

HOMEWORK 9 MATH 220

Problem 1 By Prop 0.11 (in Inner product handout), $\|f - \phi\|$ is minimal if

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x) dx = \frac{\pi}{2}; \\ a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos x dx = -\frac{4}{\pi}; \\ a_2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos 2x dx = 0; \\ b_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin x dx = 0; \\ a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin 2x dx = 0. \end{aligned}$$

Problem 2

(i) Let λ be an eigenvalue and v be a nonzero eigenvector corresponding to λ . If A is positive, then

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Av, v \rangle \geq 0.$$

Since $\langle v, v \rangle > 0$, $\lambda \geq 0$.

(ii) Let $f, g \in D$. First, we show that A is symmetric. By integration by part and the fact that $f(0) = f'(l) = 0$ and $g(0) = g'(l) = 0$, we obtain that

$$\begin{aligned} \langle Af, g \rangle &= \int_0^l \left(-\frac{d^2}{dx^2} f(x) \right) \overline{g(x)} dx \\ &= \left(-\frac{d}{dx} f(x) \right) \overline{g(x)} \Big|_{x=0}^{x=l} - \int_0^l \left(-\frac{d}{dx} f(x) \right) \left(\frac{d}{dx} \overline{g(x)} \right) dx \\ &= \int_0^l \left(\frac{d}{dx} f(x) \right) \left(\frac{d}{dx} \overline{g(x)} \right) dx \\ &= f(x) \left(\frac{d}{dx} \overline{g(x)} \right) \Big|_{x=0}^{x=l} - \int_0^l f(x) \left(\frac{d^2}{dx^2} \overline{g(x)} \right) dx \\ &= \int_0^l f(x) \left(-\frac{d^2}{dx^2} \overline{g(x)} \right) dx = \langle f, Ag \rangle. \end{aligned}$$

Next we show that A is positive. By the same argument as above, we have

$$\langle Af, f \rangle = \int_0^l \left(\frac{d}{dx} f(x) \right) \left(\frac{d}{dx} \overline{f(x)} \right) dx = \int_0^l \left| \frac{d}{dx} f(x) \right|^2 dx \geq 0.$$

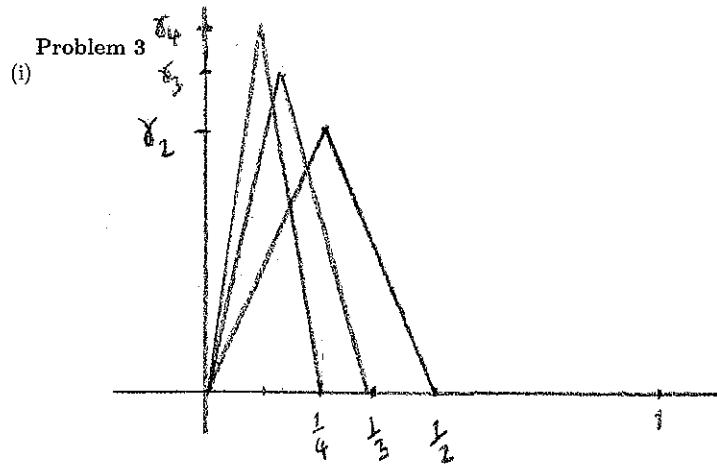
(iii) Let $f, g \in D$. First, we show that A is symmetric. By integration by part and

the fact that $f(0) = f'(0) = f(l) = f'(l) = 0$ and $g(0) = g'(0) = g(l) = g'(l) = 0$, we obtain that

