MATH 220: MIDTERM – SOLUTIONS
NOVEMBER 1, 2012

This is a closed book, closed notes, no computers/calculators exam. There are 5 problems. Solve Problems 1-3 and one of Problems 4 and 5. Write your solutions to problems 1 and 2 in blue book #1, and your solutions to problems 3, 4 and 5 in blue book #2. Within each book, you may solve the problems in any order. Total score: 100 points.

Problem 1. Solve the PDE
\[(\cos y)u_x + u_y = u, \quad u(x,0) = x^2,\]
on \(\mathbb{R}^2\). Do all characteristics intersect the \(x\)-axis, and do so in a unique point, non-tangentially? What does this mean for the solutions of the initial value problem of the PDE? Sketch the characteristics.

Solution. The characteristic ODEs are
\[
\frac{dx}{ds} = \cos y, \quad x(r,0) = r,
\]
\[
\frac{dy}{ds} = 1, \quad y(r,0) = 0,
\]
\[
\frac{dz}{ds} = z, \quad z(r,0) = r^2.
\]
From the ODE for \(y\), \(y(r,s) = s\), so the \(x\) ODE becomes \(\frac{dx}{ds} = \cos s\), hence \(x(r,s) = \sin s + r\).
Finally the \(z\) ODE gives \(z(r,s) = r^2 e^s\).
Since \(s = y\) and \(r = x - \sin s = x - \sin y\),
\[u(x,y) = (x - \sin y)^2 e^y.\]
The characteristics we found are the curves \(\gamma_r\) given by \(x = \sin y + r\); for every \((x,y) \in \mathbb{R}^2\) there is one of these characteristics through that point (namely \(\gamma_r\) with \(r = x - \sin y\)), so we indeed found all characteristics (not just those that the parameterization gives). As the characteristics are \(C^\infty\) graphs over the \(y\) axis, in particular they intersect the \(x\)-axis, given by \(y = 0\), and do so in a unique point and non-tangentially. This means that the solution of the initial value problem for the PDE is globally defined as long as the ODE along the characteristics (the \(z\)-ODE) has a global solution; in any case there is at most one solution on \(\mathbb{R}^2\), which indeed exists as our solution shows (since the \(z\)-ODE has a global solution). (Sketch not shown.)

Problem 2. Consider the (real-valued) damped wave equation on \([0,\ell] \times [0,\infty)\) with Robin boundary conditions:
\[u_{tt} + a(x)u_t = (c(x)^2u_x)_x, \quad u_x(0,t) = \alpha u(0,t), \quad u_x(\ell,t) = -\beta u(\ell,t)\]
where \(\alpha, \beta \geq 0\) are constants, \(a \geq 0\) and \(c > 0\) depend on \(x\) only, and there are constants \(c_1, c_2 > 0\) such that \(c_1 \leq c(x) \leq c_2\) for all \(x\). (Note that if \(\alpha = 0\) and \(\beta = 0\) then this is just the Neumann boundary condition! In general, this BC would hold for example for a string if its ends were attached to springs.) Assume throughout that \(u\) is \(C^2\). Let
\[E(t) = \frac{1}{2} \int_0^\ell (u_t(x,t)^2 + c(x)^2u_x(x,t)^2) \, dx + \frac{1}{2}(c(0)^2\alpha u(0,t)^2 + c(\ell)^2\beta u(\ell,t)^2).\]
(i) Show that if \(a \equiv 0\) then \(E\) is constant.
(ii) Show that if \( a \geq 0 \) then \( E \) is a decreasing (i.e. non-increasing) function of \( t \), and that the solution of the damped wave equation (under the conditions mentioned above) with given initial condition is unique.

**Solution.** Differentiation gives

\[
E'(t) = \int_0^\ell (u_t u_{tt} + c(x)^2 u_x(x,t) u_{xt}) \, dx + (c(0)^2 \alpha u(0,t) u_t(0,t) + c(\ell)^2 \beta u(\ell,t) u_t(\ell,t)).
\]

Using the PDE to rewrite \( E'(t) \),

\[
E'(t) = \int_0^\ell (u_t(c(x)^2 u_x)_x - a(x) u_t^2 + c(x)^2 u_x(x,t) u_{xt}) \, dx
+ (c(0)^2 \alpha u(0,t) u_t(0,t) + c(\ell)^2 \beta u(\ell,t) u_t(\ell,t))
\]

\[
= \int_0^\ell ((c(x)^2 u_x) u_t) \, dx + (c(0)^2 \alpha u(0,t) u_t(0,t) + c(\ell)^2 \beta u(\ell,t) u_t(\ell,t)).
\]

