueness is straightforward. Any operator $A = I(a)$, $a \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$, has a Schwartz kernel, $K_A \in \mathcal{S}'$ (as it is a continuous linear map $\mathcal{S} \to \mathcal{S}$, thus $\mathcal{S} \to \mathcal{S}'$). When $m < -n$, the Schwartz kernel satisfies

(31)
$$\begin{aligned} K_A(\phi \otimes u) &= \int (Au)(z)\phi(z)\,dz = (2\pi)^{-n} \int e^{i\zeta\cdot(z-z')} a(z,z',\zeta)\,u(z')\,\phi(z)\,d\zeta\,dz'\,dz \\ &= \int (\mathcal{F}_\zeta^{-1} a)(z,z',z-z')u(z')\phi(z)\,dz'\,dz, \end{aligned}$$

where $\mathcal{F}_\zeta^{-1}$ is the inverse Fourier transform in the third variable, ζ. ($\mathcal{F}_3^{-1}$ is a logically better, but less self-explanatory, notation.) Thus, for such a, K_A is the polynomially bounded function (hence tempered distribution) given by

$$F_a(z,z') = (\mathcal{F}_\zeta^{-1} a)(z,z',z-z') = (\mathcal{F}_3^{-1} a)(z,z',z-z'). \tag{32}$$

If $a \in S^{m,\ell}_{\infty,\delta}$, then, with 2 denoting that the inverse Fourier transform is in the second slot, we have

$$F_{\pi_L^* a}(z,z') = (\mathcal{F}_2^{-1} a)(z,z-z') = (G^* \mathcal{F}_2^{-1} a)(z,z')$$

where $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the invertible linear map $G(z,z') = (z,z-z')$, thus one can pull back tempered distributions by it. Thus,

$$K_{I(\pi_L^* a)} = G^* \mathcal{F}_2^{-1} a,$$

and correspondingly

$$a = \mathcal{F}_2 (G^{-1})^* K_{I(\pi_L^* a)},$$

first for $m < -n$, but then as both sides are continuous maps $S^{m,\ell}_{\infty,\delta} \to \mathcal{S}'$, this identity holds in general. In particular, given $\tilde{a} \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$ there exists at most one $a \in S^{m,\ell_1+\ell_2}_{\infty,\delta}$ such that $I(\pi_L^* a) = I(\tilde{a})$, for

$$a = \mathcal{F}_2 (G^{-1})^* K_{I(\tilde{a})} \tag{33}$$

then.

Now for existence. In principle (33) solves this problem, but then one needs to show that the a it provides, i.e. a_L in the notation of the proposition, is not merely a tempered distribution, but is in an appropriate symbol class. So we proceed differently.

With the notation of the proposition, one expands a in Taylor series in z' around the diagonal, with the integral remainder term:

(34)
$$\begin{aligned} a(z,z',\zeta) &= \sum_{|\alpha|\le N-1} \frac{(z-z')^\alpha}{\alpha!} ((\partial_{z'})^\alpha a)(z,z,\zeta) + R_N(z,z'\zeta) \\ R_N(z,z'\zeta) &= \sum_{|\alpha|=N} N \frac{(z-z')^\alpha}{\alpha!} \int_0^1 (1-t)^{N-1} ((\partial_{z'})^\alpha a)(z,(1-t)z+tz',\zeta)\,dt. \end{aligned}$$

Now, for $m < -n$, as $(z_j - z_j')e^{i\zeta\cdot(z-z')} = D_{\zeta_j} e^{i\zeta\cdot(z-z')}$,

$$\begin{aligned} (I((z_j - z_j')a)u)(z) &= (2\pi)^{-n} \int D_{\zeta_j} e^{i\zeta\cdot(z-z')} a(z,z',\zeta)\,u(z')\,dz'\,d\zeta \\ &= (2\pi)^{-n} \int e^{i\zeta\cdot(z-z')} (D_{\zeta_j} a)(z,z',\zeta)\,u(z')\,dz'\,d\zeta = (I(D_{\zeta_j} a)u)(z), \end{aligned}$$

so as

$$S^{m,\ell_1,\ell_2}_{\infty,\delta} \times \mathcal{S} \ni (a,u) \mapsto I((z_j - z'_j)a)u \in \mathcal{S}$$

and

$$S^{m,\ell_1,\ell_2}_{\infty,\delta} \times \mathcal{S} \ni (a,u) \mapsto I(D_{\zeta_j}a)u \in \mathcal{S}$$

are both continuous bilinear maps, the density of $S^{m,\ell_1,\ell_2}_{\infty,\delta}$ in the topology of $S^{m,\ell_1,\ell'_2}_{\infty,\delta}$ for $m' > m$ shows that

$$I((z-z')^\alpha a) = I(D^\alpha_\zeta a)$$

for all m and $a \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$.

Thus, for a as in (34)

$$I(a) = \sum_{|\alpha|\le N-1} \frac{1}{\alpha!} I((D_\zeta)^\alpha \iota^* \partial^\alpha_{z'} a) + I(R'_N),$$

$$R'_N(z,z'\zeta) = \sum_{|\alpha|=N} N\frac{1}{\alpha!}\int_0^1 (1-t)^{N-1} (D^\alpha_\zeta (\partial_{z'})^\alpha a)(z,(1-t)z+tz',\zeta)\,dt.$$

But

$$(D_\zeta)^\alpha \iota^* \partial^\alpha_{z'} a \in S^{m-(1-2\delta)|\alpha|,\ell_1,\ell_2}_{\infty,\delta},\ R'_N \in S^{m-(1-2\delta)N,\ell_1,\ell_2}_{\infty,\delta},$$

with the map

$$S^{m,\ell_1,\ell_2}_{\infty,\delta} \ni a \to (D_\zeta)^\alpha \iota^* \partial^\alpha_{z'} a \in S^{m-(1-2\delta)|\alpha|,\ell_1+\ell_2}_{\infty,\delta}$$

continuous, and similarly with R_N. Since $(D_\zeta)^\alpha \iota^* \partial^\alpha_{z'} a$ is independent of z', this proves the following weaker version of Proposition 0.5: for all $a \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$ and for all N there exist $a_N \in S^{m,\ell_1+\ell_2}_{\infty,\delta}$ such that

$$I(a) - I(a_N) = I(R'_N),\ R'_N \in S^{m-(1-2\delta)N,\ell_1,\ell_2}_{\infty,\delta}.$$

Notice that if $a \in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ then writing $a = a_1 + a_2$, we already know by (24) that for any m', ℓ'_1, ℓ'_2 we can write $I(a_2) = I(b_2)$, $b_2 \in S^{m',\ell'_1,\ell'_2}_{\delta,\delta'}$, while for a_1 the analogous conclusions to the $S^{m,\ell_1,\ell_2}_{\infty,\delta}$ setting hold but with

$$(D_\zeta)^\alpha \iota^* \partial^\alpha_{z'} a_1 \in S^{m-(1-2\delta)|\alpha|,\ell_1+\ell_2-(1-2\delta)|\alpha|},$$
$$R'_{1,N} \in S^{m-(1-2\delta)N,\ell_1-(1-2\delta')N,\ell_2}.$$

An asymptotic summation argument allows one to improve this. This means the following: suppose $a_j \in S^{m-(1-2\delta)j,\ell}_{\infty,\delta}$ for $j \in \mathbb{N}$. Then there exists $a \in S^{m,\ell}_{\infty,\delta}$ such that

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-(1-2\delta)N,\ell}_{\infty,\delta}. \tag{35}$$

To see this, we take $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\chi(\zeta) = 1$ for $|\zeta| \ge 2$, $\chi(\zeta) = 0$ for $|\zeta| \le 1$. For $0 < \epsilon_j < 1$ to be determined, but with $\epsilon_j \to 0$, consider

$$a(z,\zeta) = \sum_{j=0}^{\infty} \chi(\epsilon_j \zeta) a_j(z,\zeta);$$

the sum is finite for (z,ζ) with $|\zeta|\leq R$, with $\leq R+1$ terms. Thus, a is $\mathcal{C}^\infty$; the question is convergence in $S^{m,\ell}_{\infty,\delta}$, and the property (35). But by Leibniz' rule,

$$(D^\alpha_\zeta D^\beta_z a)(z,\zeta)=\sum_{j=0}^\infty\sum_{\gamma\leq\alpha}C_{\alpha\gamma}\epsilon_j^{|\gamma|}(D^\gamma\chi)(\epsilon_j\zeta)(D^{\alpha-\gamma}_\zeta D^\beta_z a_j)(z,\zeta).$$

To get convergence of the tail in $S^{m-(1-2\delta)N,\ell}_{\infty,\delta}$, we need to estimate the sup norm of

$$\begin{aligned}&\langle\zeta\rangle^{-m+(1-2\delta)N-\delta(|\alpha|+|\beta|)+|\alpha|}\langle z\rangle^{-\ell}\Big(D^\alpha_\zeta D^\beta_z\Big(\sum_{j=N}^\infty\chi(\epsilon_j\zeta)a_j(z,\zeta)\Big)\Big)\\&=\sum_{j=N}^\infty\sum_{\gamma\leq\alpha}C_{\alpha\gamma}\langle\zeta\rangle^{-\delta|\gamma|}\epsilon_j^{(j-N)(1-2\delta)}\Big(\langle\zeta\rangle^{|\gamma|+(N-j)(1-2\delta)}\epsilon_j^{(N-j)(1-2\delta)+|\gamma|}(D^\gamma\chi)(\epsilon_j\zeta)\Big)\\&\qquad\Big(\langle\zeta\rangle^{-m+(1-2\delta)j+(1-\delta)(|\alpha|-|\gamma|)-\delta|\beta|}\langle z\rangle^{-\ell}\langle z'\rangle^{-\ell}(D^{\alpha-\gamma}_\zeta D^\beta_z a_j)(z,\zeta)\Big);\end{aligned}$$

we used the above expansion. For $\gamma=0$, we use $|\zeta|\geq\epsilon_j^{-1}$ on $\operatorname{supp}\chi(\epsilon_j.)$, so for $j\geq N$ (as $\delta\in[0,1/2)$),

$$\epsilon_j^{(N-j)(1-2\delta)}\langle\zeta\rangle^{(N-j)(1-2\delta)}=(\epsilon_j^2+\epsilon_j^2|\zeta|^2)^{(1-2\delta)(N-j)/2}\leq 1,$$

while for $\gamma\neq 0$ we use $\epsilon_j^{-1}\leq|\zeta|\leq 2\epsilon_j^{-1}$ on $\operatorname{supp}(D^\gamma\chi)(\epsilon_j.)$, so

$$1\leq\langle\zeta\rangle\epsilon_j=(\epsilon_j^2+\epsilon_j^2|\zeta|^2)^{1/2}\leq 5^{1/2}$$

on $\operatorname{supp}(D^\gamma\chi)(\epsilon_j.)$ for all $\gamma\neq 0$, and thus for $j\geq N$,

$$\langle\zeta\rangle^{|\gamma|+(N-j)(1-2\delta)}\epsilon_j^{(N-j)(1-2\delta)+|\gamma|}\leq 5^{|\gamma|/2}$$

there. Thus, adding up the terms with $|\alpha|+|\beta|=M$, there are constants $C_M>0$ such that the series is absolutely summable, and hence convergent, if for all M

$$\sum_{j\geq N+(1-2\delta)^{-1}}^\infty C_M\epsilon_j\|a_j\|_{S^{m-(1-2\delta)j,\ell_1}_{\infty,\delta},M}$$

converges. Now, if $\|a_j\|_{S^{m-(1-2\delta)j,\ell}_{\infty,\delta},M}\leq R_{j,M}$, where $R_{j,M}$ are specified constants, then one can arrange the convergence by for instance requiring that for $j>M$, the summand is $\leq 2^{-j}$, i.e. that for $j>M$,

$$\epsilon_j\leq 2^{-j}C_M^{-1}R_{j,M}^{-1}.$$

Note that for each j this is finitely many constraints (as only the values of M with $M<j$ matter), which can thus be satisfied. Correspondingly, the tail of the series converges for each N in $S^{m-(1-2\delta)N,\ell}_{\infty,\delta}$, and thus $a\in S^{m,\ell}_{\infty,\delta}$ and also (35) holds. This gives a continuous asymptotic summation map on arbitrary bounded subsets of the product of the symbol spaces. (One can make the map globally defined and continuous by letting ϵ_j to be the minimum of, say,

$$2^{-j}C_M^{-1}(1+\|a_j\|_{S^{m-(1-2\delta)j,\ell}_{\infty,\delta},M})^{-1},$$

over $M=0,1,\ldots,j-1$, but this is actually not important below.)

