MATH 220: PRACTICE MIDTERM, SOLUTIONS

This is a closed book, closed notes, no calculators exam.
There are 5 problems. Solve all of them. Total score: 100 points.

Problem 1. (i) (13 points) *Find the general C^1 solution of the PDE*

$$3y^2 u_x + u_y = 0.$$

(ii) (6 points) *Solve the initial value problem with initial condition*

$$u(x, 0) = x^2.$$

Solution. The characteristic ODEs are

$$\frac{dx}{ds} = 3y^2,\ \frac{dy}{ds} = 1,\ \frac{dz}{ds} = 0,$$

so the projected characteristic curves satisfy $\frac{dx}{dy} = 3y^2$, so $x = y^3 + C$, i.e. $x - y^3$ constant are exactly the projected characteristic curves. Moreover, as $\frac{dz}{ds} = 0$, u is constant along these curves, so the general solution is $u(x, y) = f(x - y^3)$, f an arbitrary C^1 function.

To satisfy the initial condition, let $x^2 = u(x, 0) = f(x)$, so $u(x, y) = (x - y^3)^2$.

An alternative method is to solve the characteristic ODEs directly. Notice that this being a linear (or more generally semilinear) PDE, the equations for the projected characteristic curves decouple, and then one sees that the projected characteristic curves through the x axis give all projected characteristic curves. So we could impose the initial condition at the x axis to obtain the general solution: $u(x, 0) = f(x)$, f to be specified. The initial condition for the characteristic ODEs is then $(x(r, 0), y(r, 0), z(r, 0)) = (r, 0, f(r))$. The solution for the characteristic ODEs is

$$y(r, s) = s + c_2(r),\ z(r, s) = c_3(r),$$

hence $\frac{dx}{ds} = 3(s + c_2(r))^2$, and

$$x(r, s) = (s + c_2(r))^3 + c_1(r).$$

The initial condition gives $c_2(r) = 0$, $c_3(r) = f(r)$, $c_1(r) = r$, so

$$x(r, s) = s^3 + r,\ y(r, s) = s,\ z(r, s) = f(r).$$

Correspondingly, $s = y$, $r = x - y^3$, and $u(x, y) = f(x - y^3)$. If $f(x) = x^2$, this gives $u(x, y) = (x - y^3)^2$.

Problem 2. (i) (10 points) *Find the general C^2 solution of the PDE*

$$u_{xx} + 3u_{xt} + 2u_{tt} = 0.$$

(ii) (10 points) *Solve the initial value problem with initial condition*

$$u(x, -x) = \phi(x),\ u_t(x, -x) = \psi(x),$$

with ϕ, ψ given.

Solution. Factoring the partial differential operator as

$$\partial_x^2 + 3\partial_x\partial_t + 2\partial_t^2 = (\partial_x + \partial_t)(\partial_x + 2\partial_t),$$

so to find the general solution, we need to solve

$$(\partial_x + \partial_t)v = 0,\ (\partial_x + 2\partial_t)u = v.$$

This can be done directly; the main point being that solutions of the *homogeneous* versions of these equations would be $f(x-t)$, resp. $g(2x-t)$.

For the sake of variety, we change into corresponding characteristic coordinates. For $\partial_x + \partial_t$, the projected characteristic curves satisfy

$$\frac{dx}{ds} = 1,\ \frac{dt}{ds} = 1 \Rightarrow \frac{dt}{dx} = 1,$$

so they are $t = x + C'$, i.e. $x - t = C$. Similarly, for $\partial_x + 2\partial_t$,

$$\frac{dx}{ds} = 1,\ \frac{dt}{ds} = 2 \Rightarrow \frac{dt}{dx} = 2,$$

so we get $2x - t = C$. We change into characteristic coordinates $\xi = x - t$, $\eta = 2x - t$, so $x = \eta - \xi$, $t = \eta - 2\xi$, yields $-\partial_\xi = \partial_x + 2\partial_t$, $\partial_\eta = \partial_x + \partial_t$, so the PDE is $-\partial_\xi\partial_\eta u = 0$, hence $u = f(\xi) + g(\eta)$ with f, g arbitrary, as shown in class. Therefore,

$$u(x,t) = f(x-t) + g(2x-t),$$

f, g arbitrary C^2 functions is the general solution.