Using the fundamental theorem of calculus to evaluate the integral of the derivative,

\[
E'(t) = c(x)^2 u_x u_t |_0^\ell - \int_0^\ell a(x) u_t^2 \, dx + (c(0)^2 \alpha u(0,t) u_t(0,t) + c(\ell)^2 \beta u(\ell,t) u_t(\ell,t))
\]

\[
= - \int_0^\ell a(x) u_t^2 \, dx + c(\ell)^2 u_x(\ell,t) u_t(\ell,t) - c(0)^2 u_x(0,t) u_t(0,t)
+ (c(0)^2 \alpha u(0,t) u_t(0,t) + c(\ell)^2 \beta u(\ell,t) u_t(\ell,t)).
\]

Using the boundary conditions thus yields

\[
E'(t) = - \int_0^\ell a(x) u_t^2 \, dx - \beta c(\ell)^2 u_x(\ell,t) u_t(\ell,t) - \alpha c(0)^2 u(0,t) u_t(0,t)
+ (c(0)^2 \alpha u(0,t) u_t(0,t) + c(\ell)^2 \beta u(\ell,t) u_t(\ell,t))
\]

\[
= - \int_0^\ell a(x) u_t^2 \, dx.
\]

If \( a \equiv 0 \), we deduce that \( E'(t) = 0 \), i.e. \( E \) is independent of \( t \). In general, if \( a \geq 0 \), we deduce that \( E'(t) \leq 0 \), hence \( E(t) \leq E(0) \) for \( t \geq 0 \). In particular, if \( E(0) = 0 \) then, taking into account that \( E(t) \) is non-negative in view of its definition (as \( \alpha, \beta \geq 0 \)), \( 0 \leq E(t) \leq 0 \), thus \( E(t) \) vanishes identically. But this gives that \( u_t \) and \( u_x \) vanish identically from the definition of \( E \), and so \( u \) is a constant. So now suppose that \( u_1 \) and \( u_2 \) solve the PDE with the same initial condition, and let \( u = u_1 - u_2 \). Thus, \( u(x,0) = 0 \) and \( u_t(x,0) = 0 \), so also \( u_x(x,0) = 0 \), and thus \( E(0) = 0 \). We deduce that \( u \) is a constant, so as \( u \) vanishes when \( t = 0 \), this constant is 0. Thus, \( u_1 \equiv u_2 \), showing the claimed uniqueness.

**Problem 3.**

(i) Find the general \( C^2 \) solution of the PDE

\[
 u_{xx} - 4u_{xt} - 5u_{tt} = 0.
\]

(ii) Solve the initial value problem with initial condition

\[
u(x,3x) = \phi(x), \quad u_t(x,3x) = \psi(x),
\]

with \( \phi, \psi \) given.

**Solution.** We factor the operator as

\[
\partial_x^2 - 4 \partial_x \partial_t - 5 \partial_t^2 = (\partial_x + \partial_t)(\partial_x - 5 \partial_t).
\]

The characteristics of \( \partial_x + \partial_t \) are \( x - t \) constant, while those of \( \partial_x - 5 \partial_t \) are \( 5x + t \) constant, so

\[
u(x,t) = f(x - t) + g(5x + t)
\]

certainly solves the PDE. That this is the general solution, would follow by a change of coordinates reducing it to the wave equation, in which case we have already shown the
We say that

\[ \phi(x) = u(x, 3x) = f(-2x) + g(8x), \quad \psi(x) = u_t(x, 3x) = -f'(2x) + g'(8x). \]

The first equation gives

\[ \phi'(x) = -2f'(2x) + 8g'(8x), \]

so

\[ -6f'(2x) = 8\psi(x) - \phi'(x), \]

\[ 6g'(8x) = \phi'(x) - 2\psi(x) \]

and thus

\[ f'(s) = -\frac{4}{3}\psi(-s/2) + \frac{1}{6}\phi'(-s/2) \]

\[ g'(s) = \frac{1}{6}\phi'(s/8) - \frac{1}{3}\psi(s/8). \]

