Problem 1. By proposition 0.11 in the Inner product spaces handout, we know that $\|f - \phi\|$ is minimal if we take the generalized Fourier coefficients. Meaning, if we call $c_n(x) = \cos(nx)$ and $s_n(x) = \sin(nx)$:

$$
\begin{aligned}
  a_0 &= \langle \phi, c_0 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x)dx = \frac{\pi}{2}, \\
  a_1 &= \langle \phi, c_1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos(x)dx = -\frac{4}{\pi}, \\
  a_2 &= \langle \phi, c_2 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos(2x)dx = 0, \\
  b_1 &= \langle \phi, s_1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin(x)dx = 0, \\
  b_2 &= \langle \phi, s_2 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin(2x)dx = 0.
\end{aligned}
$$

And hence $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos(x)$.

Remark. We knew a priori that the $b_n$ coefficients were zeros since $\phi$ is even.

Problem 2.

(i) Assume $A$ is symmetric positive. Let $\lambda$ be an eigenvalue of $A$ with associated (non-zero) eigenfunction $v$. Then we have that $\lambda$ is real since

$$
\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Av, v \rangle = \langle v, Av \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle,
$$

which implies that $\bar{\lambda} = \lambda$ since $\langle v, v \rangle \neq 0$.

Now we have

$$
\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Av, v \rangle \geq 0,
$$

which gives that $\lambda \geq 0$ since $\langle v, v \rangle > 0$. 

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(ii) Let \( f, g \in D \). Let’s show that \( A \) is symmetric. By integration by part, and using the fact that \( f(0) = f'(l) = 0 \) and \( g(0) = g'(l) = 0 \), we get that
\[
\langle Af, g \rangle = \int_0^l (-f''(x)) g(x) \, dx \\
= (-f'(x)) g(x) \bigg|_{x=0}^l - \int_0^l (-f'(x)) g''(x) \, dx \\
= \int_0^l f'(x) g'(x) \, dx \\
= f(x) g(x) \bigg|_{x=0}^l - \int_0^l f(x) g''(x) \, dx \\
= \int_0^l f'(x) g(x) \, dx
\]
(4)

Now let’s prove that \( A \) is positive. We get that easily from the previous third equality, that is,
\[
\langle Af, f \rangle = \int_0^l f'(x) g(x) \, dx = \int_0^l |f'(x)|^2 \, dx \geq 0.
\]
(5)

(iii) We proceed in the same fashion as before. Let \( f, g \in D \). Let’s show that \( A \) is symmetric. By integration by part, and using the fact that \( f(0) = f'(0) = f'(l) = 0 \) and \( g(0) = g'(0) = g(l) = g'(l) = 0 \), we get that
\[
\langle Af, g \rangle = \int_0^l f^{(4)}(x) g(x) \, dx \\
= f'''(x) g(x) \bigg|_{x=0}^l - \int_0^l f'''(x) g''(x) \, dx \\
= -\int_0^l f'''(x) g'(x) \, dx \\
= -f''(x) g'(x) \bigg|_{x=0}^l + \int_0^l f''(x) g'''(x) \, dx \\
= \int_0^l f''(x) g''(x) \, dx \\
= f'(x) g''(x) \bigg|_{x=0}^l - \int_0^l f'(x) g'''(x) \, dx \\
= -\int_0^l f'(x) g''(x) \, dx \\
= -f(x) g'''(x) \bigg|_{x=0}^l + \int_0^l f(x) g''''(x) \, dx \\
= \int_0^l f(x) g''''(x) \, dx
\]
(6)
\[= \langle f, Ag \rangle. \]
Now let’s prove that $A$ is positive. We get that easily from the fifth equality above, that is,

$$
\langle Af, f \rangle = \int_0^1 f''(x) \overline{f''(x)} \, dx = \int_0^1 |f''(x)|^2 \, dx \geq 0. \quad (7)
$$

Problem 3.

(i) Here are sketched some of the $f_n$’s (say $f_2$, $f_3$ and $f_5$):

![Diagram with sketches of $f_2$, $f_3$, and $f_5$]

(ii) We need to show that for a fixed $x_0 \in [0, 1]$, we have $f_n(x_0) \xrightarrow{n \to +\infty} 0$. If $x_0 = 0$, then $f_n(x_0) = 0$ for each $n$. If $0 < x_0 \leq 1$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x_0$, and for all $n \geq n_0$, we get $f_n(x_0) = 0$ by definition of $f_n$. Therefore we have the result.

(iii) The convergence to zero is uniform if $\sup_{x \in [0, 1]} |f_n(x)| \xrightarrow{n \to +\infty} 0$. However, we see that

$$
\sup_{x \in [0, 1]} |f_n(x)| = \gamma_n \xrightarrow{n \to +\infty} +\infty, \quad (8)
$$

which indeed does not converge to 0 as $n \to +\infty$.