For the initial condition, we just substitute in $(x,t) = (x,-x)$, and note that $u_t(x,t) = -f'(x-t) - g'(2x-t)$, so

$$f(2x) + g(3x) = \phi(x),\ -f'(2x) - g'(3x) = \psi(x).$$

Differentiating the first equation with respect to x we get a system that we can solve for $f'(2x)$ and $g'(3x)$:

$$2f'(2x) + 3g'(3x) = \phi'(x),\ -f'(2x) - g'(3x) = \psi(x).$$

hence

$$f'(2x) = -3\psi(x) - \phi'(x),\ g'(3x) = \phi'(x) + 2\psi(x).$$

Integration from 0 to x yields for some constants A, B,

$$\frac{1}{2}f(2x) = -3\int_0^x \psi(y)\,dy - \phi(x) + A,$$
$$\frac{1}{3}g(3x) = \phi(x) + 2\int_0^x \psi(y)\,dy + B.$$

Multiplying through and replacing x by $x/2$, resp. $x/3$ yields f and g, so

$$\begin{aligned} u(x,t) = &-2\phi\left(\frac{x-t}{2}\right) + 3\phi\left(\frac{2x-t}{3}\right) \\ &- 6\int_0^{(x-t)/2} \psi(y)\,dy + 6\int_0^{(2x-t)/3} \psi(y)\,dy + 2A + 3B. \end{aligned}$$

Substituting in $t = -x$, $u(x,t) = \phi(x) + 2A + 3B$, so $2A + 3B = 0$, and the solution is

$$u(x,t) = -2\phi\left(\frac{x-t}{2}\right) + 3\phi\left(\frac{2x-t}{3}\right) + 6\int_{(x-t)/2}^{(2x-t)/3} \psi(y)\,dy.$$

As mentioned, one can also proceed directly, solving the PDE by solving two first order ones, and one can even get the initial conditions satisfied immediately. Namely, let $v = u_x + 2u_t$. As $u(x, -x) = \phi(x)$, we deduce that $u_x(x, -x) - u_t(x, -x) = \phi'(x)$. As $u_t(x, -x) = \psi(x)$, v satisfies the initial condition $v(x, -x) = \phi'(x) + 3\psi(x)$. Thus the characteristic ODEs for v, which solves $v_x + v_t = 0$, are

$$\frac{dx}{ds} = 1,\ \frac{dt}{ds} = 1,\ \frac{dz}{ds} = 0,\ x(r, 0) = r,\ t(r, 0) = -r,\ z(r, 0) = \phi'(r) + 3\psi(r).$$

This gives

$$x = s + r,\ t = s - r,\ z = \phi'(r) + 3\psi(r),$$

so

$$s = \frac{x+t}{2},\ r = \frac{x-t}{2},\ v(x, y) = \phi'\left(\frac{x-t}{2}\right) + 3\psi\left(\frac{x-t}{2}\right).$$

Having solved for v, now consider the PDE for u: $u_x + 2u_t = v$, with initial condition $u(x, -x) = \phi(x)$. The characteristic ODEs are

$$\begin{aligned} &\frac{dx}{ds} = 1,\ \frac{dt}{ds} = 2,\ \frac{dz}{ds} = v = \phi'\left(\frac{x-t}{2}\right) + 3\psi\left(\frac{x-t}{2}\right), \\ &x(r, 0) = r,\ t(r, 0) = -r,\ z(r, 0) = \phi(r). \end{aligned}$$

The ODEs for x and t give

$$x = s + r,\ t = 2s - r \Rightarrow s = \frac{x+t}{3},\ r = \frac{2x-t}{3},$$

so

$$\frac{dz}{ds} = \phi'\left(\frac{-s+2r}{2}\right) + 3\psi\left(\frac{-s+2r}{2}\right),$$

so by the fundamental theorem of calculus,

$$z(r, s) = \phi(r) + \int_0^s \left(\phi'\left(\frac{-\sigma+2r}{2}\right) + 3\psi\left(\frac{-\sigma+2r}{2}\right)\right) d\sigma.$$