$$\begin{aligned}
 \langle Af, g \rangle &= \int_0^l \left(\frac{d^4}{dx^4} f(x) \right) \overline{g(x)} dx \\
 &= \left(\frac{d^3}{dx^3} f(x) \right) \overline{g(x)} \Big|_{x=0}^{x=l} - \int_0^l \left(\frac{d^3}{dx^3} f(x) \right) \left(\frac{d}{dx} \overline{g(x)} \right) dx \\
 &= - \left(\frac{d^2}{dx^2} f(x) \right) \left(\frac{d}{dx} \overline{g(x)} \right) \Big|_{x=0}^{x=l} + \int_0^l \left(\frac{d^2}{dx^2} f(x) \right) \left(\frac{d^2}{dx^2} \overline{g(x)} \right) dx \\
 &= \left(\frac{d}{dx} f(x) \right) \left(\frac{d^2}{dx^2} \overline{g(x)} \right) \Big|_{x=0}^{x=l} - \int_0^l \left(\frac{d}{dx} f(x) \right) \left(\frac{d^3}{dx^3} \overline{g(x)} \right) dx \\
 &= -f(x) \left(\frac{d^3}{dx^3} \overline{g(x)} \right) \Big|_{x=0}^{x=l} + \int_0^l f(x) \left(\frac{d^4}{dx^4} \overline{g(x)} \right) dx \\
 &= \int_0^l f(x) \left(\frac{d^4}{dx^4} \overline{g(x)} \right) dx = \langle f, Ag \rangle.
 \end{aligned}$$

Finally we show that A is positive. Following the arguments above, we have

$$\langle Af, f \rangle = \int_0^l \left(\frac{d^2}{dx^2} f(x) \right) \left(\frac{d^2}{dx^2} \overline{f(x)} \right) dx = \int_0^l \left| \frac{d^2}{dx^2} f(x) \right|^2 dx \geq 0.$$



(ii) We need to show that for a fixed x , $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$. If $x = 0$, $f_n(0) = 0$ for each n . If $x > 0$, $f_n(x) = 0$ for $x > 1/n$.

(iii) The convergence is uniform if $\sup_{x \in [0,1]} |f_n(x) - 0| \rightarrow 0$ as $n \rightarrow \infty$. However $\sup_{x \in [0,1]} |f_n(x)| = \gamma_n$ which does not converge to 0 as $n \rightarrow \infty$.

(iv)

$$\|f_n\|_{L^2}^2 = \int_0^1 f_n(x)^2 dx = (2n\gamma_n)^2 \frac{(2n)^{-3}}{3} \cdot 2 = \frac{\gamma_n^2}{6n} = \frac{1}{2\sqrt{n}} \rightarrow 0$$

as $n \rightarrow \infty$.

$$(v) \|f_n\|_{L^2}^2 = \frac{2n^2}{6n} = 6n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Problem 4

(i) Let the Fourier sine series of ϕ be $\sum_{n=1}^{\infty} A_n \sin(n\pi x/l)$. We can compute

$$A_n = \frac{2}{l} \int_0^l \phi(x) \sin(n\pi x/l) dx = \frac{4(1 - (-1)^n)}{l} \left(\frac{l}{n\pi}\right)^3.$$

From p.4 on handout (convergence of Fourier series), the Fourier sine series converges to an odd $2l$ -periodic extension function of ϕ since $\phi(0) = \phi(l) = 0$.

(ii) Let the Fourier cosine series of ϕ be $\sum_{n=1}^{\infty} B_n \cos(n\pi x/l)$. We can compute

$$B_0 = \frac{1}{l} \int_0^l \phi(x) dx = \frac{l^3}{6}.$$

$$B_n = \frac{2}{l} \int_0^l \phi(x) \cos(n\pi x/l) dx = -(1 + (-1)^n) \frac{l^3}{n^2 \pi^2}.$$

The Fourier cosine series converges to an even $2l$ -periodic extension function of ϕ .

(iii) A_n decays as n^{-3} while B_n decays as n^{-2} . The former one is faster. The reason is that the odd extension of ϕ is continuously differentiable and the even extension of ϕ is merely continuous. The smoother the function is, the faster the Fourier coefficients decay.

Problem 5 For Fourier sine series, we first make odd extension for $\phi(x)$. Now our ϕ is defined on $[-l, l]$.

