MATH 220: PRACTICE FINAL – SOLUTIONS

This is a closed book, closed notes, no calculators exam.
There are 8 problems. Solve all of them. Total score: 200 points.

Problem 1. (i) *(15 points)* Solve

$$u_x + 2xu_y = y,\ u(0,y) = y^2.$$

(ii) *(15 points)* Solve

$$uu_x + yu_y = x,\ u(0,y) = y,$$

for $|x|$ small.

Solution 1. (i) This is a linear first order PDE. The characteristic ODEs are

$$\frac{\partial x}{\partial s} = 1,\ \frac{\partial y}{\partial s} = 2x,\ \frac{\partial z}{\partial s} = y,$$

with initial condition

$$x(r,0) = 0,\ y(r,0) = r,\ z(r,0) = r^2.$$

Solving the x ODE first, substituting the result into the y ODE, solving it, and finally substituting the result into the z ODE one has:

$$\begin{aligned}
&x(r,s) = s,\\
&\frac{\partial y}{\partial s} = 2s \Rightarrow y(r,s) = s^2 + r\\
&\frac{\partial z}{\partial s} = s^2 + r \Rightarrow z(r,s) = \frac{s^3}{3} + rs + r^2.
\end{aligned}$$

Inverting the map $(r,s) \mapsto (x,y)$ yields $s = x$, $r = y - s^2 = y - x^2$. Thus,

$$u(x,y) = z(r(x,y), s(x,y)) = \frac{x^3}{3} + x(y - x^2) + (y - x^2)^2.$$

(ii) This is a quasilinear first order PDE. The characteristic ODEs and initial conditions are

$$\begin{aligned}
&\frac{\partial x}{\partial s} = z,\ x(r,0) = 0,\\
&\frac{\partial y}{\partial s} = y,\ y(r,0) = r,\\
&\frac{\partial z}{\partial s} = x,\ z(r,0) = r.
\end{aligned}$$

The y ODE yields

$$y(r,s) = re^s.$$

The x and z ODEs do not decouple, and cannot be solved one by one. However, differentiating the x ODE once more with respect to s and using the z ODE yields

$$\frac{\partial^2 x}{\partial s^2} = \frac{\partial z}{\partial s} = x,\ x(r,0) = 0,\ \frac{\partial x}{\partial s}(r,0) = z(r,0) = r.$$

The general solution of the ODE is a linear combination of e^s and e^{-s}, or more conveniently $\cosh s$ and $\sinh s$. Thus,

$$x(r,s) = A(r)\cosh s + B(r)\sinh(s).$$

Substituting the initial conditions yields $A(r) = 0$, $B(r) = r$, so

$$x(r,s) = r\sinh s,\ z(r,s) = \frac{\partial x}{\partial s} = r\cosh s.$$

We now need to express r, s in terms of (x, y). Recall that $\sinh s = \frac{e^s - e^{-s}}{2}$, so we have

$$x(r,s) = \frac{r}{2}e^s - \frac{r}{2}e^{-s}.$$

Thus, $x = \frac{y}{2} - \frac{r}{2}e^{-s}$, hence

$$re^{-s} = y - 2x,\ re^s = y.$$

While we could solve for r and s now (for e^s: take the quotient, for r, multiply these two equations), this is not needed, since

$$z = \frac{r}{2}e^s + \frac{r}{2}e^{-s} = \frac{y}{2} + \frac{y-2x}{2} = y - x.$$

Thus, the solution of the PDE is

$$u(x,y) = y - x,$$

which indeed solves the PDE as we can easily check.

Problem 2. (i) *(9 points)* State the maximum principle for solutions of the heat equation $u_t = ku_{xx}$, $k > 0$, on $[0, \ell]$.

(ii) *(7 points)* If u solves $u_t = ku_{xx}$ with Dirichlet boundary condition $u(0,t) = 0 = u(\ell, t)$ for all t, and $u(x, 0) = x(\ell - x)$, find the maximum value of u in $[0, \ell]_x \times [0, \infty)_t$.

(iii) *(9 points)* Show that the maximum principle does not hold for the wave equation $u_{tt} = c^2 u_{xx}$, $c > 0$, on $[0, \ell]_x \times [0, \infty)_t$. (Hint: write down a solution of the wave equation with Dirichlet boundary condition and which vanishes at $t = 0$).

