# MATH 220: PRACTICE MIDTERM

This is a closed book, closed notes, no calculators exam.
There are 5 problems. Solve all of them. Total score: 100 points.

**Problem 1.** (i) *(13 points)* Find the general $C^1$ solution of the PDE

$$3y^2 u_x + u_y = 0.$$

(ii) *(6 points)* Solve the initial value problem with initial condition

$$u(x,0) = x^2.$$

**Problem 2.** (i) *(10 points)* Find the general $C^2$ solution of the PDE

$$u_{xx} + 3u_{xt} + 2u_{tt} = 0.$$

(ii) *(10 points)* Solve the initial value problem with initial condition

$$u(x,-x) = \phi(x),\ \ u_t(x,-x) = \psi(x),$$

with $\phi, \psi$ given.

**Problem 3.** *(20 points)* Find the *bounded* solution of Laplace's equation on the infinite strip $\mathbb{R}_x \times (0,1)_y$:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, && x \in \mathbb{R},\ y \in (0,1), \\ &u(x,0) = \phi(x), && x \in \mathbb{R}, \\ &u(x,1) = 0, && x \in \mathbb{R}, \end{aligned}$$

where $\phi(x) = xe^{-x}$ for $x \geq 0$, $\phi(x) = 0$ for $x < 0$. You may leave your solution as the inverse Fourier transform of a function that you have evaluated explicitly.

**Problem 4.** *(17 points)* Consider the damped wave equation on $\mathbb{R}_x \times [0,\infty)_t$:

$$u_{tt} + a(x)u_t = (c(x)^2 u_x)_x,$$

where $a \geq 0$, $c > 0$ depending on $x$ only, and there are constants $c_1, c_2 > 0$ such that $c_1 \leq c(x) \leq c_2$ for all $x$. Suppose that $u(x,0)$ and $u_t(x,0)$ vanish for $|x| \geq R$. Let

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} (u_t(x,t)^2 + c(x)^2 u_x(x,t)^2)\, dx.$$

Show that $E$ is a decreasing (i.e. non-increasing) function of $t$, and that the solution of the damped wave equation (under the conditions mentioned above) is unique. You may use that the damped wave equation also has finite propagation speed $\leq c_2$. (This would be proved as in your homework problem.)

**Problem 5.** Consider Burger's equation

$$u_t + uu_x = 0,\ \ u(x,0) = \phi(x),\ \ t \geq 0,$$

with initial condition

$$\phi(x) = \begin{cases} 0, & x < 0, \\ h, & 0 < x < 1, \\ 0, & x > 1, \end{cases}$$

$h > 0$ constant.

(i) *(3 points)* State the definition of $u$ being a weak solution of this PDE.

(ii) *(3 points)* For piecewise $C^1$ functions $u$, with a jump along a $C^1$ curve $x = \xi(t)$, state an equivalent condition that $u$ solves the PDE in a weak sense in $t > 0$.

(iii) *(3 points)* What is the entropy condition?

(iv) *(15 points)* Find the weak solution $u$ that satisfies the entropy condition.