

**MATH 220: PROBLEM SET 9**  
**DUE THURSDAY, DECEMBER 3, 2009**

**Problem 1.** Let  $\phi(x) = |x|$  on  $[-\pi, \pi]$ . Let

$$f(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + b_1 \sin x + b_2 \sin(2x),$$

With what choice of the coefficients  $a_j$  and  $b_j$  is the  $L^2$  error  $\|f - \phi\|$  minimal? (Here  $\|f - \phi\|^2 = \int_{-\pi}^{\pi} |f(x) - \phi(x)|^2 dx$ .)

**Problem 2.** Let  $V = C([0, \ell])$  with the inner product  $\langle f, g \rangle = \int_0^\ell f(x) \overline{g(x)} dx$ . If  $D \subset V$  is a subspace, and  $A : D \rightarrow V$  is symmetric, we say that  $A$  is positive if  $\langle Av, v \rangle \geq 0$  for all  $v \in D$ .

- (i) Show that if  $A$  is positive then all eigenvalues  $\lambda$  of  $A$  are  $\geq 0$ .
- (ii) Show that  $A = -\frac{d^2}{dx^2}$  with domain

$$D = \{f \in C^2([0, \ell]) : f(0) = f'(\ell) = 0\}$$

is symmetric and is positive.

- (iii) Show that  $A = \frac{d^4}{dx^4}$  with domain

$$D = \{f \in C^4([0, \ell]) : f(0) = f'(0) = f(\ell) = f'(\ell) = 0\}$$

is symmetric and is positive.

**Problem 3.** Let  $\gamma_n$  be a sequence of constants tending to  $\infty$ . Consider the following continuous functions  $f_n$ ,  $n \geq 2$ , on  $[0, 1]$ :

$$f_n(x) = \begin{cases} 2n\gamma_n x, & 0 \leq x \leq \frac{1}{2n}, \\ 2n\gamma_n(\frac{1}{n} - x), & \frac{1}{2n} \leq x \leq \frac{1}{n}, \\ 0, & \frac{1}{n} \leq x \leq 1. \end{cases}$$

- (i) Sketch the graph of a few  $f_n$ .
- (ii) Show that  $f_n \rightarrow 0$  pointwise.
- (iii) Show that the convergence is not uniform.
- (iv) Show that  $f_n \rightarrow 0$  in  $L^2$  if  $\gamma_n = n^{1/4}$ .
- (v) Show that  $f_n$  does not converge in  $L^2$  if  $\gamma_n = n$ .

**Problem 4.** Let  $\phi(x) = x(\ell - x)$  on  $[0, \ell]$ .

- (i) Find the Fourier sine series of  $\phi$ , and state what it converges to for  $x \in \mathbb{R}$ .
- (ii) Find the Fourier cosine series of  $\phi$ , and state what it converges to for  $x \in \mathbb{R}$ .
- (iii) Compare the decay rates of the coefficients of the two series as  $n \rightarrow \infty$ . Why do the coefficients decay faster in one of the cases?

**Problem 5.** For both of the following functions discuss whether the Fourier sine series converges uniformly or in  $L^2$ :

- (i)  $\phi(x) = x$  on  $[0, \ell]$ ,
- (ii)  $\phi(x) = x(\ell - x)^2$  on  $[0, \ell]$ ,

Justify your answer by quoting the relevant convergence theorems. You do not need to compute the respective Fourier series.

**Problem 6.** In this problem we consider the inhomogeneous wave equation on  $[0, \ell]$  with inhomogeneous Dirichlet boundary conditions:

$$u_{tt} - c^2 u_{xx} = f, \quad u(0, t) = h(t), \quad u(\ell, t) = j(t), \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x).$$

- (i) Suppose  $h(t) = 0 = j(t)$  for all  $t$ . For each  $t$ , we can expand  $u(x, t)$  (which is a function of  $x$  only then) in Fourier sine series in  $x$ , with coefficients depending on  $t$ :

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x/\ell),$$

and can obtain a similar expansion for  $f$ .

- (ii) Assuming that you can differentiate term by term, i.e. that the odd  $2\ell$ -periodic extension of  $u$  is well-behaved, find the Fourier sine series of  $u_{xx}$  and  $u_{tt}$ .
- (iii) Derive an ODE for  $u_n(t)$  by substituting the Fourier series into the PDE. Find the initial conditions satisfied by  $u_n(t)$ , and solve the ODE.
- (iv) Now do not assume that  $h(t) = 0 = j(t)$ . We could solve the problem by subtracting from  $u$  a function  $F$  that satisfies the boundary conditions to place ourselves into the previous scenario. Instead, we proceed as follows.

As  $u$  does *not* satisfy the homogeneous boundary conditions that sine does, the Fourier coefficients will decay slowly, and term by term differentiation is not allowed to calculate  $u_{xx}$ . Instead, let  $w(x, t) = u_{xx}(x, t)$ , and expand  $w$  in Fourier sine series:

$$w(x, t) = \sum_{n=1}^{\infty} w_n(t) \sin(n\pi x/\ell).$$

Calculate the coefficients  $w_n(t)$  in terms of  $w$ , and then integrate by parts twice to obtain them in terms of  $u_n(t)$ ,  $h(t)$  and  $j(t)$ .

- (v) Obtain an ODE for  $u_n(t)$ , and solve it.