Solution 2. (i) The maximum principle states that for each $T > 0$ the maximum of u on $[0, \ell]_x \times [0, T]_t$ is attained either at $t = 0$ or at $x = 0$ or at $x = \ell$.

(ii) The maximum of u at $t = 0$ is, as

$$x(\ell - x) = -x^2 + \ell x = -(x - \ell/2)^2 + \ell^2/4,$$

$\ell^2/4 > 0$, attained at $x = \ell/2$. Since u vanishes at $x = 0$ and $x = \ell$, we conclude that for each $T > 0$ the maximum on $[0, \ell]_x \times [0, T]_t$ is $\ell^2/4$. Thus, the maximum value of u in $[0, \ell]_x \times [0, \infty)_t$ is $\ell^2/4$.

(iii) Consider $u(x,t) = \sin(\pi x/\ell)\sin(\pi ct/\ell)$. Then u vanishes at $x = 0$, $x = \ell$ as well as at $t = 0$. Thus, if the maximum principle held, with $T = \ell/(2c)$, then the maximum of u on $[0, \ell]_x \times [0, T]_t$ would be 0 – but $u(\ell/2, T) = 1$! This shows that the maximum principle does not hold for the wave equation.

Problem 3. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the half-line, i.e. on $[0, \infty)_x \times [0, \infty)_t$, with homogeneous Neumann boundary condition $u_x(0, t) = 0$, and with initial conditions $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ for $x \geq 0$.

(i) *(10 points)* Find u.

(ii) *(8 points)* Suppose ϕ, ψ are constant near 0, and are C^∞ away from a point $x_0 > 0$. Where can you say for sure that u is C^∞?

(iii) *(7 points)* Suppose that $\phi \equiv 0$, and $\psi(x) = 1$ for $x < 1$, $\psi(x) = 0$ for $x > 1$. Find $u(x, t)$ explicitly for $t \geq 0$. (Hint: it is best to consider different cases depending on where (x, t) lies.) Does the location of the singularities (lack of being C^∞) agree with what you found in (ii)?

You may use in any part of the problem that if v solves $v_{tt} - c^2 v_{xx} = f$ on Δ, the backward characteristic triangle from (x,t), then

$$v(x,t) = \frac{v(x-ct,0)+v(x+ct,0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_t(x',0)\,dx' + \frac{1}{2c}\int_\Delta f.$$

Solution 3. (i) Let v be the even extension of u, i.e. $v(x,t) = u(x,t)$ for $x \geq 0$, $v(x,t) = u(-x,t)$ for $x \leq 0$, and let ϕ_e, ψ_e be the even extensions of ϕ, ψ to $\mathbb{R}$. Then v solves the wave equation with initial data ϕ_e, ψ_e, so

$$v(x,t) = \frac{\phi_e(x-ct)+\phi_e(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_e(y)\,dy.$$

Then for $x \geq 0$, $u(x,t) = v(x,t)$, so for $x > ct$,

$$u(x,t) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy,$$

while for $0 \leq x < ct$, $\phi_e(x-ct) = \phi(ct-x)$, and

$$\int_{x-ct}^{x+ct} \psi_e(y)\,dy = \int_{x-ct}^{0} \psi(-y)\,dy + \int_0^{x+ct} \psi(y)\,dy$$
$$= \int_0^{ct-x} \psi(y)\,dy + \int_0^{x+ct} \psi(y)\,dy = 2\int_0^{ct-x} \psi(y)\,dy + \int_{x-ct}^{x+ct} \psi(y)\,dy,$$

so

$$u(x,t) = \frac{\phi(ct-x)+\phi(x+ct)}{2} + \frac{1}{2c}\left(2\int_0^{ct-x} \psi(y)\,dy + \int_{x-ct}^{x+ct} \psi(y)\,dy\right).$$

(ii) Under these assumptions, ϕ_e and ψ_e are C^∞ except at $\pm x_0$, so $v(x,t)$ is C^∞ except on the characteristics through these two points, i.e. $x \pm ct = x_0$ and $x \pm ct = -x_0$, hence u, being the restriction of v, is C^∞ except at where these characteristics intersect $x \geq 0$, i.e. along the broken characteristics emanating from x_0.