Now, let

$$\tilde a\sim\sum_\alpha\frac{1}{\alpha!}(D_\zeta)^\alpha\iota^*\partial^\alpha_{z'}a\in S^{m,\ell_1+\ell_2}_{\infty,\delta};$$

asymptotic summation can be done so that the map $a \mapsto \tilde{a}$ is continuous. Then $\tilde{a} - a_N \in S^{m-(1-2\delta)N,\ell_1,\ell_2}_{\infty,\delta}$ for all N, and thus

$$I(a) - I(\tilde{a}) \in \cap_N I(S^{m-(1-2\delta)N,\ell_1,\ell_2}_{\infty,\delta}).$$

If $a \in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ then with

$$\tilde{a} \sim \sum_\alpha \frac{1}{\alpha!}(D_\zeta)^\alpha \iota^* \partial^\alpha_{z'} a \in S^{m,\ell_1+\ell_2}_{\delta,\delta'},$$

asymptotic sum both in the z and in the ζ variables,

$$I(a) - I(\tilde{a}) \in \cap_N I(S^{m-(1-2\delta)N,\ell_1,\ell_2-(1-2\delta')N}_{\delta,\delta'}).$$

The following lemma then finishes the proof of Proposition 0.5:

Lemma 0.7. *Suppose $b \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$ satisfies $I(b) \in \cap_N I(S^{m-N,\ell_1,\ell_2}_{\infty,\delta})$, i.e. for all $N \in \mathbb{N}$ there is $b_N \in S^{m-N,\ell_1,\ell_2}_{\infty,\delta}$ such that $I(b) = I(b_N)$. Then there exists $c \in S^{-\infty,\ell_1+\ell_2}_{\infty,\delta}$ such that $I(c) = I(b)$. Moreover, if there are continuous maps $j_N : b \to b_N$, then the map $b \to c$ is continuous.*

If instead $b \in S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ satisfies $I(b) \in \cap_N I(S^{m-N,\ell_1,\ell_2-N}_{\delta,\delta'})$, i.e. for all $N \in \mathbb{N}$ there is $b_N \in S^{m-N,\ell_1,\ell_2-N}_{\delta,\delta'}$ such that $I(b) = I(b_N)$. Then there exists $c \in S^{-\infty,-\infty}$ such that $I(c) = I(b)$. Moreover, if there are continuous maps $j_N : b \to b_N$, then the map $b \to c$ is continuous.

The idea of the proof is to use (33), as in the present setting the Schwartz kernel can be shown to be well-behaved, so (33) immediately gives the appropriate symbolic properties of c. Thus, we note that for all N there is $b_N \in S^{m-N,\ell_1,\ell_2}_{\infty,\delta}$ such that $I(b) = I(b_N)$, so taking N such that $m - N < -n$, (31)-(32) give that the Schwartz kernel (which is independent of N) is the continuous polynomially bounded function

$$K_{I(b_N)}(z,z') = (\mathcal{F}^{-1}_\zeta b_N)(z,z',z-z');$$

taking $m - N < -n - k$, this is in fact C^k with polynomial bounds up to the kth derivatives. Correspondingly, it satisfies, for $|\alpha|+|\beta|+\delta|\gamma| \le k$, and writing D^α_j for the αth derivative in the jth slot, M^α_j for the multiplication by the αth coordinate in the jth slot,

$$\begin{aligned}
&\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}(z-z')^\gamma D^\alpha_z D^\beta_{z'} K_{I(b_N)}(z,z') \\
&\quad = \big(\langle . \rangle_1^{-\ell_1}\langle . \rangle_2^{-\ell_2} M^\gamma_3 (D_1 + D_3)^\alpha (D_2 - D_3)^\beta (\mathcal{F}^{-1}_3 b_N)\big)(z,z',z-z') \\
&\quad = \big(\mathcal{F}^{-1}_3 \langle . \rangle_1^{-\ell_1}\langle . \rangle_2^{-\ell_2} D^\gamma_3 (D_1 + M_3)^\alpha (D_2 - M_3)^\beta b_N\big)(z,z',z-z').
\end{aligned}$$

As

$$\langle . \rangle_1^{-\ell_1}\langle . \rangle_2^{-\ell_2} D^\gamma_3 (D_1 + M_3)^\alpha (D_2 - M_3)^\beta b_N$$

is bounded in $C_\infty(\mathbb{R}^n \times \mathbb{R}^n; L^1(\mathbb{R}^n_\zeta))$ by a seminorm of b_N as $|\alpha| + |\beta| + \delta|\gamma| \le k$, $m - N < -n - k$, where C_∞ stands for bounded continuous functions,

$$\mathcal{F}^{-1}_3 \langle . \rangle_1^{-\ell_1}\langle . \rangle_2^{-\ell_2} D^\gamma_3 (D_1 + M_3)^\alpha (D_2 - M_3)^\beta b_N$$

is bounded in $C_\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ by a seminorm of b_N, hence the same holds for the pullback by the map $(z,z') \mapsto (z,z',z-z')$. Since N is arbitrary, we can take arbitrary α, β, γ and deduce that

$$\sup |\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}(z-z')^\gamma (D^\alpha_z D^\beta_{z'} K_{I(b)})(z,z')| < \infty.$$

Using (23) and that γ is arbitrary, we deduce that

$$\sup |\langle z\rangle^{-\ell_1-\ell_2}(z-z')^\gamma D_z^\alpha D_{z'}^\beta K_{I(b)}| < \infty. \tag{36}$$

Since we want $K_{I(c)} = K_{I(b)}$, we need

$$(\mathcal{F}_2^{-1}c)(z, z-z') = K_{I(b)}(z, z'),$$

i.e. with $w = z - z'$,

$$(\mathcal{F}_2^{-1}c)(z, w) = K_{I(b)}(z, z-w).$$

Now, a linear change of variables for $K_{I(b)}$ gives that

$$\sup |\langle z\rangle^{-\ell_1-\ell_2} w^\gamma (D_z^\alpha D_w^\beta \mathcal{F}_2^{-1}c)(z, w)| < \infty,$$

so $\langle z\rangle^{-\ell_1-\ell_2} D_z^\alpha \mathcal{F}_2^{-1}c$ is Schwartz in w, uniformly in z, and thus $\langle z\rangle^{-\ell_1-\ell_2} D_z^\alpha c$ is Schwartz in the second variable, ζ, uniformly in z, i.e. $c \in S^{-\infty,\ell_1+\ell_2}_{\infty,\delta}$. This also shows that any seminorm of c depends only on the seminorms of b_N for some N, and does so continuously, and thus depends on b continuously.

The argument in the case of $S^{m,\ell_1,\ell_2}_{\delta,\delta'}$ is completely analogous, but now even

$$\begin{aligned}
&\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}(z')^\mu (z-z')^\gamma D_z^\alpha D_{z'}^\beta K_{I(b_N)}(z, z') \\
&\quad = \big(\langle .\rangle_1^{-\ell_1}\langle .\rangle_2^{-\ell_2} M_3^\gamma (D_1+D_3)^\alpha (D_2-D_3)^\beta (\mathcal{F}_3^{-1}b_N)\big)(z, z', z-z') \\
&\quad = \big(\mathcal{F}_3^{-1}\langle .\rangle_1^{-\ell_1}\langle .\rangle_2^{-\ell_2} M_2^\mu D_3^\gamma (D_1+M_3)^\alpha (D_2-M_3)^\beta b_N\big)(z, z', z-z'),
\end{aligned}$$

with the result that

$$\sup |\langle z\rangle^{-\ell_1}\langle z'\rangle^{-\ell_2}(z')^\mu (z-z')^\gamma (D_z^\alpha D_{z'}^\beta K_{I(b)})(z, z')| < \infty.$$

Using (23) and that γ, μ are arbitrary, we deduce that

$$\sup |(z')^\mu (z-z')^\gamma D_z^\alpha D_{z'}^\beta K_{I(b)}| < \infty.$$

This gives $K_{I(b)} \in \mathcal{S}(\mathbb{R}^{2n})$, and the argument is finished as before. This completes the proof of Lemma 0.7.

As a corollary of the lemma, we note that elements of $\Psi^{-\infty,\ell}_{\infty,\delta}$ have a $\mathcal{C}^\infty$ Schwartz kernel, of the form $\mathcal{C}^\infty(\mathbb{R}^n_z; \mathcal{S}(\mathbb{R}^n_{z'}))$, and thus give continuous linear maps $\mathcal{S}' \to \mathcal{C}^\infty(\mathbb{R}^n)$, i.e. are *smoothing*. Note that this does not mean decay at infinity. On the other hand, elements of $\Psi^{-\infty,-\infty}$ are *completely regularizing*, as their Schwartz kernel is in $\mathcal{S}(\mathbb{R}^{2n})$, and thus they give maps $\mathcal{S}' \to \mathcal{S}$. Note that maps $\mathcal{S}' \to \mathcal{S}$ are actually compact on all polynomially weighted Sobolev spaces $H^{r,s}$.

The isomorphism $q_L : S^{m,\ell}_{\infty,\delta} \to \Psi^{m,\ell}_{\infty,\delta}$ can be used to topologize $\Psi^{m,\ell}_{\infty,\delta}$. Since $q_R^{-1} \circ q_L$, $q_L^{-1} \circ q_R$ are continuous, this is the same topology as that induced by q_R.

Note that if $a \in S^{m,\ell_1,\ell_2}_{\infty,\delta}$ then $\iota^* a - a_L, \iota^* a - a_R \in S^{m-1+2\delta,\ell_1+\ell_2}_{\infty,\delta}$, while if $a \in S^{m,\ell_1,\ell_2}$ then $\iota^* a - a_L, \iota^* a - a_R \in S^{m-1+2\delta,\ell_1+\ell_2-1+2\delta'}_{\infty,\delta}$. We thus make the following definition:

Definition 2. The principal symbol $\sigma_{\infty,m,\ell}(q_L(a))$ in $\Psi^{m,\ell}_{\infty,\delta}$ of $q_L(a)$, $a \in S^{m,\ell}_{\infty,\delta}$, is the equivalence class $[a]_\infty$ of a in $S^{m,\ell}_{\infty,\delta}/S^{m-1+2\delta,\ell}_{\infty,\delta}$.

The joint principal symbol $\sigma_{m,\ell}(q_L(a))$ in $\Psi^{m,\ell}$ of $q_L(a)$, $a \in S^{m,\ell}_{\delta,\delta'}$, is the equivalence class $[a]$ of a in $S^{m,\ell}_{\delta,\delta'}/S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}$.

In case the orders are variable, the principal symbols $\sigma_{\infty,\mathsf{m},\mathsf{l}}(q_L(a))$ and $\sigma_{\mathsf{m},\mathsf{l}}(q_L(a))$ are defined analogously in $S^{\mathsf{m},\mathsf{l}}_{\infty,\delta}/S^{\mathsf{m}-1+2\delta,\mathsf{l}}_{\infty,\delta}$, resp. $S^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}/S^{\mathsf{m}-1+2\delta,\mathsf{l}-1+2\delta'}_{\delta,\delta'}$.

Thus, the principal symbol also satisfies

$$\sigma_{\infty,m,\ell}(q_R(a)) = [a]_\infty,\ \sigma_{m,\ell}(q_R(a)) = [a],$$

with analogues for variable orders.

For $a \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) \subset S^{0,0}$, there is a natural identification of the equivalence class, namely the restriction of a to $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ can be identified with its equivalence class, namely changing a by any element of $\mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ which vanishes in the boundary, and thus is in $S^{-1,-1}$ does not affect the equivalence class, so the map $a \mapsto [a]$ descends to $a|_{\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})} \to [a]$, and is injective. Note that $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ is a manifold with corners with two boundary hypersurfaces, $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ and $\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$, so equivalently one can restrict to each of these separately, and keep in mind that the restrictions must agree at the corner, $\partial\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$.

In the case of σ_∞, a common way of understanding it is in terms of the $\mathbb{R}^+$-action by dilations on the second factor of $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$:

$$\mathbb{R}^+ \times \overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\}) \ni (t, z, \zeta) \mapsto (z, t\zeta) \in \overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\}).$$

The quotient of $\mathbb{R}^n \setminus \{0\}$ by the $\mathbb{R}^+$ action can be identified with the unit sphere $\mathbb{S}^{n-1}$: every orbit of the $\mathbb{R}^+$-action intersects the sphere in exactly one point. A different identification of this quotient (which is actually more relevant from the perspective of where our analysis actually takes place) is the sphere at infinity, $\partial\overline{\mathbb{R}^n}$. Thus, homogeneous degree zero $\mathcal{C}^\infty$ functions on $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$ are identified with either $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ or $\mathcal{C}^\infty(\mathbb{R}^n \times \partial\overline{\mathbb{R}^n})$. So one can correspondingly identify the principal symbol of $A = q_L(a_L)$, $a_L \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, as a function on $\overline{\mathbb{R}^n} \times \mathbb{S}^{n-1}$, or instead as a homogeneous degree zero function on $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$.