Thus,

$$\begin{aligned} u(x, y) &= \phi\left(\frac{2x-t}{3}\right) + \int_0^{(x+t)/3} \left(\phi'\left(\frac{-\sigma}{2} + \frac{2x-t}{3}\right) + 3\psi\left(\frac{-\sigma}{2} + \frac{2x-t}{3}\right)\right) d\sigma \\ &= \phi\left(\frac{2x-t}{3}\right) - 2\phi\left(\frac{-\sigma}{2} + \frac{2x-t}{3}\right)|_0^{(x+t)/3} + 6\int_{(x-t)/2}^{(2x-t)/3} \psi(y)\, dy \\ &= 3\phi\left(\frac{2x-t}{3}\right) - 2\phi\left(\frac{x-t}{2}\right) + 6\int_{(x-t)/2}^{(2x-t)/3} \psi(y)\, dy, \end{aligned}$$

where in the penultimate step we used the fundamental theorem of calculus on ϕ and changed the variable of integration for ψ by letting $y = \frac{-\sigma}{2} + \frac{2x-t}{3}$, so $\sigma = 0$ gives $y = \frac{2x-t}{3}$ and $\sigma = (x+t)/3$ gives $y = \frac{x-t}{2}$.

Problem 3. (20 points) *Find the* bounded *solution of Laplace's equation on the infinite strip* $\mathbb{R}_x \times (0, 1)_y$*:*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, &&x \in \mathbb{R},\ y \in (0, 1), \\ &u(x, 0) = \phi(x), &&x \in \mathbb{R}, \\ &u(x, 1) = 0, &&x \in \mathbb{R}, \end{aligned}$$

where $\phi(x) = xe^{-x}$ *for* $x \geq 0$, $\phi(x) = 0$ *for* $x < 0$. *You may leave your solution as the inverse Fourier transform of a function that you have evaluated explicitly.*

Solution. Take the partial Fourier transform of the PDE in x, and writing $\hat{u}(\xi, y) = (\mathcal{F}_x u)(\xi, y)$, we obtain

$$\begin{aligned} &-\xi^2 \hat{u} + \partial_y^2 \hat{u} = 0, \qquad \xi \in \mathbb{R},\ y \in (0,1),\\ &\hat{u}(\xi, 0) = \mathcal{F}\phi(\xi), \qquad \xi \in \mathbb{R},\\ &\hat{u}(\xi, 1) = 0, \qquad \xi \in \mathbb{R}, \end{aligned}$$

For each $\xi \in \mathbb{R}$ we thus have an ODE in y. The general solution of this ODE is

$$\hat{u}(\xi, y) = A(\xi) e^{\xi y} + B(\xi) e^{-\xi y},$$

if $\xi \neq 0$, and

$$\hat{u}(0, y) = C + Dy.$$

Matching the boundary conditions for $\xi \neq 0$ yields

$$A(\xi) + B(\xi) = \mathcal{F}\phi(\xi),\ A(\xi)e^{\xi} + B(\xi)e^{-\xi} = 0.$$

From the second of these, $B(\xi) = -e^{2\xi} A(\xi)$, hence the first yields

$$A(\xi) = (1 - e^{2\xi})^{-1} \mathcal{F}\phi(\xi),\ \xi \neq 0.$$

Note that we are *not* dividing by 0. Thus, for $\xi \neq 0$,

$$\begin{aligned} \hat{u}(\xi, y) &= (1 - e^{2\xi})^{-1} \mathcal{F}\phi(\xi)(e^{\xi y} - e^{2\xi} e^{-\xi y})\\ &= (e^{-\xi} - e^{\xi})^{-1} \mathcal{F}\phi(\xi)(e^{\xi(y-1)} - e^{-\xi(y-1)}) = \frac{\sinh(\xi(1-y))}{\sinh \xi} \mathcal{F}\phi(\xi). \end{aligned}$$