(iii) It is easier to find v in $x > 0$ directly rather than apply the formula we derived above. Thus, $\psi_e(x) = 1$ if $|x| < 1$, 0 otherwise, so the answer depends on where $x \pm ct$ are relative to the interval $(-1,1)$. If $x \geq 0$, we have the following cases:

$$u(x,t) = \begin{cases} 0, & 1 < x-ct, \\ \frac{1+ct-x}{2c}, & -1 < x-ct < 1 < x+ct, \\ t, & -1 < x-ct < x+ct < 1, \\ \frac{1}{c}, & x-ct < -1 < 1 < x+ct. \end{cases}$$

This is certainly smooth except where two regions meet, which is exactly along the characteristic lines emanating from ± 1.

Problem 4. *(20 points)* In $\{(x,y) :\ x \geq 0\}$, find the bounded solution of

$$u_{xx} + u_{yy} - u = 0,\ u_x(0,y) = h(y),$$

where h is a given Schwartz function. Write your answer as a partial convolution in y. You may leave the inverse Fourier transform of an explicit function in your answer without calculating it.

Solution 4. Since in x we are only working with the half-line, we take the partial Fourier transform in y only and get, with $\hat{u}(x,\eta) = (\mathcal{F}_y u)(x,\eta)$,

$$\hat{u}_{xx} - (\eta^2+1)\hat{u} = 0,\ \hat{u}_x(0,\eta) = \hat{h}(\eta).$$

Solving the ODE, the general solution is

$$\hat{u}(x,\eta) = A(\eta)e^{\sqrt{\eta^2+1}x} + B(\eta)e^{-\sqrt{\eta^2+1}x}.$$

Since we want the solution to be bounded in $x > 0$, we must have $A(\eta) = 0$ for all η, and thus

$$\hat{h}(\eta) = -\sqrt{\eta^2+1}B(\eta)e^{-\sqrt{\eta^2+1}x}|_{x=0},$$

and so

$$B(\eta) = -\frac{\hat{h}(\eta)}{\sqrt{\eta^2+1}}.$$

Thus,

$$u(x,y) = \mathcal{F}_\eta^{-1}\left(-\frac{\hat{h}(\eta)}{\sqrt{\eta^2+1}}e^{-\sqrt{\eta^2+1}x}\right).$$

Rewriting this as a partial convolution, letting $S(x,\eta) = -(\eta^2+1)^{-1/2}e^{-\sqrt{\eta^2+1}x}$,

$$u(x,y) = \int_{\mathbb{R}} (\mathcal{F}_\eta^{-1}S)(x, y-z)h(z)\,dz.$$

Problem 5. (i) *(8 points)* Consider the following eigenvalue problem on $[0,\ell]$:

$$-X'' = \lambda X,\ X'(0) = 0,\ X(\ell) = 0.$$

Find all eigenvalues and eigenfunctions, and show that eigenfunctions corresponding different eigenvalues are orthogonal to each other.

(ii) *(8 points)* Using separation of variables, find the general 'separated' solution of the wave equation

$$u_{tt} = c^2 u_{xx},\ u_x(0,t) = 0,\ u(\ell,t) = 0.$$

(iii) *(5 points)* Solve the wave equation with initial conditions

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

i.e. give a formula for the series coefficients in part (ii) in terms of ϕ and ψ.

(iv) *(4 points)* Now suppose $\phi(x) = 0$, $\psi(x) = \cos(3\pi x/(2\ell)) + \cos(7\pi x/(2\ell))$. Find u explicitly.

Solution 5. (i) As both the Dirichlet and Neumann boundary conditions are symmetric, the operator A given by $AX = -X''$ on the domain $\{X \in C^2([0,\ell]) :\ X'(0) = 0,\ X(\ell) = 0\}$ is symmetric. Thus, all eigenvalues of A are real, and eigenfunctions corresponding to different eigenvalues are orthogonal to each other. To find the eigenfunctions, note that the general solution of $-X'' = \lambda X$ is, for $\lambda \neq 0$,

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

so $X'(0) = 0$ gives $B = 0$, while $X(\ell) = 0$ gives (assuming $A \neq 0$)

$$\sqrt{\lambda}\ell = \left(n + \frac{1}{2}\right)\pi,$$

with n an integer. As the corresponding eigenfunctions are

$$X_n(x) = \cos(\frac{(n+\frac{1}{2})\pi x}{\ell}),$$

allowing $n < 0$ gives the same functions as $n \geq 0$, so we may restrict to $n \geq 0$. For $\lambda = 0$ the general solution is $X(x) = A + Bx$, and the boundary conditions give $B = 0$, and then $A = 0$, so 0 is not an eigenvalue.