For

$$a = \langle z\rangle^\ell \langle\zeta\rangle^m \tilde{a},\ \tilde{a} \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}),$$

one cannot simply restrict a to the boundary, though as (given ℓ and m) a and $\tilde{a}$ are in a bijective correspondence, one could restrict $\tilde{a}$ and call it the principal symbol, i.e. the actual principal symbol, as we defined it, is given by any $\mathcal{C}^\infty$ extension of this restriction times $\langle z\rangle^\ell\langle\zeta\rangle^m$. In a more geometric context this is not quite natural (depends on the differentials of choices of boundary defining functions, here $\langle z\rangle^{-1}$ and $\langle\zeta\rangle^{-1}$, at the boundary). Taking $\ell = 0$ as it is the most common case, in terms of the $\mathbb{R}^+$ action on the second factor, it is more common then to view the part of the principal symbol corresponding to $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ as a *homogeneous degree* m function on $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$. In terms of $\tilde{a}$ and its identification with a homogeneous degree zero function on $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$, the part of the principal symbol corresponding to $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ is

$$\sigma_{\mathrm{fiber},m,0}(A) = |\zeta|^m \tilde{a}.$$

On the other hand, the part of the principal symbol corresponding to $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ can be described by simply restricting to $\partial\overline{\mathbb{R}^n} \times \mathbb{R}^n$, with the result being symbolic in the second variable:

$$\sigma_{\mathrm{base},m,0}(A) = \langle\zeta\rangle^m \tilde{a}|_{\partial\overline{\mathbb{R}^n} \times \mathbb{R}^n}.$$

Concretely, if A is a differential operator, $A = \sum a_\alpha D^\alpha$, $a_\alpha \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$, then the two parts of the principal symbol under this identification are

$$\sigma_{\mathrm{fiber},m,0}(A)(z,\zeta) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha,\ (z,\zeta) \in \overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\}), \tag{37}$$

and

$$\sigma_{\mathrm{base},m,0}(A)(z,\zeta)=\sum_{|\alpha|\le m} a_\alpha(z)\zeta^\alpha,\ (z,\zeta)\in\partial\overline{\mathbb{R}^n}\times\mathbb{R}^n. \tag{38}$$

As an example, if g is a Riemannian metric on $\mathbb{R}^n$ with $g_{ij}\in\mathcal{C}^\infty(\overline{\mathbb{R}^n})$, then for $V\in\langle z\rangle^{-1}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$,

$$H=\Delta_g+V-\sigma \tag{39}$$

has principal symbol in these two senses given by

$$\sigma_{\mathrm{fiber},2,0}=\sum g_{ij}\zeta_i\zeta_j,\ \sigma_{\mathrm{base},2,0}=\sum g_{ij}\zeta_i\zeta_j-\sigma.$$

In the case of σ, one could apply a similar construction for the restriction of the symbol of $A=q_L(a_L)$ to $\partial\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$; it is then either a homogeneous degree zero function on $(\mathbb{R}^n\setminus\{0\})\times\overline{\mathbb{R}^n}$ where the action is in the first factor, or a function on $\mathbb{S}^{n-1}\times\overline{\mathbb{R}^n}$; the last version would be rarely considered. Thus, two different point of views would be needed for describing σ in terms of homogeneous functions, which is the reason for this being a less useful point of view in this case than in that of σ_∞.

Proposition 0.8. *The sequences*

$$0\to\Psi^{m-1+2\delta,\ell}_{\infty,\delta}\to\Psi^{m,\ell}_{\infty,\delta}\to S^{m,\ell}_{\infty,\delta}/S^{m-1+2\delta,\ell}_{\infty,\delta}\to 0,$$

resp.

$$0\to\Psi^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}\to\Psi^{m,\ell}_{\delta,\delta'}\to S^{m,\ell}_{\delta,\delta'}/S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}\to 0,$$

are short exact sequences of topological vector spaces.

Here $\iota:\Psi^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}\to\Psi^{m,\ell}_{\delta,\delta'}$ *is the inclusion map and*

$$\sigma_{m,\ell}:\Psi^{m,\ell}_{\delta,\delta'}\to S^{m,\ell}_{\delta,\delta'}/S^{m-1+2\delta,\ell-1+2\delta}_{\delta,\delta'}$$

is the principal symbol map, with analogous definitions in the case of $\Psi_{\infty,\delta}$.

The analogous statements also hold if $m=\mathsf{m}$, $\ell=\mathsf{l}$ *are variable.*

This is essentially tautological, given the short exact sequence

$$0\to S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}\to S^{m,\ell}_{\delta,\delta'}\to S^{m,\ell}_{\delta,\delta'}/S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}\to 0,$$

and the isomorphisms $q_{L,m',\ell'}:S^{m',\ell'}_{\delta,\delta'}\to\Psi^{m',\ell'}_{\delta,\delta'}$ with $m'=m,m-1+2\delta$, $\ell'=\ell,\ell-1+2\delta'$, and that these are consistent with the inclusion $\iota_S:S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}\to S^{m,\ell}_{\delta,\delta'}$, i.e. that one has a commutative diagram $q_{L,m,\ell}\circ\iota_S=\iota\circ q_{L,m-1+2\delta,\ell-1+2\delta'}$.

We also define operator wave front sets, for which variable orders are irrelevant. We first start with the microsupport of symbols:

Definition 3. Suppose $a\in S^{m,\ell}_{\delta,\delta'}(\mathbb{R}^n;\mathbb{R}^n)$. We say that $\alpha\in\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ is not in $\mathrm{esssupp}(a)$ if there is a neighborhood U of α in $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$ such that $a|_{U\cap(\mathbb{R}^n\times\mathbb{R}^n)}$ is $\mathcal{S}=S^{-\infty,-\infty}$ (i.e. satisfies Schwartz estimates in U).

Similarly, for $a\in S^{m,\ell}_{\infty,\delta}(\mathbb{R}^n;\mathbb{R}^n)$ we say that $\alpha\in\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n}$ is not in $\mathrm{esssupp}_{\infty,\ell}(a)$ if there is a neighborhood U of α in $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$ such that $a|_{U\cap(\mathbb{R}^n\times\mathbb{R}^n)}$ is $S^{-\infty,\ell}$ (i.e. satisfies the corresponding symbol estimates in U).

In either case, esssupp is called the microsupport, or essential support, of a.

Now for operators we define the wave front set in terms of the microsupport of their left amplitudes a_L.

Definition 4. Suppose that $A \in \Psi^{m,\ell}_{\delta,\delta'}$, $A = q_L(a_L)$. We write

$$\mathrm{WF}'(A) = \mathrm{esssupp}(a),$$

i.e. we say that $\alpha \in \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ is not in $\mathrm{WF}'(A)$ if there is a neighborhood U of α in $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ such that $a_L|_{U\cap(\mathbb{R}^n\times\mathbb{R}^n)}$ is $\mathcal{S} = S^{-\infty,-\infty}$ (i.e. satisfies Schwartz estimates in U).

Similarly, for $A \in \Psi^{m,\ell}_{\infty,\delta}$, we write $\mathrm{WF}'_{\infty,\ell}(A) = \mathrm{esssupp}_{\infty,\ell}(A)$.

Note that directly from the definition, the complement of esssupp, and thus the wave front set, is open, i.e. the wave front set itself is closed. Further, even for $\mathrm{WF}'_{\infty,\ell}$, ℓ is only relevant for $\alpha \in \partial\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$; one commonly simply writes WF'_∞, or indeed WF'. While the principal symbol captures the leading order behavior of a pseudodifferential operator, the (complement of the) wave front set captures where it is 'trivial'.

As an example, if $a \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, $A = q_L(a)$, then $\mathrm{WF}'(A) \subset \operatorname{supp} a \cap \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, since certainly in the complement of $\operatorname{supp} a$, a vanishes, and is thus a symbol of order $-\infty, -\infty$. However, notice that the containment is not an equality, as e.g. $a \in \mathcal{S}(\mathbb{R}^{2n})$ which never vanishes on $\mathbb{R}^{2n}$ (e.g. a Gaussian) has support everywhere, but $\mathrm{WF}'(q_L(a)) = \emptyset$. Thus, the more precise statement is that $\alpha \notin \mathrm{WF}'(A)$ for such a, A, if α has a neighborhood U in $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ on which the full Taylor series of a vanishes.

Again, as in the case of the principal symbol, one could consider $\mathrm{WF}'_{\infty,\ell}$ a subset of $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$ which is invariant under the $\mathbb{R}^+$-action (dilations in the second factor), i.e. which is *conic*; this is the standard point of view. The corresponding statement for WF' is, as in the case of the principal symbol, more awkward, and is thus less common.

In view of Proposition 0.5, one could also use a_R with $A = q_R(a_R)$ in place of a_L in the definition. Also, as $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ and $\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$ are compact, so symbol estimates corresponding to an open cover imply symbol estimates everywhere, so

Lemma 0.9. *If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ and $\mathrm{WF}'(A) = \emptyset$, then $A \in \Psi^{-\infty,-\infty}$.*

If $A \in \Psi^{m,\ell}_{\infty,\delta}$ and $\mathrm{WF}'_{\infty,\ell}(A) = \emptyset$, then $A \in \Psi^{-\infty,\ell}_{\infty}$.

The analogues also hold in variable order spaces.

We also have from (20) that

Proposition 0.10. *If $A \in \Psi^{m,\ell}_{\infty,\delta}$ then $A^* \in \Psi^{m,\ell}_{\infty,\delta}$ and*

$$\sigma_{\infty,m,\ell}(A^*) = \overline{\sigma_{\infty,m,\ell}(A)},\ \mathrm{WF}'_\infty(A^*) = \mathrm{WF}'_\infty(A).$$

If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ then $A^* \in \Psi^{m,\ell}_{\delta,\delta'}$ and

$$\sigma_{m,\ell}(A^*) = \overline{\sigma_{m,\ell}(A)},\ \mathrm{WF}'(A^*) = \mathrm{WF}'(A).$$

The analogues also hold in variable order spaces.

We can also strengthen the surjectivity part of Proposition 0.8:

Proposition 0.11. *For $a \in S^{m,\ell}_{\infty,\delta}$ there exists $A \in \Psi^{m,\ell}_{\infty,\delta}$ with $\sigma_{\infty,m,\ell}(A) = [a]$ and $\mathrm{WF}'_\infty(A) \subset \mathrm{esssupp}_\infty a$.*

Similarly, for $a \in S^{m,\ell}_{\delta,\delta'}$ there exists $A \in \Psi^{m,\ell}_{\delta,\delta'}$ with $\sigma_{m,\ell}(A) = [a]$ and $\mathrm{WF}'(A) \subset \mathrm{esssupp}\, a$.

The analogues also hold in variable order spaces.

Indeed, taking $A = q_L(a)$ or $A = q_R(a)$ will do the job.

The most important part of a treatment of pseudodifferential operators is their properties under composition and commutators:

Proposition 0.12. *If* $A \in \Psi^{m,\ell}_{\infty,\delta}$*,* $B \in \Psi^{m',\ell'}_{\infty,\delta}$*, then* $AB \in \Psi^{m+m',\ell+\ell'}_{\infty,\delta}$*,*

$$\sigma_{\infty,m+m',\ell+\ell'}(AB) = \sigma_{\infty,m,\ell'}(A)\sigma_{\infty,m',\ell'}(B),$$

and

$$\mathrm{WF}'_\infty(AB) \subset \mathrm{WF}'_\infty(A) \cap \mathrm{WF}'_\infty(B).$$

If $A \in \Psi^{m,\ell}_{\delta,\delta'}$*,* $B \in \Psi^{m',\ell'}_{\delta,\delta'}$*, then* $AB \in \Psi^{m+m',\ell+\ell'}_{\delta,\delta'}$*, and*

$$\sigma_{m+m',\ell+\ell'}(AB) = \sigma_{m,\ell'}(A)\sigma_{m',\ell'}(B),$$

and

$$\mathrm{WF}'(AB) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B).$$

The analogues also hold in variable order spaces.

Thus, Ψ_∞ and Ψ are order-filtered $*$-algebras, and in case of Ψ_∞, composition is commutative to leading order in terms of the differential order, m, while in the case of Ψ, it is commutative to leading order in both the differential and the growth orders m and ℓ.