(Note: we could have found this faster if we started by writing the general solution of the ODE as $A'(\xi)\sinh(\xi(1-y)) + B'(\xi)\cosh(\xi(1-y))$.) For $\xi = 0$, we get from the boundary conditions

$$C = \mathcal{F}\phi(0),\ C + D = 0,$$

so

$$\hat{u}(0, y) = \mathcal{F}\phi(0)(1 - y).$$

Note that by L'Hopital's rule, this is indeed $\lim_{\xi \to 0} \hat{u}(\xi, y)$, so $\hat{u}$ is continuous at $\xi = 0$. We thus simply write

$$\hat{u}(\xi, y) = \frac{\sinh(\xi(1-y))}{\sinh \xi} \mathcal{F}\phi(\xi),$$

with the understanding that the value at $\xi = 0$ is obtained by letting $\xi \to 0$. We now compute $\mathcal{F}\phi$, integrating by parts (differentiating x):

$$\begin{aligned} \mathcal{F}\phi(\xi) &= \int_0^\infty x e^{-ix\xi} e^{-x}\, dx = \int_0^\infty x e^{-(i\xi+1)x}\, dx\\ &= \frac{1}{i\xi + 1} \int_0^\infty e^{-(i\xi+1)x}\, dx = \frac{1}{(i\xi+1)^2}. \end{aligned}$$

We thus deduce that

$$\hat{u}(\xi, y) = \frac{\sinh(\xi(1-y))}{\sinh \xi} \frac{1}{(i\xi+1)^2},$$

and thus

$$u(x, y) = \mathcal{F}_\xi^{-1}\left(\frac{\sinh(\xi(1-y))}{\sinh \xi} \frac{1}{(i\xi+1)^2}\right).$$

Note here that $\frac{\sinh(\xi(1-y))}{\sinh\xi}$ is actually bounded even as $\xi \to \infty$; indeed, it decreases rapidly if $0 < y < 1$, so this indeed gives a bounded solution u of the PDE. Namely, to see the rapid decrease as $\xi \to +\infty$, rewrite

$$\frac{\sinh(\xi(1-y))}{\sinh\xi} = \frac{e^{\xi(1-y)} - e^{-\xi(1-y)}}{e^{\xi} - e^{-\xi}} = \frac{e^{-\xi y} - e^{-\xi}e^{-\xi(1-y)}}{1 - e^{-2\xi}},$$

and note that all factors and terms except 1 in both the numerator and the denominator go to 0 as $\xi \to +\infty$. The argument as $\xi \to -\infty$ is similar.

Problem 4. (17 points) *Consider the damped wave equation on* $\mathbb{R}_x \times [0,\infty)_t$*:*

$$u_{tt} + a(x)u_t = (c(x)^2 u_x)_x,$$

where $a \geq 0$*,* $c > 0$ *depending on* x *only, and there are constants* $c_1, c_2 > 0$ *such that* $c_1 \leq c(x) \leq c_2$ *for all* x*. Suppose that* $u(x,0)$ *and* $u_t(x,0)$ *vanish for* $|x| \geq R$*. Let*

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} (u_t(x,t)^2 + c(x)^2 u_x(x,t)^2)\,dx.$$

Show that E *is a decreasing (i.e. non-increasing) function of* t*, and that the solution of the damped wave equation (under the conditions mentioned above) is unique. You may use that the damped wave equation also has finite propagation speed* $\leq c_2$*. (This would be proved as in your homework problem.)*

Solution. Because of the finite speed of propagation, all integrals considered below converge, and the last boundary term vanishes. We now compute

$$\begin{aligned}\frac{dE}{dt} &= \int_{\mathbb{R}} (u_t u_{tt} + c^2 u_x u_{xt})\,dx = \int_{\mathbb{R}} (u_t(-au_t + (c^2 u_x)_x) + c^2 u_x u_{xt})\,dx \\ &= -\int_{\mathbb{R}} a u_t^2\,dx + \int_{\mathbb{R}} (u_t c^2 u_x)_x\,dx \leq u_t c^2 u_x|_{x=-\infty}^{+\infty} = 0,\end{aligned}$$

so E is indeed decreasing. Here we used that $\int a u_t^2\,dx \geq 0$ as $a \geq 0$.