(ii) Writing $u(x,t) = X(x)T(t)$, substituting into the PDE yields $XT'' = c^2X''T$, so $\frac{-T''}{c^2T} = \frac{-X''}{X}$, and as the left hand side is independent of x, the right hand side is independent of t, they are both constants, λ. We must also have, from the homogeneous boundary conditions, $X'(0) = 0 = X(\ell)$, so X is one of the eigenfunctions computed in the previous part, with eigenvalue

$$\lambda = \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{\ell}\right)^2.$$

On the other hand, the T ODE gives then

$$T(t) = A\cos(\sqrt{\lambda}ct) + B\sin(\sqrt{\lambda}ct),$$

so the general separated solution is

$$u(x,t) = \sum_{n=0}^{\infty}(A_n\cos(\frac{(n+\frac{1}{2})\pi ct}{\ell}) + B_n\sin(\frac{(n+\frac{1}{2})\pi ct}{\ell}))\cos(\frac{(n+\frac{1}{2})\pi x}{\ell}).$$

(iii) Letting $t = 0$ gives

$$\phi(x) = \sum_{n=0}^{\infty} A_n\cos(\frac{(n+\frac{1}{2})\pi x}{\ell}),$$

while differentiating with respect to t and letting $t = 0$ gives

$$\psi(x) = \sum_{n=0}^{\infty} B_n\frac{(n+\frac{1}{2})\pi c}{\ell}\cos(\frac{(n+\frac{1}{2})\pi x}{\ell}).$$

As the cosines are orthogonal to each other, as remarked in the first part, and as

$$\int_0^{\ell}\cos(\frac{(n+\frac{1}{2})\pi x}{\ell})^2\,dx = \frac{\ell}{2},$$

we deduce that

$$A_n = \frac{2}{\ell}\int_0^{\ell}\phi(x)\cos(\frac{(n+\frac{1}{2})\pi x}{\ell})\,dx$$

$$B_n = \frac{2}{(n+\frac{1}{2})\pi c}\int_0^{\ell}\psi(x)\cos(\frac{(n+\frac{1}{2})\pi x}{\ell})\,dx.$$

(iv) In this case the coeffiecients of the Fourier expansion can be read off directly, so all A_n and B_n are 0 except B_1 and B_3 which give

$$B_1 = \frac{2\ell}{3\pi c},\ B_3 = \frac{2\ell}{7\pi c},$$

hence

$$u(x,t) = \frac{2\ell}{3\pi c}\sin(\frac{3\pi ct}{2\ell})\cos(\frac{3\pi x}{2\ell}) + \frac{2\ell}{7\pi c}\sin(\frac{7\pi ct}{2\ell})\cos(\frac{7\pi x}{2\ell}).$$

Problem 6. (i) *(15 points)* For both of the following functions f on $[0,\ell]$, state whether the Fourier cosine series on $[0,\ell]$ converges in each of the following senses: uniformly, in L^2. State what the Fourier series converges to at each point in $\mathbb{R}$. Make sure that you give the reasoning that led you to the conclusions.

(a) $f(x) = x(\sin(\pi x/\ell))^2$,

(b) $f(x) = 0$, for $0 \le x \le \ell/2$, and $f(x) = 1$ for $\ell/2 < x \le \ell$.

(ii) *(10 points)* For the function f in (b) above, we wish to approximate f by a function g of the form $a_1\cos(\pi x/\ell) + a_3\cos(3\pi x/\ell)$ on $[0,\ell]$. Find the constants a_1 and a_3 that minimize the L^2 error, $\int_0^{\ell}|f-g|^2\,dx$, of the approximation.

Solution 6. (i) The function f in (a) is C^2 on $[0,\ell]$ and satisfies Neumann boundary conditions (as the cosines in the cosine series do), so the Fourier cosine series converges uniformly to f, and hence also in L^2. The function f in (b) is discontinuous so the Fourier cosine series does not converge uniformly (the uniform limit of continuous functions in continuous), but f is piecewice C^1, so the Fourier series converges in L^2; in either case the uniform, resp. L^2, limit of the series on $\mathbb{R}$ is the even 2ℓ-periodic extension of f.