This proposition is proved easily using Proposition 0.5, taking advantage of (28) and (29). To do so, first assume $a, b \in S^{-\infty,-\infty}_\infty$, then

$$\begin{aligned}
&(q_L(a)q_R(b)u)(z)\\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\zeta\cdot z} a(z,\zeta)\,(\mathcal{F}\mathcal{F}^{-1}(\zeta' \mapsto \int_{\mathbb{R}^n} e^{-iz'\cdot\zeta'} b(z',\zeta')\,u(z')\,dz'))\,d\zeta\\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\zeta\cdot(z-z')} a(z,\zeta) b(z',\zeta)\,u(z')\,dz'\,d\zeta = (I(c)u)(z),
\end{aligned}$$

with

$$c(z,z',\zeta) = a(z,\zeta)b(z',\zeta) \in S^{-\infty,-\infty,-\infty}_\infty.$$

However, with $c = c(a,b)$ so defined, the map

$$S^{m,\ell}_{\infty,\delta} \times S^{m',\ell'}_{\infty,\delta} \ni (a,b) \mapsto c \in S^{\ell,\ell',m+m'}_{\infty,\delta}$$

is continuous, so as both trilinear maps

$$(a,b,u) \mapsto q_L(a)q_R(b)u,\ \ (a,b,u) \mapsto I(c(a,b))u$$

are continuous

$$S^{m,\ell}_{\infty,\delta} \times S^{m',\ell'}_{\infty,\delta} \times \mathcal{S} \to \mathcal{S}$$

for all m, m', ℓ, ℓ', it follows that

$$q_L(a)q_R(b) = I(c(a,b)).$$

Since q_L, q_R are isomorphisms, the closedness of $\Psi^{m,\ell}_{\infty,\delta}$ under composition is immediate, as is the continuity of composition. As for the principal symbol, this follows as if $B \in \Psi^{m',\ell'}_{\infty,\delta}$, $B = q_R(b)$, then $\sigma_{\infty,m',\ell'}(B) = b$, and then by (27), $I(c(a,b)) = q_L(c_L)$ with $c_L - ab \in S^{m+m'-1+2\delta,\ell+\ell'}_{\infty,\delta}$. The wave front set statement is also immediate in view of (27).

In the case of Ψ, the same arguments go through, but corresponding to the improvement in (27), $c_L - ab \in S^{m+m'-1+2\delta,\ell+\ell'-1+2\delta'}_{\delta,\delta'}$.

Going one order further in the expansions, one obtains the principal symbol of the commutators. Here we recall the Poisson bracket on $\mathbb{R}^n_z \times \mathbb{R}^n_\zeta$, identified with $T^*\mathbb{R}^n$:

$$\{a, b\} = \sum_{j=1}^{n} \left((\partial_{\zeta_j} a)(\partial_{z_j} b) - (\partial_{z_j} a)(\partial_{\zeta_j} b)\right).$$

Proposition 0.13. *If $A \in \Psi^{m,\ell}_{\infty,\delta}$, $B \in \Psi^{m',\ell'}_{\infty,\delta}$, then $[A, B] \in \Psi^{m+m'-1+2\delta,\ell+\ell'}_{\infty,\delta}$, and*

$$\sigma_{\infty,m+m'-1+2\delta,\ell+\ell'}(AB) = \frac{1}{i}\{\sigma_{\infty,m,\ell'}(A), \sigma_{\infty,m',\ell'}(B)\}.$$

If $A \in \Psi^{m,\ell}_{\delta,\delta'}$, $B \in \Psi^{m',\ell'}_{\delta,\delta'}$, then $[A, B] \in \Psi^{m+m'-1+2\delta,\ell+\ell'-1+2\delta'}$, and

$$\sigma_{m+m'-1+2\delta,\ell+\ell'-1+2\delta'}(AB) = \frac{1}{i}\{\sigma_{m,\ell'}(A), \sigma_{m',\ell'}(B)\}.$$

The analogues also hold in variable order spaces.

We now turn to the simplest consequences of the machinery we built up, such as the parametrix construction for elliptic operators.

Definition 5. We say that A is elliptic in $\Psi^{m,\ell}_{\infty,\delta}$, resp. $\Psi^{m,\ell}_{\delta,\delta'}$, if $[a]_\infty$, resp. $[a]$, is invertible, i.e. if there exists $[b]_\infty \in S^{m,\ell}_{\infty,\delta}/S^{m-1+2\delta,\ell}_{\infty,\delta}$, resp. $[b] \in S^{m,\ell}_{\delta,\delta'}/S^{m-1+2\delta,\ell-1+2\delta'}_{\delta,\delta'}$ with $[a]_\infty[b]_\infty = [1]$ in $S^{0,0}_{\infty,\delta}/S^{-1+2\delta,0}_{\infty,\delta}$, resp. $[a][b] = [1]$ in $S^{0,0}_{\delta,\delta'}/S^{-1+2\delta,-1+2\delta'}_{\delta,\delta'}$.

More generally, we make the analogous definition if $m = \mathsf{m}$, $l = \mathsf{l}$ are variable.

These definitions are equivalent to the statements that there exist $c > 0$, $R > 0$ such that

$$|a| \geq c\langle z\rangle^\ell \langle\zeta\rangle^m,\ c > 0, |\zeta| > R, \tag{40}$$

resp.

$$|a| \geq c\langle z\rangle^\ell \langle\zeta\rangle^m,\ c > 0, |\zeta| + |z| > R; \tag{41}$$

indeed, if a satisfies this, the reciprocal is easily seen to satisfy the appropriate conditions in $|\zeta| > R$, resp. $|z|+|\zeta| > R$, and the multiplying by a cutoff, identically 1 near infinity, in ζ, resp. (z, ζ), gives b. Conversely, if b exists, upper bounds for $|b|$ give the desired lower bounds for $|a|$.

Concretely, if $A = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ as in (1), then under the identification of the part of the principal symbol at $\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$ with a homogeneous degree m function on $\overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\})$, while identifying the principal symbol at $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ as an mth order symbol on $\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$, ellipticity means:

$$z \in \overline{\mathbb{R}^n}, \zeta \neq 0 \Rightarrow \sum_{|\alpha|=m} a_\alpha \zeta^\alpha \neq 0,$$

and

$$z \in \partial\overline{\mathbb{R}^n}, \zeta \in \overline{\mathbb{R}^n} \Rightarrow \sum_{|\alpha|\leq m} a_\alpha \zeta^\alpha \neq 0.$$

For $H = \Delta_g + V - \sigma$ as in (39), ellipticity does means

$$\begin{aligned} &(z, \zeta) \in \overline{\mathbb{R}^n} \times (\mathbb{R}^n \setminus \{0\}),\ \zeta \neq 0 \Rightarrow \sum g_{ij}(z)\zeta_i\zeta_j, \neq 0,\\ &(z, \zeta) \in \partial\overline{\mathbb{R}^n} \times \mathbb{R}^n \Rightarrow \sum g_{ij}\zeta_i\zeta_j - \sigma \neq 0. \end{aligned} \tag{42}$$

Now the first is just the statement that g is a Riemannian metric on $\mathbb{R}^n$ in the uniform sense we discussed; the second holds if and only if $\sigma \notin [0,\infty)$. Note that if $V \in S^{-\rho}(\mathbb{R}^n)$ instead, $\rho > 0$, then V does affect the principal symbol in the second sense, but it does *not* affect ellipticity.

If A is elliptic in $\Psi^{m,\ell}_{\delta,\delta'}$ (with the variable order case going through without changes), say, then one can construct a parametrix B with a residual, or completely regularizing, error, i.e. $B \in \Psi^{-m,-\ell}_{\delta,\delta'}$ such that

$$AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty,-\infty}.$$

Indeed, one takes any B_0 with $\sigma_{-m,-\ell}(B_0)$ being the inverse for $\sigma_{m,\ell}(A)$, so

$$\sigma_{0,0}(AB_0 - \mathrm{Id}) = \sigma_{m,\ell}(A)\sigma_{-m,-\ell}(B_0) - 1 = 0,$$

thus $E_0 = AB_0 - \mathrm{Id} \in \Psi^{-1+2\delta,-1+2\delta'}_{\delta,\delta'}$. Now, $AB_0 = \mathrm{Id} + E_0$, so one wants to invert $\mathrm{Id} + E_0$ approximately; this can be done by a finite Neumann series, $\mathrm{Id} + \sum_{j=1}^N (-1)^j E_0^j$, then

$$(\mathrm{Id} + E_0)(\mathrm{Id} + \sum_{j=1}^N (-1)^j E_0^j) - \mathrm{Id} \in \Psi^{-(1-2\delta)(N+1),-(1-2\delta')(N+1)}_{\delta,\delta'}.$$

This can be improved by writing $E_0^j = q_L(e_j)$, then computing the asymptotic sum

$$\tilde{e} \sim \sum_{j=1}^\infty (-1)^j e_j \in S^{-1+2\delta,-1+2\delta'}_{\delta,\delta'},$$

taking $\tilde{E} = q_L(\tilde{e})$, $(\mathrm{Id} + E_0)(\mathrm{Id} + \tilde{E}) - \mathrm{Id} \in \Psi^{-\infty,-\infty}$, so $B = B_0(\mathrm{Id} + \tilde{E})$ provides a right parametrix: $E = AB - \mathrm{Id} \in \Psi^{-\infty,-\infty}$. A left parametrix B' can be constructed similarly, and the standard identities showing the identity of left and right inverses in a semigroup, as applied to the quotient by completely regularizing operators, shows that $B - B' \in \Psi^{-\infty,-\infty}$, so one may simply replace B' by B. Indeed, if $B'A = \mathrm{Id} + E'$,

$$\begin{aligned} B' &= B'(AB - E) = (B'A)B - B'E = B - E'B - B'E, \\ B'E&, EB' \in \Psi^{-\infty,-\infty}. \end{aligned} \tag{43}$$

Notice that all of the constructions can be done uniformly as long (41) is satisfied for a fixed c and R, i.e. one can construct the maps $A \mapsto B, E$ such that they are continuous from the set of elliptic operators to $\Psi^{-m,-\ell}_{\delta,\delta'}$ resp. $\Psi^{-\infty,-\infty}$.

If $A \in \Psi^{m,\ell}_{\infty,\delta}$ then the same argument only gains in the first order, m, so one obtains a parametrix $B \in \Psi^{-m,-\ell}_{\infty,\delta}$ with errors $E, E' \in \Psi^{-\infty,0}_\infty$.

We have thus proved:

Proposition 0.14. *If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ is elliptic then there exists $B \in \Psi^{-m,-\ell}_{\delta,\delta'}$ such that $AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty,-\infty}$. Further, the maps $A \mapsto B \in \Psi^{-m,-\ell}_{\delta,\delta'}$ and $A \mapsto AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty,-\infty}$ can be taken to be continuous from the set of elliptic operators in $\Psi^{m,\ell}_{\delta,\delta'}$ (an open subset of $\Psi^{m,\ell}_{\delta,\delta'}$), equipped with the $\Psi^{m,\ell}_{\delta,\delta'}$ topology.*

If $A \in \Psi^{m,\ell}_{\infty,\delta}$ is elliptic then there exists $B \in \Psi^{-m,-\ell}_{\infty,\delta}$ such that $AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty,0}_{\infty,\delta}$. Again, the maps $A \mapsto B \in \Psi^{-m,-\ell}_{\infty,\delta}$ and $A \mapsto AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty,0}_{\infty,\delta}$ can be taken to be continuous from the set of elliptic operators in $\Psi^{m,\ell}_{\infty,\delta}$.

The analogous variable order statements also hold.

If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ elliptic is invertible in the weak sense that there exist $G : \mathcal{S} \to \mathcal{S}'$ continuous such that $GA = \mathrm{Id} : \mathcal{S} \to \mathcal{S}$ and $AG = \mathrm{Id} : \mathcal{S} \to \mathcal{S}$ then, with B a parametrix for A, $BA - \mathrm{Id} = E_L$, $AB - \mathrm{Id} = E_R$,

$$G = G(AB + E_R) = B + GE_R = B + (BA + E_L)GE_R = B + BE_R + E_L GE_R,$$

with the first two terms on the right in $\Psi^{-m,-\ell}_{\delta,\delta'}$, resp. $\Psi^{-\infty,-\infty}$, and the last term is residual as well since it is a continuous linear map $\mathcal{S}' \to \mathcal{S}$, and thus has Schwartz kernel in $\mathcal{S}(\mathbb{R}^{2n})$, thus is in $\Psi^{-\infty,-\infty}$. Hence $G \in \Psi^{-m,-\ell}$, and $G - B \in \Psi^{-\infty,-\infty}$. Thus, the inverses of actually invertible elliptic operators are pseudodifferential operators themselves.

