MATH 220: PRACTICE FINAL

This is a closed book, closed notes, no calculators exam.
There are 8 problems. Solve all of them. Total score: 200 points.

Problem 1. (i) *(15 points)* Solve

$$u_x + 2xu_y = y,\ u(0,y) = y^2.$$

(ii) *(15 points)* Solve

$$uu_x + yu_y = x,\ u(0,y) = y,$$

for $|x|$ small.

Problem 2. (i) *(9 points)* State the maximum principle for solutions of the heat equation $u_t = ku_{xx}$, $k > 0$, on $[0,\ell]$.

(ii) *(7 points)* If u solves $u_t = ku_{xx}$ with Dirichlet boundary condition $u(0,t) = 0 = u(\ell,t)$ for all t, and $u(x,0) = x(\ell-x)$, find the maximum value of u in $[0,\ell]_x \times [0,\infty)_t$.

(iii) *(9 points)* Show that the maximum principle does not hold for the wave equation $u_{tt} = c^2u_{xx}$, $c > 0$, on $[0,\ell]_x \times [0,\infty)_t$. (Hint: write down a solution of the wave equation with Dirichlet boundary condition and which vanishes at $t = 0$).

Problem 3. Consider the wave equation $u_{tt} = c^2u_{xx}$ on the half-line, i.e. on $[0,\infty)_x \times [0,\infty)_t$, with homogeneous Neumann boundary condition $u_x(0,t) = 0$, and with initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for $x \geq 0$.

(i) *(10 points)* Find u.

(ii) *(8 points)* Suppose ϕ, ψ are constant near 0, and are C^∞ away from a point $x_0 > 0$. Where can you say for sure that u is C^∞?

(iii) *(7 points)* Suppose that $\phi \equiv 0$, and $\psi(x) = 1$ for $x < 1$, $\psi(x) = 0$ for $x > 1$. Find $u(x,t)$ explicitly for $t \geq 0$. (Hint: it is best to consider different cases depending on where (x,t) lies.) Does the location of the singularities (lack of being C^∞) agree with what you found in (ii)?

You may use in any part of the problem that if v solves $v_{tt} - c^2v_{xx} = f$ on Δ, the backward characteristic triangle from (x,t), then

$$v(x,t) = \frac{v(x-ct,0) + v(x+ct,0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_t(x',0)\,dx' + \frac{1}{2c}\int_\Delta f.$$

Problem 4. *(20 points)* In $\{(x,y) :\ x \geq 0\}$, find the bounded solution of

$$u_{xx} + u_{yy} - u = 0,\ u_x(0,y) = h(y),$$

where h is a given Schwartz function. Write your answer as a partial convolution in y. You may leave the inverse Fourier transform of an explicit function in your answer without calculating it.

Problem 5. (i) *(8 points)* Consider the following eigenvalue problem on $[0,\ell]$:

$$-X'' = \lambda X,\ X'(0) = 0,\ X(\ell) = 0.$$

Find all eigenvalues and eigenfunctions, and show that eigenfunctions corresponding different eigenvalues are orthogonal to each other.

(ii) *(8 points)* Using separation of variables, find the general 'separated' solution of the wave equation

$$u_{tt} = c^2u_{xx},\ u_x(0,t) = 0,\ u(\ell,t) = 0.$$

(iii) *(5 points)* Solve the wave equation with initial conditions

$$u(x,0)=\phi(x),\ u_t(x,0)=\psi(x),$$

i.e. give a formula for the series coefficients in part (ii) in terms of ϕ and ψ.

(iv) *(4 points)* Now suppose $\phi(x)=0$, $\psi(x)=\cos(3\pi x/(2\ell))+\cos(7\pi x/(2\ell))$. Find u explicitly.

Problem 6. (i) *(15 points)* For both of the following functions f on $[0,\ell]$, state whether the Fourier cosine series on $[0,\ell]$ converges in each of the following senses: uniformly, in L^2. State what the Fourier series converges to at each point in $\mathbb{R}$. Make sure that you give the reasoning that led you to the conclusions.

(a) $f(x)=x(\sin(\pi x/\ell))^2$,

(b) $f(x)=0$, for $0\le x\le \ell/2$, and $f(x)=1$ for $\ell/2<x\le\ell$.

(ii) *(10 points)* For the function f in (b) above, we wish to approximate f by a function g of the form $a_1\cos(\pi x/\ell)+a_3\cos(3\pi x/\ell)$ on $[0,\ell]$. Find the constants a_1 and a_3 that minimize the L^2 error, $\int_0^\ell |f-g|^2\,dx$, of the approximation.

Problem 7. Let $c=c(x)>0$ be a C^1 function on $[0,\ell]$, and consider the differential operator

$$A=-\frac{d}{dx}(c(x)^2\frac{d}{dx})$$

on functions f on $[0,\ell]$ which satisfy Dirichlet boundary conditions $f(0)=f(\ell)=0$. That is, let $D=\{f\in C^2([0,\ell]):\ f(0)=f(\ell)=0\}$, and let $A:D\to C^0([0,\ell])$ be defined by $Af=-(c(x)^2f'(x))'$. Let $\langle f,g\rangle=\int_0^\ell f(x)\overline{g(x)}\,dx$ denote the standard inner product on $C^0([0,\ell])$.

(i) *(7 points)* Show that A is symmetric: $\langle Af,g\rangle=\langle f,Ag\rangle$ for all functions $f,g\in D$.

(ii) *(7 points)* What can you say about the eigenvalues and eigenfunctions of A based on (i)?

(iii) *(6 points)* We say that a symmetric operator B is positive if $\langle Bf,f\rangle\ge 0$ for all $f\in D$. Show that A is positive.

(iv) *(5 points)* Show that if B is positive then every eigenvalue of B is non-negative.

Problem 8. (i) *(15 points)* Using separation of variables, solve the Klein-Gordon equation:

$$u_{tt}+\gamma^2u=c^2u_{xx},\ u(0,t)=0=u(\ell,t),$$
$$u(x,0)=\phi(x),\ u_t(x,0)=\psi(x),$$

where $\gamma>0$ constant. (Find the solution in terms of ϕ, ψ!)

(ii) *(10 points)* Recall that Duhamel's principle states the following: if $U=U(t)$, $F=F(t)$ are vector valued function with values in a vector space V, Φ is an element of V, and A is a linear operator on V, then the solution U of

$$U_t+AU=F,\ U(0)=\Phi,$$

is

$$U(t)=S(t)\Phi+\int_0^t S(t-s)F(s)\,ds,$$

where $S(t)$ is the solution operator for the associated homogeneous problem ($U_t+AU=0$, $U(0)=\Phi$). Use this to solve the inhomogeneous Klein-Gordon equation:

$$u_{tt}+\gamma^2u-c^2u_{xx}=f,\ u(0,t)=0=u(\ell,t),$$
$$u(x,0)=\phi(x),\ u_t(x,0)=\psi(x),$$

$f=f(x,t)$, ϕ and ψ given.