Integrating yields

\[ f(s) = -\frac{1}{3}\phi(-s/2) - \frac{4}{3}\int_{0}^{s} \psi(-\sigma/2) \, d\sigma + A = -\frac{1}{3}\phi(-s/2) + \frac{8}{3}\int_{0}^{-s/2} \psi(\sigma) \, d\sigma + A, \]

\[ g(s) = \frac{4}{3}\phi(s/8) - \frac{1}{3}\int_{0}^{s} \psi(s/8) \, d\sigma + B = \frac{4}{3}\phi(s/8) - \frac{8}{3}\int_{0}^{s/8} \psi(\sigma) \, d\sigma + B. \]

Substituting into \( \phi(x) = f(-2x) + g(8x) \) gives \( B = -A \). Thus,

\[ u(x,t) = -\frac{1}{3}\phi((t-x)/2) + \frac{4}{3}\phi((5x+t)/8) - \frac{8}{3}\int_{(t-x)/2}^{(5x+t)/8} \psi(\sigma) \, d\sigma. \]

**Problem 4.** Consider Burgers’ equation

\[ u_t + uu_x = 0, \quad u(x,0) = \phi(x), \quad t \geq 0, \]

with initial condition

\[ \phi(x) = \begin{cases} 1, & x < -1, \\ -x, & -1 < x < 0, \\ 0, & x > 0. \end{cases} \]

(i) State the definition of \( u \) being a weak solution of this PDE.

(ii) What does the Rankine-Hugoniot condition

\[ (u_-(\xi(t),t) - u_+(\xi(t),t))\xi'(t) = f(u_-(\xi(t),t)) - f(u_+(\xi(t),t)) \]

refer to for a conservation law \( u_t + f(u)_x = 0 \)? Make a precise statement connecting this to part (i) (but you do not need to prove it).

(iii) Find a weak solution \( u \) of Burgers’ equation above valid for all \( t \geq 0 \).

**Solution.**

(i) We say that \( u \) is a weak solution of this PDE if for all \( \psi \in C_c^\infty(\mathbb{R} \times [0,\infty)) \)

\[ \int_{\mathbb{R} \times [0,\infty)} (u(x,t)\psi_t(x,t) + f(u(x,t))\psi_x(x,t)) \, dx \, dt + \int_{\mathbb{R}} \phi(x,0)\psi(x,0) \, dx = 0, \]

where \( f(z) = \frac{z^2}{2} \), so \( f'(z) = z \), and \( f(u)_x = uu_x \).

(ii) If \( x = \xi(t) \), \( \xi \) a \( C^1 \) function divides \( \mathbb{R} \times [0,\infty) \) into two regions, \( \Omega_+ \) and \( \Omega_- \), corresponding to \( x > \xi(t) \) and \( x < \xi(t) \), with \( u = u_+ \) on \( \Omega_+ \), and \( u_\pm \) extend to be \( C^1 \) on \( \Omega_\pm \), then \( u \) is a weak solution of the PDE if and only if it is a classical solution in \( \Omega_+ \cup \Omega_- \), satisfies the initial condition and satisfies the Rankine-Hugoniot condition.
(iii) For our Burgers’ equation, the characteristic equations are
\[
\frac{dx}{ds} = z, \quad x(r, 0) = r,
\]
\[
\frac{dt}{ds} = 1, \quad y(r, 0) = 0,
\]
\[
\frac{dz}{ds} = 0, \quad z(r, 0) = \phi(r).
\]
From the ODE for \( t, t(r, s) = s \), from the \( z \) ODE, \( z(r, s) = \phi(r) \) and from the \( x \) ODE becomes \( \frac{dx}{dt} = \phi(r) \), so \( x(r, s) = \phi(r)s + r \), and thus \( s = t \) and \( r \) is given by the implicit relation \( x = \phi(r)t + r \), so the projected characteristics have constant speed, given by the initial condition. Thus, for \( r < -1 \), we have \( x = t+r \), so \( r = x-t \), and hence \( x - t < -1 \), i.e. \( x < -1 + t \), and \( u(x, t) = \phi(r) = 1 \) in this region as long as the solution remains valid (no discontinuities). For \( r > 0 \) and analogous argument gives \( x = r \), so \( u(x, t) = 0 \), and this is valid when \( x > 0 \) and the solution is valid.