As a corollary we have elliptic regularity:

Proposition 0.15. *If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ (or more generally $A \in \Psi^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}$) is elliptic and $Au \in \mathcal{S}$ for some $u \in \mathcal{S}'$ then $u \in \mathcal{S}$.*

Proof. Let B be a parametrix for A with $BA - \mathrm{Id} = E \in \Psi^{-\infty,-\infty}$. Then

$$u = \mathrm{Id}\, u = (BA + E)u = B(Au) + Eu,$$

and $Eu \in \mathcal{S}$ as E is completely regularizing while $Au \in \mathcal{S}$ by assumption, hence $B(Au) \in \mathcal{S}$ as well. □

We can now discuss Hörmander's proof of L^2-boundedness of elements of $\Psi^{0,0}_{\delta,\delta'}$, or indeed $\Psi^{0,0}_{\infty,\delta}$, via a square root construction.

Lemma 0.16. *Suppose that $A \in \Psi^{0,0}_{\infty,\delta}$ is elliptic, symmetric ($A^* = A$) with principal symbol that has a positive (bounded below) representative a. Then there exists $B \in \Psi^{0,0}_{\infty,\delta}$ such that B is symmetric and $A = B^2 + E$ with $E \in \Psi^{-\infty,0}_{\infty}$. The maps $A \mapsto B \in \Psi^{0,0}_{\infty,\delta}$ and $A \mapsto E \in \Psi^{-\infty,0}_{\infty}$ can be taken continuous from the set of A satisfying these constraints (equipped with the $\Psi^{0,0}_{\infty,\delta}$ topology).*

The same result holds with the (∞,δ) subscript replaced by (δ,δ'), but with $E \in \Psi^{-\infty,-\infty}$.

Proof. Let $b_0 = \sqrt{a}$; one easily checks that $b_0 \in S^{0,0}_{\infty}$. Let $\tilde{B}_0 \in \Psi^{0,0}_{\infty,\delta}$ have principal symbol b_0, and let $B_0 = \frac{1}{2}(\tilde{B}_0 + \tilde{B}_0)^*$, so B_0 still has principal symbol b_0 and is symmetric. Then $A - B_0^2$ has vanishing principal symbol, so $E_0 = A - B_0^2 \in \Psi^{-1+2\delta,0}_{\infty,\delta}$, providing the first step in the construction.

In general, for $j \in \mathbb{N}$, suppose one has found $B_j \in \Psi^{0,0}_{\infty,\delta}$ symmetric such that $E_j = A - B_j^2 \in \Psi^{-(1-2\delta)(j+1),0}_{\infty,\delta}$; we have shown this for $j = 0$. Let e_j be the principal symbol of E_j, and let $b_{j+1} = -\frac{1}{2b_0} e_j \in S^{-(1-2\delta)(j+1),0}_{\infty,\delta}$; this uses b_0 elliptic. Let $\tilde{B}_{j+1} \in \Psi^{-(1-2\delta)(j+1),0}_{\infty,\delta}$ have principal symbol b_{j+1}, $B'_{j+1} = \frac{1}{2}(\tilde{B}_{j+1} + \tilde{B}^*_{j+1})$, $B_{j+1} = B_j + B'_{j+1}$, so B_{j+1} is symmetric. Further, the principal symbol of

$$\begin{aligned} A - B_{j+1}^2 &= A - (B_j + B'_{j+1})^2 = A - B_j^2 - B_j B'_{j+1} - B'_{j+1} B_j - (B'_{j+1})^2 \\ &= E_j - B_j B'_{j+1} - B'_{j+1} B_j - (B'_{j+1})^2 \in \Psi^{-(1-2\delta)(j+1),0}_{\infty,\delta} \end{aligned}$$

is $e_j - 2b_0 b_{j+1} = 0$, so $E_{j+1} = A - B_{j+1}^2 \in \Psi^{-(1-2\delta)(j+2),0}_{\infty,\delta}$, providing the inductive steps. One can finish up by asymptotically summing, as in the elliptic case. □

Proposition 0.17. *Elements $A \in \Psi^{0,0}_{\infty,\delta}$ are bounded on L^2.*

Further, if a is a representative of $\sigma_{\infty,0,0}(A)$ and $C > \inf_{r\in S^{-1+2\delta,0}_{\infty,\delta}} \sup|a+r|$ then there exists $E \in \Psi^{-\infty,0}_{\infty}$ such that

$$\|Au\|_{L^2} \leq C\|u\|_{L^2} + |\langle Eu, u\rangle|.$$

Moreover, the map $A \mapsto E \in \Psi^{-\infty,0}_{\infty}$ can be taken to be continuous, and thus the inclusion $\Psi^{0,0}_{\infty,\delta} \to \mathcal{L}(L^2)$ is continuous.

Proof. We reduce the proof to the boundedness of elements of $\Psi^{-\infty,0}_{\infty}$ on L^2, which is in easy consequence of Schur's lemma since by (36), the Schwartz kernel of elements of this space is a bounded continuous function in z with values in $\mathcal{S}(\mathbb{R}^n_z)$ (hence with values in $L^1(\mathbb{R}^n_{z'})$), and similarly with z and z' interchanged.

Now, suppose that $A \in \Psi^{0,0}_{\infty,\delta}$, so its principal symbol has a bounded representative a; let $M > \sup|a|$. Then $M^2 - |a|^2 \in S^{0,0}_{\infty,\delta}$ is bounded below by a positive constant, and is thus elliptic. By Lemma 0.16, there exists $B \in \Psi^{0,0}_{\infty,\delta}$ symmetric such that $M^2 - A^*A = B^2 + E$, $E \in \Psi^{-\infty,0}_{\infty}$. Then, first for $u \in \mathcal{S}$, with inner products and norms the standard L^2 ones,

$$\langle M^2 u, u\rangle = \|Au\|^2 + \|Bu\|^2 + \langle Eu, u\rangle,$$

i.e. with $\|E\|_{\mathcal{L}(L^2)}$ the L^2 bound of E, which is finite as discussed above,

$$\|Au\|^2 \leq M^2\|u\|^2 + \|E\|_{\mathcal{L}(L^2)}\|u\|^2.$$

Since $\mathcal{S}$ is dense in L^2, this implies that A has a unique continuous extension to L^2; one still denotes it by L^2. Since $\mathcal{S}$ is also dense in $\mathcal{S}'$, and the inclusion $L^2 \to \mathcal{S}'$ is continuous, this extension is the restriction of A acting on $\mathcal{S}'$. This proves the first part of the proposition.

For the second part we simply replace a by $a + r$, choosing $r \in S^{-1+2\delta,0}_{\infty,\delta}$ such that $C > \sup|a+r|$, then we can take $M = C$ in the argument above to complete the proof. □

While elements of $\Psi^{0,0}_{\delta,\delta'}$ are in $\Psi^{0,0}_{\infty,\delta}$ for $\delta' = 0$ and are thus L^2-bounded, it is useful to make the bound more explicit there as well, in addition to generalizing to $\delta' > 0$:

Proposition 0.18. *Elements $A \in \Psi^{0,0}_{\delta,\delta'}$ are bounded on L^2.*

Further, if a is a representative of $\sigma_{0,0}(A)$ and $C > \inf_{r\in S^{-1+2\delta,-1+2\delta'}_{\delta,\delta'}} \sup|a+r|$ then there exists $E \in \Psi^{-\infty,-\infty}$ such that

$$\|Au\|_{L^2} \leq C\|u\|_{L^2} + |\langle Eu, u\rangle|. \tag{44}$$

Moreover, the map $A \mapsto E \in \Psi^{-\infty,-\infty}$ can be taken to be continuous.

Concretely, if $A = q_L(a)$ with $a \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, then for any

$$C > \sup\left|a|_{\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})}\right|,$$

(44) *holds.*

Proof. This is the same argument as above, but constructing B in $\Psi^{0,0}_{\delta,\delta'}$. □

We now recall that the weighted Sobolev spaces are

$$H^{s,r} = \{u \in \mathcal{S}' :\ \langle z\rangle^r u \in H^s\},\ \|u\|_{H^{s,r}} = \|\langle z\rangle^r u\|_{H^s}$$

Further, with

$$\Lambda_s = \mathcal{F}^{-1}\langle\zeta\rangle^s\mathcal{F} \in \Psi^{s,0} \subset \Psi^{s,0}_\infty,$$

$H^s = \{u :\ \Lambda_s u \in L^2\}$ with $\|u\|_{H_s} = \|\Lambda^s u\|_{L^2}$. We note here $\cup_{M,N\in\mathbb{R}} H^{M,N} = \mathcal{S}'$. Thus, $\Lambda_{s,r} = \Lambda_s\langle z\rangle^r : H^{s,r} \to L^2$ is an isometry, with inverse $\Lambda'_{-s,-r} = \langle z\rangle^{-r}\Lambda_{-s} : L^2 \to H^{s,r}$. Hence, the boundedness of some $A \in \Psi^{m,\ell}_{\infty,\delta}$ as a map $H^{s,r} \to H^{s',r'}$ is equivalent to the boundedness on L^2 of $\Lambda_{s',r'}A\Lambda'_{-s,-r}$ as $A = \Lambda'_{-s',-r'}(\Lambda_{s',r'}A\Lambda'_{-s,-r})\Lambda_{s,r}$. But $\Lambda_{s',r'}A\Lambda'_{-s,-r} \in \Psi^{m+s'-s,\ell+r'-r}_{\infty,\delta}$, so we conclude that

Proposition 0.19. *An operator $A \in \Psi^{m,\ell}_{\infty,\delta}$ is bounded $H^{s,r} \to H^{s',r'}$ if $m = s - s'$ and $\ell = r - r'$ (thus if $m \le s - s'$ and $\ell \le r - r'$).*

This gives a quantified version of elliptic regularity:

Proposition 0.20. *If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ is elliptic and $Au \in H^{s,r}$ for some $u \in \mathcal{S}'$ then $u \in H^{s+m,r+\ell}$. In fact, for any M, N there is $C > 0$ such that*

$$\|u\|_{H^{s+m,r+\ell}} \le C(\|Au\|_{H^{s,r}} + \|u\|_{H^{M,N}}).$$

If $A \in \Psi^{m,\ell}_{\infty,\delta}$ is elliptic and $Au \in H^{s,r}$ for some $u \in H^{k,r+\ell}$, $k \in \mathbb{R}$, then $u \in H^{s+m,r+\ell}$. In fact, for any k there is $C > 0$ such that

$$\|u\|_{H^{s+m,r+\ell}} \le C(\|Au\|_{H^{s,r}} + \|u\|_{H^{k,r+\ell}}).$$

The point of the quantitative estimate is to allow M, N very negative, so e.g. $H^{s+m,r+\ell} \to H^{M,N}$ is compact. One thinks of $\|u\|_{H^{M,N}})$ as a 'trivial' term correspondingly.

In the case of $\Psi^{m,\ell}_{\infty,\delta}$ ellipticity is too weak of a notion to gain decay at infinity; one simply has a uniform gain of Sobolev regularity.

Proof. Suppose $A \in \Psi^{m,\ell}_{\delta,\delta'}$. Let $B \in \Psi^{-m,-\ell}_{\delta,\delta'}$ be a parametrix for A with $BA - \mathrm{Id} = E \in \Psi^{-\infty,-\infty}$. Then

$$u = \mathrm{Id}\, u = (BA + E)u = B(Au) + Eu,$$

and $Eu \in \mathcal{S}$ while $Au \in H^{s,r}$ by assumption, hence $B(Au) \in H^{s+m,r+\ell}$, as claimed. The bound in the proposition follows from $E : H^{M,N} \to H^{s+m,r+\ell}$ being bounded.

If $A \in \Psi^{m,\ell}_{\infty,\delta}$, and $B \in \Psi^{-m,-\ell}_{\infty,\delta}$ is a parametrix, so $BA - \mathrm{Id} = E \in \Psi^{-\infty,0}_\infty$ then the same argument gives, using $E : H^{k,r+\ell} \to H^{s+m,r+\ell}$ bounded, the conclusion that $u \in H^{s+m,r+\ell}$, as well as the estimate. □

An immediate corollary is:

Proposition 0.21. *Any elliptic $A \in \Psi^{m,\ell}_{\delta,\delta'}$ is Fredholm as a map $H^{s,r} \to H^{s-m,r-\ell}$ for all $m, \ell, s, r \in \mathbb{R}$, i.e. has closed range, finite dimensional nullspace and the range has finite codimension. Further, the nullspace is a subspace of $\mathcal{S}$, while the annihilator of the range in $H^{s-m,r-\ell}$ in the dual space $H^{-s+m,-r+\ell}$ is also in $\mathcal{S}$. Correspondingly, the nullspace of A as well as the annihilator of its range is independent of r, s; if A is invertible for one value of r, s, then it is invertible for all.*

Proof. If B is a parametrix for A, then $B \in \mathcal{L}(H^{s-m,r-\ell}, H^{s,r})$ and $E_L = BA - \mathrm{Id}, E_R = AB - \mathrm{Id} \in \Psi^{-\infty,\infty}$, thus map $H^{s,r}$, resp. $H^{s-m,r-\ell}$ to $\mathcal{S}$ continuously, and are thus compact as maps in $\mathcal{L}(H^{s,r})$, resp. $\mathcal{L}(H^{s-m,r-\ell})$. Then standard arguments give the Fredholm property.