(ii) The choice of a_1 and a_3 minimizing $\int_0^\ell |f-g|^2\,dx$ are the ones given by the orthogonal projection of f to the span of $\cos(\pi x/\ell)$ and $\cos(3\pi x/\ell)$, which in turn are simply the Fourier cosine coefficients. These are

$$\frac{2}{\ell}\int_0^\ell f(x)\,\cos(n\pi x/\ell)\,dx = \frac{2}{\ell}\int_{\ell/2}^\ell \cos(n\pi x/\ell)\,dx = \frac{2}{n\pi}\sin(n\pi x/\ell)|_{x=\ell/2}^{\ell}$$

$n=1,3$, so $a_1=-\frac{2}{\pi}$, $a_3=\frac{2}{3\pi}$.

Problem 7. Let $c=c(x)>0$ be a C^1 function on $[0,\ell]$, and consider the differential operator

$$A=-\frac{d}{dx}(c(x)^2\frac{d}{dx})$$

on functions f on $[0,\ell]$ which satisfy Dirichlet boundary conditions $f(0)=f(\ell)=0$. That is, let $D=\{f\in C^2([0,\ell]):\ f(0)=f(\ell)=0\}$, and let $A: D\to C^0([0,\ell])$ be defined by $Af=-(c(x)^2f'(x))'$. Let $\langle f,g\rangle=\int_0^\ell f(x)\overline{g(x)}\,dx$ denote the standard inner product on $C^0([0,\ell])$.

(i) *(7 points)* Show that A is symmetric: $\langle Af,g\rangle=\langle f,Ag\rangle$ for all functions $f,g\in D$.

(ii) *(7 points)* What can you say about the eigenvalues and eigenfunctions of A based on (i)?

(iii) *(6 points)* We say that a symmetric operator B is positive if $\langle Bf,f\rangle\geq 0$ for all $f\in D$. Show that A is positive.

(iv) *(5 points)* Show that if B is positive then every eigenvalue of B is non-negative.

Solution 7. (i) We check that A is symmetric. For $f,g\in D$, we integrate by parts twice:

$$\langle Af,g\rangle=\int_0^\pi(-(c^2f')')\overline{g}\,dx=-c^2f'\overline{g}|_0^\pi+\int_0^\pi c^2f'\overline{g'}\,dx=c^2f\overline{g'}|_0^\pi-\int_0^\pi f\overline{(c^2g')'}\,dx=\langle f,Ag\rangle,$$

where we used that $f(0)=0=f(\pi)$, $g(0)=0=g(\pi)$, and that differentiation commutes with taking complex conjugates.

(ii) (1) implies that all eigenvalues of A are real, and eigenfunctions corresponding to different eigenvalues are orthogonal to each other.

(iii) By the middle expression in the above calculation,

$$\langle Af,f\rangle=\int_0^\pi c^2f'\overline{f'}\,dx=\int_0^\pi c^2|f'|^2\,dx\geq 0,$$

so A is a positive operator.

(iv) Suppose B is positive, and λ is an eigenvalue of B with corresponding eigenfunction $f\neq 0$. Then λ is real and

$$0\leq\langle Bf,f\rangle=\langle\lambda f,f\rangle=\lambda\|f\|^2,$$

so dividing through by $\|f\|^2>0$ gives $\lambda\geq 0$.

Problem 8. (i) *(15 points)* Using separation of variables, solve the Klein-Gordon equation:

$$u_{tt} + \gamma^2 u = c^2 u_{xx},\ u(0,t) = 0 = u(\ell, t),$$
$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

where $\gamma > 0$ constant. (Find the solution in terms of ϕ, ψ!)

(ii) *(10 points)* Recall that Duhamel's principle states the following: if $U = U(t)$, $F = F(t)$ are vector valued function with values in a vector space V, Φ is an element of V, and A is a linear operator on V, then the solution U of

$$U_t + AU = F,\ U(0) = \Phi,$$

is

$$U(t) = S(t)\Phi + \int_0^t S(t-s)F(s)\,ds,$$

where $S(t)$ is the solution operator for the associated homogeneous problem ($U_t + AU = 0$, $U(0) = \Phi$). Use this to solve the inhomogeneous Klein-Gordon equation:

$$u_{tt} + \gamma^2 u - c^2 u_{xx} = f,\ u(0,t) = 0 = u(\ell, t),$$
$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

$f = f(x,t)$, ϕ and ψ given.