Finally, for \( -1 < r < 0 \), \( x = -tr+r = (1-t)r \), so \( r = \frac{x}{1-t} \), and \( u(x, t) = -r = \frac{-x}{1-t} \), valid when \( -1 < \frac{x}{1-t} < 0 \), i.e. \( t-1 < x < 0 \), until \( t = 1 \) (when the solution becomes singular and the region collapses). Note that for \( t < 1 \) the so defined solution is indeed continuous; \( u(x, 1) = 1 \) for \( x < 0 \), \( u(x, 1) = 0 \) for \( x > 1 \). From here we obtain a weak solution for \( t > 1 \) using the Rankine-Hugoniot condition, which states that we need (as \( u_+ = 1, u_- = 0 \))
\[
(1 - 0)\xi'(t) = \frac{1^2}{2} - \frac{0^2}{2},
\]
\[
i.e. \ \xi'(t) = 1/2. \quad \text{Since } \xi(1) = 0 \text{ to go through the point } (0, 1), \text{ we get } \xi(t) = \frac{t-1}{2}.
\]
Thus, the solution for \( t > 1 \) is 1 if \( x < \frac{t-1}{2} \), 0 if \( x > \frac{t-1}{2} \).

Problem 5.

(i) State the definition of a function \( f \) being Schwartz: \( f \in S(\mathbb{R}^n) \).

(ii) On \( \mathbb{R}^n \), solve the PDE
\[
\Delta^2 u + u = e^{-|x|^2}
\]
(\( \Delta u = \sum_{j=1}^n \partial_j^2 u \) and \( \Delta^2 u = \Delta(\Delta u) \)). You may leave your answer as the inverse Fourier transform of a function. Is there a unique Schwartz function solving this PDE? You may use that the Fourier transform of \( f(x) = e^{-a|x|^2}, a > 0 \), is \((\pi/a)^{n/2} e^{-||\xi||^2/(4a)}\).

(iii) Now consider the PDE
\[
\Delta^2 u - u = f
\]
with \( f \) a given Schwartz function. If \( f = 0 \), an \( u \) is Schwartz, is \( u \) necessarily 0? If \( f(x) = e^{-|x|^2} \), does a Schwartz solution \( u \) exist?

Solution. (i) A Schwartz function \( f \) is a \( C^\infty \) function on \( \mathbb{R}^n \) such that for all multi-indices \( \alpha, \beta \in \mathbb{N}^n, x^\alpha D_\xi^\beta f \) is bounded.

(ii) We take the Fourier transform in \( x \), and denote the corresponding variable by \( \xi \). Since \( \mathcal{F}D_{x_j} = i\xi_j \mathcal{F} \), and \( -\Delta = \sum_{j=1}^n \partial_j^2, \Delta^2 = (-\Delta)^2 \), this gives
\[
(|\xi|^4 + 1)\mathcal{F}u = \mathcal{F}((\Delta^2 + 1)u) = \mathcal{F}e^{-|\xi|^2} = \pi^{n/2} e^{-||\xi||^2/4}.
\]
Thus,
\[
\mathcal{F}u = \frac{\pi^{n/2}}{|\xi|^4 + 1} e^{-||\xi||^2/4},
\]
so
\[
u = \mathcal{F}^{-1} \left( \frac{\pi^{n/2}}{|\xi|^4 + 1} e^{-||\xi||^2/4} \right).
\]
In particular, the solution is unique, and is indeed Schwartz as \( \frac{\pi^{n/2}}{|\xi|^4 + 1} e^{-||\xi||^2/4} \) is Schwartz, and the inverse Fourier transform preserves this class.
(iii) Now
\[(|\xi|^2 - 1)\mathcal{F}u = \mathcal{F}((\Delta^2 - 1)u) = \mathcal{F}f,\]
which is Schwartz. If \(f = 0\), and \(u\) is Schwartz, then \(\mathcal{F}u\) is Schwartz, so \((|\xi|^4 - 1)\mathcal{F}u = 0\) implies \((\mathcal{F}u)(\xi) = 0\) when \(|\xi| \neq 1\). Since \(\mathcal{F}u\) is continuous, it then vanishes for all \(\xi\).

If \(f = e^{-|x|^2}\), so \((\mathcal{F}f)(\xi) = \pi^{n/2}e^{-|\xi|^2/4}\), and \(u\) is Schwartz, then \(\mathcal{F}u\) is Schwartz and \((\mathcal{F}u)(\xi) = \pi^{n/2}e^{-|\xi|^2/4}/|\xi|^4 - 1\) when \(|\xi| \neq 1\). But the right hand side does not have a limit as \(|\xi| \to 1\), contradicting the continuity of \(\mathcal{F}u\). Thus, there are no Schwartz functions \(u\) solving this equation.