The property of the nullspace being in $\mathcal{S}$ is elliptic regularity. If v is in the annihilator as stated, i.e. $\langle v, Au\rangle = 0$ for all $u \in H^{s,r}$ then $\langle A^* v, u\rangle = 0$ for all $u \in H^{s,r}$, so $A^* v = 0$ in $H^{-s,-r}$. As A^* has principal symbol $\bar{a}$, elliptic regularity shows that $v \in \mathcal{S}$. □

Corollary 0.22. *Suppose $m, \ell > 0$, $A \in \Psi^{m,\ell}_{\delta,\delta'}$ is symmetric on L^2 and is elliptic. Then A is self-adjoint with domain $H^{m,\ell}$.*

Proof. It suffices to show that $A - \sigma : H^{m,\ell} \to L^2$ are invertible for $\sigma \in \mathbb{C} \setminus \mathbb{R}$. As $m, \ell > 0$, these are elliptic regardless of σ, thus Fredholm as stated, with nullspace and annihilator of the cokernel in $\mathcal{S}$. But the symmetry of A shows that for u in the kernel, $0 = \mathrm{Im}\langle (A-\sigma)u, u\rangle = -\mathrm{Im}\,\sigma \|u\|^2$, so $u = 0$, hence the kernel is trivial. Thus, the kernel of $A^* = A$ is also trivial, so A is surjective, thus the desired invertibility follows. □

Corollary 0.23. *Suppose $m \geq 0$, $\ell \geq 0$, $A \in \Psi^{m,\ell}_{\delta,\delta'}$ is symmetric on L^2 and is elliptic. Then A is self-adjoint with domain $H^{m,\ell}$.*

Proof. We have already dealt with $m, \ell > 0$; $m, \ell = 0$ is standard, so it remains to deal with $m > 0$, $\ell = 0$ as $m = 0$, $\ell > 0$ is similar. Again, it suffices to show that $A - \sigma : H^{m,\ell} \to L^2$ are invertible for $\sigma \in \mathbb{C} \setminus \mathbb{R}$. As the principal symbol has a real representative a,

$$|a - \sigma|^2 = |a - \mathrm{Re}\,\sigma|^2 + |\mathrm{Im}\,\sigma|^2 \geq c\langle\zeta\rangle^{2m},\ c > 0,$$

since $|a| \geq c_0\langle\zeta\rangle^m$, $c > 0$, so for $|a| \geq 2|\mathrm{Re}\,\sigma|$, $|a - \mathrm{Re}\,\sigma|^2 \geq (|a| - |\mathrm{Re}\,\sigma|)^2 \geq |a|^2/4$, while for $|a| \leq 2|\mathrm{Re}\,\sigma|$, $\langle\zeta\rangle^m \leq 2c_0^{-1}|\mathrm{Re}\,\sigma|$, so ζ is bounded, and then the $\mathrm{Im}\,\sigma$ term gives the desired positivity. So $A - \sigma$ is elliptic when $\mathrm{Im}\,\sigma \neq 0$, thus Fredholm as stated, with nullspace and annihilator of the cokernel in $\mathcal{S}$. Again, the symmetry of A shows that for u in the kernel, $0 = \mathrm{Im}\langle (A-\sigma)u, u\rangle = -\mathrm{Im}\,\sigma\|u\|^2$, so $u = 0$, hence the kernel of $A - \sigma$ is trivial. Thus, the kernel of $A^* = A$ is also trivial, so A is surjective, thus the desired invertibility follows. □

We summarize our results so far for the Schrödinger operators:

Proposition 0.24. *Let g be a Riemannian metric, $g_{ij} \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$, positive definite on $\overline{\mathbb{R}^n}$, $V \in S^{-\rho}(\mathbb{R}^n)$ with $\rho > 0$. Let $H = \Delta_g + V$. Then for $\sigma \in \mathbb{C} \setminus [0, \infty)$, $H - \sigma : H^{s,r} \to H^{s-2,r}$ is Fredholm for all r, s, with nullspace in $\mathcal{S}$. If V is real-valued, then H is self-adjoint.*

We can now define variable order Sobolev spaces.

Definition 6. Let $A \in \Psi^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}$ be elliptic, $\mathsf{m} \geq m$, $\mathsf{l} \geq \ell$. Let $H^{\mathsf{m},\mathsf{l}}$ be subspace of $H^{m',\ell'}$ given by

$$H^{\mathsf{m},\mathsf{l}} = \{u \in H^{m,\ell} :\ Au \in L^2\},$$

with norm

$$\|u\|^2_{H^{\mathsf{m},\mathsf{l}}} = \|u\|^2_{H^{m,\ell}} + \|Au\|^2_{L^2}.$$

Then $H^{\mathsf{m},\mathsf{l}}$ is easily seen to be a complete space, thus a Hilbert space, which in the case of m,l being constant, simply gives $H^{m,\ell}$. Indeed, if $\{u_j\}_{j=1}^\infty$ is Cauchy in $H^{\mathsf{m},\mathsf{l}}$, then it is Cauchy in $H^{m,\ell}$, so it converges to some $u\in H^{m,\ell}$; in addition Au_j is Cauchy in L^2 so converges to some $v\in L^2$. But $A:\mathcal{S}'\to\mathcal{S}'$ is continuous, so $Au_j\to Au$ in $\mathcal{S}'$, so $v=Au\in L^2$, thus $u\in H^{\mathsf{m},\mathsf{l}}$. Further, as $Au_j\to Au$ in L^2, the completeness of $H^{\mathsf{m},\mathsf{l}}$ follows.

Moreover, different choices of both A and (m,ℓ) are equivalent in the sense that they give the same space with equivalent norms: if $\tilde A\in\Psi^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}$ is elliptic as well, writing $B\in\Psi^{-\mathsf{m},-\mathsf{l}}_{\delta,\delta'}$ as a parametrix, with $E=BA-\mathrm{Id}\in\Psi^{-\infty,-\infty}_{\delta,\delta'}$,

$$\tilde A u=\tilde A(BA)-\tilde A E u=(\tilde A B)Au-(\tilde A E)u$$

with $\tilde A B\in\Psi^{0,0}_{\delta,\delta'}$, $\tilde A E\in\Psi^{-\infty,-\infty}_{\delta,\delta'}$, we deduce that $\tilde A u\in L^2$, and $\|\tilde A u\|^2\leq C(\|u\|^2_{H^{m,\ell}}+\|Au\|^2_{L^2})$, showing that the $\tilde A$-based norm is bounded by the A-based norm. A similar argument gives the converse estimate, thus the equivalence of norms.

We conclude

Proposition 0.25. *An operator $A\in\Psi^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}$ is bounded $H^{\mathsf{s},\mathsf{r}}\to H^{\mathsf{s}',\mathsf{r}'}$ if $\mathsf{m}=\mathsf{s}-\mathsf{s}'$ and $\mathsf{l}=\mathsf{r}-\mathsf{r}'$ (thus if $\mathsf{m}\leq\mathsf{s}-\mathsf{s}'$ and $\mathsf{l}\leq\mathsf{r}-\mathsf{r}'$).*

Proof. Let s,r be such that $s\leq\mathsf{s}$, $r\leq\mathsf{r}$. Such an $A\in\Psi^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}\subset\Psi^{m,\ell}_{\delta,\delta'}$ maps $H^{s,r}$ to $H^{s-m,r-\ell}$ continuously. Further, if $\tilde A\in\Psi^{\mathsf{s},\mathsf{r}}_{\delta,\delta'}$, $\tilde A'\in\Psi^{\mathsf{s}',\mathsf{r}'}_{\delta,\delta'}$ are elliptic, then with $\tilde B\in\Psi^{-\mathsf{s}',-\mathsf{r}'}_{\delta,\delta'}$, $\tilde B\tilde A-\mathrm{Id}=\tilde E\in\Psi^{-\infty,-\infty}_{\delta,\delta'}$, then

$$\tilde A' A u=(\tilde A' A\tilde B)\tilde A u-(\tilde A' A\tilde E)u,$$

with $\tilde A' A\tilde B\in\Psi^{0,0}_{\delta,\delta'}$ and $\tilde A' A\tilde E\in\Psi^{-\infty,-\infty}_{\delta,\delta'}$, thus bounded on L^2, giving the conclusion. □

One then has a Fredholm and a self-adjointness statement as above for the variable order setting.

The elliptic parametrix construction can be microlocalized, i.e. if the principal symbol of A is only assumed to be elliptic on (hence near) a closed subset K of $\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$, one still can construct a microlocal parametrix B, i.e. one whose errors $BA-\mathrm{Id}, AB-\mathrm{Id}$ as a parametrix have wave front set disjoint from K. To make this precise, first we define microlocal ellipticity:

Definition 7. We say that $A\in\Psi^{m,\ell}_{\delta,\delta'}$, $\sigma_{m,\ell}(A)=[a]$, is elliptic at $\alpha\in\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ if α has a neighborhood U in $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$ such that $a|_{U\cap\mathbb{R}^n\times\mathbb{R}^n}$ is elliptic, i.e. satisfies (41) on U. We say that A is elliptic on a subset K of $\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ if it is elliptic at each point of K. The elliptic set $\mathrm{Ell}(A)$ is the set of points at which A is elliptic; the characteristic set $\mathrm{Char}(A)$ is its complement.

We say that $A\in\Psi^{m,\ell}_{\infty,\delta}$, $\sigma_{\infty,m,\ell}(A)=[a]$, is elliptic at $\alpha\in\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n}$ if α has a neighborhood U in $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$ such that $a|_{U\cap\mathbb{R}^n\times\mathbb{R}^n}$ is elliptic, i.e. satisfies (40) on U. We say that A is elliptic on a subset K of $\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n}$ if it is elliptic at each point of K. One defines $\mathrm{Ell}_\infty(A)$, $\mathrm{Char}_\infty(A)$ analogously to the above definition.

We also make the analogous definitions if $m=\mathsf{m}$, $\ell=\mathsf{l}$ are variable.

If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ is elliptic on a closed (hence compact) K, then a covering argument shows that a satisfies (41) on a neighborhood of K. A similar statement holds for $A \in \Psi^{m,\ell}_{\infty,\delta}$.

Proposition 0.26. *If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ (or $A \in \Psi^{\mathsf{m},\mathsf{l}}_{\delta,\delta'}$) is elliptic on a compact set K then there exists $B \in \Psi^{-m,-\ell}_{\delta,\delta'}$ (resp. $B \in \Psi^{-\mathsf{m},-\mathsf{l}}_{\delta,\delta'}$) such that $E_L = BA - \mathrm{Id}$, $E_R = AB - \mathrm{Id}$ satisfy $\mathrm{WF}'(E_L) \cap K = \emptyset$, $\mathrm{WF}'(E_R) \cap K = \emptyset$.*

Proof. If A is elliptic on K, there is a neighborhood U of K in $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ such that $a|_{U \cap \mathbb{R}^n \times \mathbb{R}^n}$ is elliptic, i.e. satisfies (41) on U. We may shrink U so that $|z| + |\zeta| > R$ on U; thus $a|_U$ has a lower bound on all of U. Let $q \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ be identically 1 near K, be supported in U, and let $Q \in \Psi^{0,0}$ be given by $Q = q_L(q)$. Thus, Q has principal symbol $\sigma_{0,0}(Q) = q|_{\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})}$, and $\mathrm{WF}'(Q) \subset U$, $\mathrm{WF}'(\mathrm{Id} - Q) \cap K = \emptyset$. Now let $[a]$ be the principal symbol of A, let $b_0 = qa^{-1} \in S^{-m,-\ell}_{\delta,\delta'}$ since a is elliptic on U. Let $B_0 = q_L(b_0)$, so $\sigma_{-m,-\ell}(B_0) = b_0$ and $\mathrm{WF}'(B_0) \subset U$. Let $q_0 \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ be identically 1 near K, have disjoint support from $1 - q$, so $q_0(1 - q) = 0$, and let $Q_0 = q_L(q_0)$. Note that $\mathrm{WF}'(\mathrm{Id} - Q_0) \cap K = \emptyset$. Then $E_{0,L} = Q_0(B_0 A - \mathrm{Id}) \in \Psi^{0,0}_{\delta,\delta'}$, $E_{0,R} = (AB_0 - \mathrm{Id})Q_0 \in \Psi^{0,0}_{\delta,\delta'}$ have vanishing principal symbols, so $E_{0,L}, E_{0,R} \in \Psi^{-1+2\delta,-1+2\delta'}_{\delta,\delta'}$. As in the globally elliptic case, one may asymptotically sum the amplitudes $e_{L,j}$ of $(-1)^j E^j_{0,L}$ to obtain $\tilde{E}_L$ such that $F_N = \tilde{E}_L - \sum_{j=1}^N (-1)^j E^j_{0,L} \in \Psi^{-(1-2\delta)(N+1),-(1-2\delta')(N+1)}_{\delta,\delta'}$ for all N. Thus,

$$\begin{aligned}(\mathrm{Id} + \tilde{E}_L) Q_0 B_0 A &= (\mathrm{Id} + \tilde{E}_L)(E_{0,L} + \mathrm{Id}) + (\mathrm{Id} + \tilde{E}_L)(Q_0 - \mathrm{Id}) \\ &= (\mathrm{Id} + \sum_{j=1}^N (-1)^j E^j_{0,L} + F_N)(\mathrm{Id} + E_{0,L}) + (\mathrm{Id} + \tilde{E}_L)(Q_0 - \mathrm{Id}) \\ &= \mathrm{Id} + (-1)^{N+1} E^{N+1}_{0,L} + F_N(\mathrm{Id} + E_{0,L}) + (\mathrm{Id} + \tilde{E}_L)(Q_0 - \mathrm{Id}).\end{aligned}$$