(i) Fourier sine series converge pointwisely to ϕ in $(-l, l)$ and to $(\phi(l) + \phi(-l))/2 = 0$ at $\pm l$. $\phi(-l) = -l \neq l = \phi(l)$, so the convergence is not uniform (otherwise, the limit function should be continuous.)

Also $\int_{-l}^l \phi(x)^2 dx < \infty$. This implies $\phi \in L^2(-l, l)$. Therefore, the Fourier sine series will converge to ϕ in L^2 .

(ii) Fourier sine series converge pointwisely to ϕ in $(-l, l)$ and to $(\phi(l) + \phi(-l))/2 = 0$ at $\pm l$. Here we know that $\phi(l) = \phi(-l) = 0$ and $\phi'(l) = \phi'(-l) = 0$. Therefore extension of ϕ is in C^2 , and the Fourier sine series converges uniformly to ϕ . Moreover, it also converges in L^2 since uniform convergence implies convergence in L^2 . (This can also be shown by showing $\phi \in L^2(-l, l)$.)

Problem 6

(i) $u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x/l)$. We obtain similar expressions for f, ϕ, ψ as the followings:

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x/l),$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(n\pi x/l),$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin(n\pi x/l).$$

(ii) Assume that we can differentiate term by term.

$$u_{tt}(x, t) = \sum_{n=1}^{\infty} u_n''(t) \sin(n\pi x/l).$$

$$u_{xx}(x, t) = \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{l^2} u_n(t) \right) \sin(n\pi x/l).$$

(iii) Now we get our ODE

$$u_n''(t) + c^2 \left(\frac{n\pi}{l} \right)^2 u_n(t) = f_n(t),$$

where $u_n(0) = \phi_n$ and $u_n'(0) = \psi(n)$. Using Duhamel's principle, this ODE has solution

$$u_n(t) = \psi_n \frac{\sin(cn\pi t/l)}{cn\pi/l} + \phi_n \cos(cn\pi t/l) + \int_0^t \frac{\sin(cn\pi(t-s)/l)}{cn\pi/l} f_n(s) ds.$$

(iv) By integration by part, we obtain

$$\begin{aligned} w_n(t) &= \frac{2}{l} \int_0^l u_{xx}(x, t) \sin \frac{n\pi x}{l} dx \\ &= -\frac{2n\pi}{l^2} \int_0^l u_x(x, t) \cos \frac{n\pi x}{l} dx \\ &= -\frac{2n\pi}{l^2} \left(u(x, t) \cos \frac{n\pi x}{l} \Big|_{x=0}^{x=l} + \frac{n\pi}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx \right) \\ &= -\frac{2n\pi}{l^2} ((-1)^n j(t) - k(t)) - \frac{n^2\pi^2}{l^2} u_n(t), \end{aligned}$$

where $u_n(t) = \frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx$.

(v) The ODE is

$$u_n''(t) = c^2 w_n(t) + f_n(t) = f_n(t) - \frac{2nc^2\pi}{l^2} ((-1)^n j(t) - k(t)) - \frac{c^2 n^2 \pi^2}{l^2} u_n(t)$$

or

$$u_n''(t) + \frac{c^2 n^2 \pi^2}{l^2} u_n(t) = f_n(t) - \frac{2nc^2\pi}{l^2} ((-1)^n j(t) - k(t))$$

with initial conditions $u_n(0) = \phi_n, u_n'(0) = \psi_n$. Using Duhamel's principle, this ODE has solution

$$u_n(t) = \psi_n \frac{\sin(cn\pi t/l)}{cn\pi/l} + \phi_n \cos(cn\pi t/l) + \int_0^t \frac{\sin(cn\pi(t-s)/l)}{cn\pi/l} q_n(s) ds,$$

where $q_n(t) = f_n(t) - \frac{2nc^2\pi}{l^2} ((-1)^n j(t) - k(t))$.