Solution 8. (i) If $u(x,t) = X(x)T(t)$, the PDE gives

$$XT'' + \gamma^2 XT = c^2 X'' T,$$

so dividing through by XT,

$$\frac{T'' + \gamma^2}{c^2 T} = \frac{X''}{X} = -\lambda,$$

λ a constant, since the left hand side is independent of x and the right hand side is independent of t. We also have from the boundary conditions $X(0) = 0 = X(\ell)$. The solution of the X ODE with these conditions is

$$X(x) = X_n(x) = \sin(n\pi x/\ell),\ \lambda = \lambda_n = (n\pi/\ell)^2,$$

n a positive integer. The T ODE becomes $T'' = -(\lambda c^2 + \gamma^2)T$, so

$$T(t) = A\cos(\sqrt{\lambda c^2 + \gamma^2}\, t) + B\sin(\sqrt{\lambda c^2 + \gamma^2}\, t),$$

so the general separated solution is

$$u(x,t) = \sum_{n=0}^{\infty} (A_n \cos(\sqrt{\lambda_n c^2 + \gamma^2}\, t) + B_n \sin(\sqrt{\lambda_n c^2 + \gamma^2}\, t)) \sin(n\pi x/\ell).$$

Letting $t = 0$ gives

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin(n\pi x/\ell),$$

while differentiating with respect to t and letting $t = 0$ gives

$$\psi(x) = \sum_{n=0}^{\infty} B_n \sqrt{\lambda_n c^2 + \gamma^2} \sin(n\pi x/\ell).$$

These are just the Fourier sine series, so

$$A_n = A_n(\phi) = \frac{2}{\ell}\int_0^\ell \phi(x)\sin(n\pi x/\ell)\,dx,$$
$$B_n = B_n(\psi) = \frac{2}{\ell\sqrt{\lambda_n c^2 + \gamma^2}}\int_0^\ell \psi(x)\sin(n\pi x/\ell)\,dx.$$

(ii) We apply Duhamel's principle with $U = \begin{bmatrix} u \\ u_t \end{bmatrix}$. Namely, this solves

$$U_t = \begin{bmatrix} u_t \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_t \\ c^2 u_{xx} - \gamma^2 u + f \end{bmatrix},$$

i.e. is a solution of the system (with $v = u$, $w = u_t$)

$$\begin{bmatrix} v_t \\ w_t \end{bmatrix} = \begin{bmatrix} w \\ c^2 v_{xx} - \gamma^2 v + f \end{bmatrix} = \begin{bmatrix} w \\ c^2 v_{xx} - \gamma^2 v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Correspondingly, the homogeneous equation is the one we just solved in the first part, and our solution operator is

$$S(t) : \begin{bmatrix} \phi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} \sum_{n=0}^\infty (A_n(\phi)\cos(\sqrt{\lambda_n c^2 + \gamma^2}t) + B_n(\psi)\sin(\sqrt{\lambda_n c^2 + \gamma^2}t))\sin(n\pi x/\ell), \\ \frac{\partial}{\partial t}\left(\sum_{n=0}^\infty (A_n(\phi)\cos(\sqrt{\lambda_n c^2 + \gamma^2}t) + B_n(\psi)\sin(\sqrt{\lambda_n c^2 + \gamma^2}t))\sin(n\pi x/\ell)\right) \end{bmatrix},$$

where we explicitly wrote out that A_n depends on ϕ and B_n depends on ψ. By Duhamel's principle, the solution of the inhomogeneous problem is

$$\begin{bmatrix} v \\ w \end{bmatrix} = S(t)\begin{bmatrix} \phi \\ \psi \end{bmatrix} + \int_0^t S(t-s)\begin{bmatrix} 0 \\ f \end{bmatrix}\,ds.$$

In particular, the first component, $v = u$, is

$$\begin{aligned} u(x,t) = &\sum_{n=0}^\infty (A_n(\phi)\cos(\sqrt{\lambda_n c^2 + \gamma^2}t) + B_n(\psi)\sin(\sqrt{\lambda_n c^2 + \gamma^2}t))\sin(n\pi x/\ell) \\ &+ \int_0^t B_n(f(.,s))\sin(\sqrt{\lambda_n c^2 + \gamma^2}(t-s))\sin(n\pi x/\ell)\,ds, \end{aligned}$$

with

$$B_n(f(.,s)) = \frac{2}{\ell\sqrt{\lambda_n c^2 + \gamma^2}}\int_0^\ell f(y,s)\sin(n\pi y/\ell)\,dy.$$