Now, $(-1)^{N+1} E^{N+1}_{0,L} + F_N(\mathrm{Id} + E_{0,L}) \in \Psi^{-(1-2\delta)(N+1),-(1-2\delta')(N+1)}_{\delta,\delta'}$, and is independent of N (since it plus Id is $(\mathrm{Id} + \sum_{j=1}^N (-1)^j E^j_{0,L} + F_N)(\mathrm{Id} + E_{0,L})$) so it is in $\Psi^{-\infty,-\infty}$, and $\mathrm{WF}'((\mathrm{Id} + \tilde{E}_L)(Q_0 - \mathrm{Id})) \subset \mathrm{WF}'(Q_0 - \mathrm{Id})$, which is disjoint from K. Thus, we may take

$$B_L = (\mathrm{Id} + \tilde{E}_L) Q_0 B_0$$

as our microlocal left parametrix, and similarly obtain a microlocal right parametrix B_R. The parametrix identity (43) now shows that $\mathrm{WF}'(B_L - B_R) \cap K = \emptyset$, completing the proof.

The proof of the variable order case goes through without changes. □

One corollary is the following.

Corollary 0.27. *Suppose $u \in \mathcal{S}'$, $A \in \Psi^{m,\ell}$, and $Au \in H^{s,r}$ then for $Q \in \Psi^{0,0}_{\delta,\delta'}$ with $\mathrm{WF}'(Q) \cap \mathrm{Char}(A) = \emptyset$, $Qu \in H^{s+m,r+\ell}$. Further, for all M, N there exists $C > 0$ such that*

$$\|Qu\|_{H^{s+m,r+\ell}} \leq C(\|Au\|_{H^{s,r}} + \|u\|_{H^{M,N}}).$$

There is also an analogue with variable order spaces.

Proof. Let B be a microlocal parametrix for A near $\mathrm{WF}'(Q)$. Then $BA - \mathrm{Id} = E$ with $\mathrm{WF}'(E) \cap \mathrm{WF}'(Q) = \emptyset$. Thus,

$$Qu = Q(BA - E)u = QB(Au) - (QE)u.$$

Now, $\mathrm{WF}'(QE) = \mathrm{WF}'(Q) \cap \mathrm{WF}'(E) = \emptyset$, so $QE \in \Psi^{-\infty,-\infty}$, and thus $QEu \in \mathcal{S}$, while $QB \in \Psi^{-m,-\ell}_{\delta,\delta'}$, so the proof is finished as for global elliptic regularity. □

Here the assumption $Au \in H^{s,r}$ is too strong; it only matters that Au is such microlocally near $\mathrm{WF}'(Q)$. That is:

Corollary 0.28. *(Microlocal elliptic regularity; operator version.) Suppose $u \in \mathcal{S}'$, $A \in \Psi^{m,\ell}_{\delta,\delta'}$, and for some $Q' \in \Psi^{0,0}_{\delta,\delta'}$, $Q'(Au) \in H^{s,r}$. Then for $Q \in \Psi^{0,0}_{\delta,\delta'}$ with $\mathrm{WF}'(Q) \subset \mathrm{Ell}(A) \cap \mathrm{Ell}(Q')$, $Qu \in H^{s+m,r+\ell}$. Further, for all M, N there exists $C > 0$ such that*

$$\|Qu\|_{H^{s+m,r+\ell}} \leq C(\|Q'Au\|_{H^{s,r}} + \|u\|_{H^{M,N}}).$$

There is again an analogue with variable order spaces.

Proof. We just note that $Q'A$ is elliptic on $\mathrm{Ell}(A)\cap\mathrm{Ell}(Q')$, so the previous corollary is applicable. □

One can restate the corollary in terms of microlocalizing the distributions instead of adding the microlocalizers explicitly as operators.

Definition 8. Suppose $\alpha \in \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, $u \in \mathcal{S}'$. We say that $\alpha \notin \mathrm{WF}^{m,\ell}(u)$ if there exists $A \in \Psi^{0,0}_{\delta,\delta'}$ elliptic at α such that $Au \in H^{m,\ell}$. We say that $\alpha \notin \mathrm{WF}(u)$ if there exists $A \in \Psi^{0,0}_{\delta,\delta'}$ elliptic at α such that $Au \in \mathcal{S}$.

For $k, \ell, m \in \mathbb{R}$, $u \in H^{k,\ell}$, $\mathrm{WF}^{m,\ell}_\infty(u)$ is defined similarly: if $\alpha \in \overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$, we say $\alpha \notin \mathrm{WF}^{m,\ell}_\infty(u)$ if there exists $A \in \Psi^{0,0}_{\infty,\delta}$ elliptic at α such that $Au \in H^{m,\ell}$. We say that $\alpha \notin \mathrm{WF}_{\infty,\ell}(u)$ if there exists $A \in \Psi^{0,0}_{\infty,\delta}$ elliptic at α such that $Au \in H^{\infty,\ell}$.

We also make the analogous definition for variable order spaces.

The most important property of WF and pseudodifferential operators is microlocality:

Proposition 0.29. *If $A \in \Psi^{m,\ell}_{\delta,\delta'}$ and $u \in \mathcal{S}'$ then*

$$\mathrm{WF}^{s,r}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}^{s+m,r+\ell}(u)$$

and

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u).$$

The variable order version of this statement also holds.

Proof. We need to show that

$$\mathrm{WF}^{s,r}(Au) \subset \mathrm{WF}'(A) \text{ and } \mathrm{WF}^{s,r}(Au) \subset \mathrm{WF}^{s+m,r+\ell}(u).$$

We start with the former, which is straightforward. Suppose $\alpha \notin \mathrm{WF}'(A)$. Let $Q \in \Psi^{0,0}$ be elliptic at α but with $\mathrm{WF}'(Q)\cap\mathrm{WF}'(A) = \emptyset$; one can achieve this as $\mathrm{WF}'(A)$ is closed, so one simply needs to take $q \in \mathcal{C}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ equal to 1 near α and with essential support disjoint from $\mathrm{WF}'(A)$. Then $\mathrm{WF}'(QA) \subset \mathrm{WF}'(Q) \cap \mathrm{WF}'(A) = \emptyset$, so $QA \in \Psi^{-\infty,-\infty}$, thus $QAu \in \mathcal{S}$.

Now for the second inclusion. Suppose $\alpha \notin \mathrm{WF}^{s+m,r+\ell}(u)$. Then there exists $B \in \Psi^{0,0}_{\delta,\delta'}$ elliptic at α such that $Bu \in H^{s+m,r+\ell}$. Let $G \in \Psi^{0,0}_{\delta,\delta'}$ be a microlocal parametrix for B, so $GB = \mathrm{Id} + E$ with $\alpha \notin \mathrm{WF}'(E)$. Then $Au = AGBu - AEu$, and $AG \in \Psi^{m,\ell}_{\delta,\delta'}$, so $AGBu \in H^{s,r}$. On the other hand, $\alpha \notin \mathrm{WF}'(AE) \subset \mathrm{WF}'(E)$. So let $Q \in \Psi^{0,0}$ be elliptic at α but with $\mathrm{WF}'(Q) \cap \mathrm{WF}'(E) = \emptyset$. Then $QE \in \Psi^{-\infty,-\infty}$, and thus

$$QAu = Q(AG)(Bu) - (QAE)u \in H^{s,r},$$

so $\alpha \notin \mathrm{WF}^{s,r}(u)$, completing the proof for $\mathrm{WF}^{s,r}(Au)$. The proof for $\mathrm{WF}(Au)$ is analogous. □

Note that the last part of the proof shows more:

Lemma 0.30. *If $\alpha \notin \mathrm{WF}^{s,r}(u)$ then there is a neighborhood U of α such that for all $Q \in \Psi^{0,0}_{\delta,\delta'}$ with $\mathrm{WF}'(Q) \subset U$, $Qu \in H^{s,r}$.*

Further, with the same U, for all $Q \in \Psi^{m,\ell}_{\delta,\delta'}$ with $\mathrm{WF}'(Q) \subset U$, $Qu \in H^{s-m,r-\ell}$.

The variable order version of this statement also holds.

Thus, while the wave front set definition is a 'there exists' statement, in fact it is equivalent to a 'for all' statement, namely for all $Q \in \Psi^{0,0}_{\delta,\delta'}$ with $\mathrm{WF}'(Q)$ in a sufficiently neighborhood of α, $Qu \in H^{s,r}$. (The other direction is simply because these Q include those elliptic at α.)

Also, as immediate from the proof below, one can take U to be the elliptic set of the $B \in \Psi^{0,0}_{\delta,\delta'}$, elliptic at α, with $Bu \in H^{s,r}$, whose existence is guaranteed by $\alpha \notin \mathrm{WF}^{s,r}(u)$

Proof. Suppose $\alpha \notin \mathrm{WF}^{s,r}(u)$. Then there exists $B \in \Psi^{0,0}_{\delta,\delta'}$ elliptic at α such that $Bu \in H^{s,r}$; let $G \in \Psi^{0,0}_{\delta,\delta'}$ be a microlocal parametrix for B, so $GB = \mathrm{Id} + E$ with $\alpha \notin \mathrm{WF}'(E)$. Let U be the complement of $\mathrm{WF}'(E)$; this is a neighborhood of α. Then for any $Q \in \Psi^{0,0}_{\delta,\delta'}$ with $\mathrm{WF}'(Q) \subset U$, $QE \in \Psi^{-\infty,-\infty}$, so $Qu = QGBu - QEu \in H^{s,r}$ as $QG \in \Psi^{0,0}_{\delta,\delta'}$.

The second statement is proved the same way, noticing that $QG \in \Psi^{m,\ell}_{\delta,\delta'}$ now. □

An immediate consequence is:

Lemma 0.31. *If $u \in \mathcal{S}'$ and $\mathrm{WF}^{m,\ell}(u) = \emptyset$ then $u \in H^{m,\ell}$.*

If $u \in H^{k,\ell}$ and $\mathrm{WF}^{m,\ell}_\infty(u) = \emptyset$ then $u \in H^{m,\ell}$.

The variable order version of this statement also holds.

Proof. Suppose $u \in \mathcal{S}'$ and $\mathrm{WF}^{m,\ell}(u) = \emptyset$. Then for all $\alpha \in \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ there exists U_α open such that for all $Q \in \Psi^{0,0}$ with $\mathrm{WF}'(Q) \subset U_\alpha$, $Qu \in H^{m,\ell}$. Now

$$\{U_\alpha : \ \alpha \in \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})\}$$

is an open cover of the compact set $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, so there is a finite subcover, say $\{U_{\alpha_j} : \ j = 1, \ldots, N\}$. Let $\tilde{U}_{\alpha_j}$ be open in $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ with $\tilde{U}_{\alpha_j} \cap \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) = U_{\alpha_j}$. Then, with $\tilde{U}_{\alpha_0} = \mathbb{R}^n \times \mathbb{R}^n$,

$$\{\tilde{U}_{\alpha_j} : \ j = 0, 1, \ldots, N\}$$

is a finite open cover of $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$. Let $\sum_{j=0}^N q_j = 1$ be a subordinate partition of unity, and let $Q_j = q_L(q_j)$. Then $\sum_{j=0}^N Q_j = \mathrm{Id}$, $Q_0 \in \Psi^{-\infty,-\infty}$ since q_0 has compact support, while for $j = 1, \ldots, N$, $\mathrm{WF}'(Q_j) \subset U_{\alpha_j}$ since $\operatorname{supp} q_j \subset \tilde{U}_{\alpha_j}$. Thus, $Q_j u \in H^{m,\ell}$ for all j, and thus $u = \sum Q_j u \in H^{m,\ell}$ as claimed.