Since E is decreasing and non-negative, $E(0) = 0$ implies that $E(t) = 0$ for all $t \geq 0$. Thus, if $u(x,0)$ and $u_t(x,0)$ vanish, then u_t and u_x vanish for $t \geq 0$, so u is a constant. Since $u(x,0) = 0$, it follows that this constant is 0, hence the solution of the equation is unique (by taking the difference of two putative solutions).

Problem 5. *Consider Burger's equation*

$$u_t + uu_x = 0,\ u(x,0) = \phi(x),\ t \geq 0,$$

with initial condition

$$\phi(x) = \begin{cases} 0, & x < 0, \\ h, & 0 < x < 1, \\ 0, & x > 1, \end{cases}$$

$h > 0$ *constant.*

(i) (3 points) *State the definition of* u *being a weak solution of this PDE.*

(ii) (3 points) *For piecewise* C^1 *functions* u*, with a jump along a* C^1 *curve* $x = \xi(t)$*, state an equivalent condition that* u *solves the PDE in a weak sense in* $t > 0$*.*

(iii) (3 points) *What is the entropy condition?*

(iv) (15 points) *Find the unique weak solution* u *that satisfies the entropy condition.*

Solution. Let $f(z) = \frac{z^2}{2}$. u is a weak solution of the PDE if for all $\psi \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support,

$$\int_{\mathbb{R}\times[0,\infty)} (u\psi_t + f(u)\psi_x)\,dx\,dt + \int_{\mathbb{R}} \phi(x)\psi(x,0)\,dx = 0.$$

For piecewise C^1 functions u with a jump at $x = \xi(t)$ this is equivalent to u solving the PDE in the classical sense away from $x = \xi(t)$, and at $x = \xi(t)$ the Rankine-Hugoniot condition holds:

$$\xi'(t) = \frac{f(u_-) - f(u_+)}{u_- - u_+},$$

where u_+ and u_- are the limits of the values of u from $x > \xi(t)$, resp. $x < \xi(t)$. The entropy condition is that along a discontinuity, in addition to Rankine-Hugoniot, we have

$$f'(u_-) > \xi'(t) > f'(u_+).$$

For our Burger's equation, we'll have a shock wave from $x = 1$, $t = 0$ (as $h > 0$), and a rarefaction wave from $x = 0$, $t = 0$. The shock wave satisfies $\xi'(t) = \frac{h}{2}$, so the shock curve is $x = 1 + \frac{ht}{2}$. The rarefaction wave is $\frac{x}{t}$ in the region $0 < x < ht$. Correspondingly, these two collide when $1 + \frac{ht}{2} = ht$, i.e. when $t = \frac{2}{h}$. Thus, for $t < \frac{2}{h}$,

$$u(x,t) = \begin{cases} 0, & x < 0, \\ \frac{x}{t}, & 0 < x < ht, \\ h, & ht < x < 1 + \frac{ht}{2}, \\ 0, & x > 1 + \frac{ht}{2}. \end{cases}$$

For $t > \frac{2}{h}$ we get a shock emanating from the collision point, $(2, \frac{2}{h})$, which satisfies $\xi' = \frac{\xi}{2t}$, so $\int \frac{d\xi}{\xi} = \int \frac{dt}{2t}$, i.e. $\xi = \sqrt{Ct}$. As $\xi(\frac{2}{h}) = 2$, this gives $C = 2h$, and we deduce that for $t > \frac{2}{h}$,

$$u(x,t) = \begin{cases} 0, & x < 0, \\ \frac{x}{t}, & 0 < x < \sqrt{2ht}, \\ 0, & x > \sqrt{2ht}. \end{cases}$$