The argument for WF_∞ is analogous. □

The distributional version of microlocal elliptic regularity then is:

Corollary 0.32. *(Microlocal elliptic regularity; distributional version.) Suppose $u \in \mathcal{S}'$, $A \in \Psi^{m,\ell}_{\delta,\delta'}$. Then*

$$\mathrm{WF}^{s+m,r+\ell}(u) \subset \mathrm{Char}(A) \cup \mathrm{WF}^{s,r}(Au).$$

The variable order version of this statement also holds.

Proof. Suppose $\alpha \notin \mathrm{Char}(A) \cup \mathrm{WF}^{s,r}(Au)$, we need to show $\alpha \notin \mathrm{WF}^{s+m,r+\ell}(u)$. As $\alpha \notin \mathrm{WF}^{s,r}(Au)$ there exists $Q' \in \Psi^{0,0}_{\delta,\delta'}$ elliptic at α such that $Q'Au \in H^{s,r}$. Let $Q \in \Psi^{0,0}_{\delta,\delta'}$ be such that $\mathrm{WF}'(Q) \subset \mathrm{Ell}(A) \cap \mathrm{Ell}(Q')$, note that the set on the right is open and includes α. Then by Corollary 0.28, $Qu \in H^{s+m,r+\ell}$. Taking Q which is in addition elliptic at α completes the proof. □

The consequence of what we proved so far for Schrödinger operators is:

Proposition 0.33. *Let g be a Riemannian metric, $g_{ij} \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$, positive definite on $\overline{\mathbb{R}^n}$, $V \in S^{-\rho}(\mathbb{R}^n)$ with $\rho > 0$. Let $H = \Delta_g + V$. Then for $\sigma \subset [0,\infty)$, $(H-\sigma)u \in H^{s,r}$ implies*

$$\mathrm{WF}^{s+2,r}(u) \subset \{(z,\zeta) \in \partial\overline{\mathbb{R}^n} \times \mathbb{R}^n :\ \sum g_{ij}(z)\zeta_i\zeta_j = \sigma\}.$$

Finally we note the diffeomorphism invariance of pseudodifferential operators.

Proposition 0.34. *Suppose $F : O \to U$ is a diffeomorphism between bounded open subsets O and U of $\mathbb{R}^n$. Suppose $A \in \Psi^{m,\ell}_{\infty,\delta}(\mathbb{R}^n)$, with Schwartz kernel supported in a compact subset of $U \times U$. Then $A_F = F^* A (F^{-1})^* \in \Psi^{m,\ell}_{\infty,\delta}$. Furthermore, with $DF(z)$ the Jacobian matrix of F, i.e. with kj entry $\partial_j F_k(z)$, and with $\dagger$ denoting $\mathbb{R}^n$-adjoint (i.e. j,k reversed),*

$$\mathrm{WF}'(A_F) = \{(z,\zeta) :\ (F(z), (DF)^\dagger(z)^{-1}\zeta) \in \mathrm{WF}'(A)\},$$

and

$$\sigma_{\infty,m,\ell}(A_F)(z,\zeta) = \sigma_{\infty,m,\ell}(A)(F(z),(DF)^\dagger(z)^{-1}\zeta).$$

Remark 0.35. The principal symbol here shows why we had a single parameter δ giving the losses in $\langle\zeta\rangle$ upon differentiation in either z or ζ: differentiation of the principal symbol of A_F in z gives rise to ζ derivatives as well in that of A. Thus, to have the class diffeomorphism invariant, the losses under z derivatives have to be at least as large as those under ζ-derivatives. Thus, the ζ-derivatives (which are the derivatives tangent to the fibers of the cotangent bundle of $\mathbb{R}^n$, thus are invariantly defined) are necessarily better (in the sense of 'no worse') behaved regarding these losses than the z-derivatives. If one also wants Fourier-invariance, one needs the opposite inequality as well, hence the equality.

Remark 0.36. Notice that if one writes a covector as $\sum_k \eta_k\, dw_k$, then its pull-back under the map F (with $F(z) = w$ for clarity) is $\sum_k \eta_k\,(\partial_j F_k)(z)\,dz_j$, i.e.

$$\zeta_j = \sum_k (\partial_j F_k)(z)\eta_k = ((DF)^\dagger(z)\eta)_j,$$

so $\zeta = (DF)^\dagger(z)\eta$. This means that $(DF)^\dagger(z)^{-1}\zeta\, dw$ is the pull-back of $\zeta\, dz$ by F^{-1}, i.e. the wave front set and the principal symbol are well behaved (invariant) if we regard them as subsets of $T^*\mathbb{R}^n \setminus o$, resp. functions on $T^*\mathbb{R}^n \setminus o$: with $F^\sharp : T^*_U\mathbb{R}^n \to T^*_O\mathbb{R}^n$ the map induced by pull-back of covectors by F, and similarly for $(F^{-1})^\sharp : T^*_O\mathbb{R}^n \to T^*_U\mathbb{R}^n$, so $((F^{-1})^\sharp)^*$ maps functions on $T^*_U\mathbb{R}^n$ to those on $T^*_O\mathbb{R}^n$, then

$$\sigma_{\infty,m,\ell}(A_F) = ((F^{-1})^\sharp)^*\sigma_{\infty,m,\ell}(A),$$

and

$$\mathrm{WF}'(A_F) = ((F^{-1})^\sharp)^{-1}(\mathrm{WF}'(A)).$$

Proof. Let $G = F^{-1}$ to simplify the notation.

First we consider the off-diagonal behavior. To do so, suppose more generally that $A : \mathcal{S} \to \mathcal{S}'$ continuous linear with Schwartz kernel supported in $U \times U$ (so A need not be a ps.d.o). We claim that, with K_A the Schwartz kernel of A, the Schwartz kernel K_{A_F} of A_F is the (compactly supported) tempered distribution

$$K_{A_F} = ((F \times F)^*K_A)(\pi_R^*|\det(DF)|), \tag{45}$$

where $\pi_R : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection to the second factor. Indeed, if K_A is Schwartz (i.e. just $\mathcal{C}^\infty$, in view of the support) Schwartz kernel is continuous, then, with $A_F u$ also considered as a distribution in the second expression,

$$\begin{aligned} K_{A_F}(u \otimes v) = (A_F u)(v) &= \int (A_F u)(z)\, v(z)\, dz \\ &= \int A(G^*u)(F(z))v(z)\, dz = \int K_A(F(z), w')G^*u(w')\, v(z)\, dw'\, dz \\ &= \int K_A(F(z), w')u(G(w'))\, v(z)\, dw'\, dz \\ &= \int K_A(F(z), F(z'))u(z')\, v(z)\, |\det DF(z')|\, dz'\, dz, \end{aligned}$$

giving the above result for K_{A_F}. Since Schwartz functions with compact support in $O \times O$ are dense in tempered distributions supported in $O \times O$, and since the operations in (45) are continuous, the result follows for general tempered distributions K_A.

Applying this to the case of pseudodifferential operators A, which have $\mathcal{C}^\infty$ Schwartz kernel away from the diagonal, we conclude that A_F has $\mathcal{C}^\infty$ Schwartz kernel away from the diagonal. In particular, when considering the behavior near the diagonal, it suffices to work in a suitably small neighborhood of the diagonal.

We have from the definition of A,

$$A_F u(z) = (A(G^*u))(F(z)) = (2\pi)^{-n} \int e^{i(F(z)-w')\cdot\eta} a(F(z), w', \eta)u(G(w'))\, dw'\, d\eta.$$

Letting $z' = G(w')$, the change of variables formula for the integral gives

$$A_F u(z) = (2\pi)^{-n} \int e^{i(F(z)-F(z'))\cdot\eta} a(F(z), F(z'), \eta)u(z')\, |\det(DF)(z')|\, dz'\, d\eta.$$

This is almost of the desired form, except the appearance of $F(z) - F(z')$ instead of $z - z'$ in the exponent. To deal with this, we use the easiest case of Taylor's theorem (which really means the fundamental theorem of calculus in this context),

$$F_k(z) - F_k(z') = \sum_{j=1}^{n} (z_j - z_j')F_{kj}(z, z')$$

with

$$F_{kj}(z,z') = \int_0^1 (\partial_j F_k)(tz + (1-t)z')\,dt,$$

so

$$F_{kj}(z,z) = \partial_j F_k(z)$$

is the Jacobian matrix of F. More generally, let us write

$$\Phi(z,z') = (\partial_j F_k(z,z'))_{kj}$$

for this matrix. Thus, the exponent is

$$\sum_{k=1}^n \sum_{j=1}^n (z_j - z_j') F_{kj}(z,z')\eta_k = \sum_{j=1}^n (z_j - z_j')\zeta_j,$$

where

$$\zeta_j = \zeta_j(z,z',\eta) = \sum_{k=1}^n F_{kj}(z,z')\eta_k = (\Phi^\dagger(z,z')\eta)_j.$$

Note that the map

$$(z,z',\eta) \mapsto (z,z',\zeta(z,z',\eta))$$

is a diffeomorphism, linear in η, if (z,z') is close to the diagonal. Indeed, since F is a diffeomorphism, $\Phi(z,z)$ is invertible, and thus so is $\Phi(z,z')$ for (z,z') near the diagonal, so the inverse of the above map is simply

$$(z,z'\zeta) \mapsto (z,z',\Phi^\dagger(z,z')^{-1}\zeta).$$

Changing the variable of integration from η to ζ gives, as

$$|d\zeta| = |\det(\Phi(z,z'))^\dagger|\,|d\eta| = |\det\Phi(z,z')|\,|d\eta|,$$

$$\begin{aligned} A_F u(z) = (2\pi)^{-n} \int e^{i(z-z')\cdot\zeta} a(F(z),F(z'),(\Phi^\dagger(z,z'))^{-1}\zeta) u(z') \\ |\det\Phi(z,z')|^{-1} |\det(DF)(z')|\,dz'\,d\zeta \\ = (2\pi)^{-n} \int e^{i(z-z')\cdot\zeta} a_F(z,z',\zeta) u(z')\,dz'\,d\zeta \end{aligned}$$

with

$$a_F(z,z',\zeta) = a(F(z),F(z'),(\Phi^\dagger(z,z'))^{-1}\zeta) |\det\Phi(z,z')|^{-1} |\det(DF)(z')|.$$

Thus, checking

$$a_F \in S^{m,\ell}_{\infty,\delta}$$

completes the proof. For this purpose the two determinant factors are irrelevant as they are $\mathcal{C}^\infty$. Thus, it remains to note that D_ζ applied to $a(F(z),F(z'),(\Phi^\dagger(z,z'))^{-1}\zeta)$ again simply gives additional smooth factors, while D_z or $D_{z'}$ applied can either correspond to derivatives of a in the first or second slot, in which case they are harmless, or in the last slot when they give a factor in ζ, but also lower the symbolic order by 1, thus preserving the estimates.

The principal symbol statement follows from the cancellation of the determinant factors when one restricts to $z = z'$, and that $(\Phi^\dagger(z,z'))^{-1}$ is $(DF)^\dagger(z)^{-1}$ then; this also gives the wave front set statement. □

In fact, the same proof gives:

Proposition 0.37. *Suppose $F : O \to U$ is a diffeomorphism between open subsets O and U of $\overline{\mathbb{R}^n}$. Suppose $A \in \Psi^{m,\ell}_{\delta,\delta'}(\mathbb{R}^n)$, with Schwartz kernel supported in a compact subset of $U \times U$. Then $A_F = F^* A (F^{-1})^* \in \Psi^{m,\ell}_{\infty,\delta}$. Furthermore, with $DF(z)$ the Jacobian matrix of F, i.e. with kj entry $\partial_j F_k(z)$, and with $\dagger$ denoting $\mathbb{R}^n$-adjoint (i.e. j, k reversed),*

$$\mathrm{WF}'(A_F) = \{(z,\zeta) :\ (F(z), (DF)^\dagger(z)^{-1}\zeta) \in \mathrm{WF}'(A)\},$$

and

$$\sigma_{m,\ell}(A_F)(z,\zeta) = \sigma_{\infty,m,\ell}(A)(F(z), (DF)^\dagger(z)^{-1}\zeta).$$

The point here is that for F as stated, DF is an elliptic symbol on O of order 0, and thus the near-diagonal argument goes through: in fact, one even gets the invertibility of $\Phi(z, z')$ for (z, z') in a conic neighborhood of the diagonal (as follows by working with valid coordinates on the compactification, and noting that a neighborhood in this compactified perspective gives a conic neighborhood without the compactification). The Schwartz kernel of ps.d.o's outside such a neighborhood is Schwartz, hence the off-diagonal piece pulls back correctly as well.