(iv) For $\gamma_n = n^{1/4}$, we have

$$
\|f_n\|_{L^2}^2 = \int_0^1 f_n(x)^2 \, dx = 2(2n\gamma_n)^2 \frac{(2n)^{-3}}{3} = \frac{\gamma_n^2}{3n} = \frac{1}{3\sqrt{n}} \xrightarrow{n \to +\infty} 0. \quad (9)
$$

Therefore we have the $L^2$ convergence.
(v) For $\gamma_n = n$, we have
\[
\|f_n\|_{L^2}^2 = \frac{\gamma_n^2}{3n} = \frac{n^2}{3n} \xrightarrow{n \to \infty} +\infty.
\]
Therefore $f_n$ does not converge in $L^2$.

**Problem 4.**

(i) Let the Fourier sine series of $\phi$ be $\sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$. We can compute the coefficients as:
\[
A_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) \, dx = \frac{4(1 - (-1)^n)}{l} \left(\frac{l}{n\pi}\right)^3.
\]
Now we know from the lecture notes about convergence of Fourier series that the sine Fourier series converges to an odd $2l$-periodic extension function of $\phi$.

(ii) Similarly, let the Fourier cosine series of $\phi$ be $\sum_{n=1}^{+\infty} B_n \cos\left(\frac{n\pi x}{l}\right)$. We can compute the coefficients as:
\[
B_0 = \frac{1}{l} \int_0^l \phi(x) \, dx = \frac{l^3}{6}, B_n = \frac{2}{l} \int_0^l \phi(x) \cos\left(\frac{n\pi x}{l}\right) \, dx = \frac{(1 + (-1)^n)}{n^2\pi^2} \frac{l^3}{n^2\pi^2}, n \geq 1.
\]
Now we know from the lecture notes about convergence of Fourier series that the sine Fourier series converges to an even $2l$-periodic extension function of $\phi$.

(iii) We can see that $A_n$ decays as $\frac{1}{n^3}$ while $A_n$ decays as $\frac{1}{n^2}$, meaning that the former one is faster. The reason is that the odd extension of $\phi$ is continuously differentiable while the even one is merely continuous. And we know that the smoother the function is, the faster the Fourier coefficients decay.

**Problem 5.** For Fourier sine series, we first make an odd extension of $\phi$. Now our $\phi$ is defined on $[-l, l]$. 

4
(i) Fourier sine series converge pointwisely to $\phi$ in $(-l, l)$ and to $\frac{\phi(l) + \phi(-l)}{2} = 0$ at $x = \pm l$. However, we have $\phi(-l) = -l \neq l = \phi(l)$. Therefore the convergence is not uniform (otherwise the limit function should be continuous).

Now, we obviously have $\int_{-l}^{l} \phi(x)^2 dx < +\infty$. This implies that $\phi \in L^2([-l, l])$. Therefore we know from the course that the Fourier sine series converge to $\phi$ in $L^2$.

(ii) Using similar arguments, Fourier sine series converge pointwisely to $\phi$ in $(-l, l)$ and to $\frac{\phi(l) + \phi(-l)}{2} = 0$ at $x = \pm l$.

Moreover, here we have $\phi(l) = \phi(-l) = 0$ and $\phi'(l) = \phi'(-l) = 0$. Therefore the extension of $\phi$ is $C^2$, and the Fourier sine series converges uniformly to $\phi$ (lecture notes).

What’s more, we obviously have $\int_{-l}^{l} \phi(x)^2 dx < +\infty$. This implies that $\phi \in L^2([-l, l])$. Therefore we know from the course that the Fourier sine series converge to $\phi$ in $L^2$. Or we could have simply said that uniform convergence implies convergence in $L^2$.

Problem 6.

(i) There is no question. We know from the course that we have (at least in the $L^2$ sense) the expansions

\[
\begin{align*}
\frac{u(x, t)}{} &= \sum_{n=1}^{+\infty} u_n(t) \sin \left(\frac{n\pi x}{l}\right), \\
\frac{f(x, t)}{} &= \sum_{n=1}^{+\infty} f_n(t) \sin \left(\frac{n\pi x}{l}\right), \\
\frac{\phi(x)}{} &= \sum_{n=1}^{+\infty} \phi_n \sin \left(\frac{n\pi x}{l}\right), \\
\frac{\psi(x, t)}{} &= \sum_{n=1}^{+\infty} \psi_n \sin \left(\frac{n\pi x}{l}\right).
\end{align*}
\]
(ii) If we assume that we can differentiate term by term the Fourier sine series of $u$, we get:

$$u_{tt}(x,t) = \sum_{n=1}^{+\infty} u''_{n}(t) \sin \left( \frac{n\pi x}{l} \right),$$

$$u_{xx}(x,t) = \sum_{n=1}^{+\infty} \left( -\frac{n^2\pi^2}{l^2} u_{n}(t) \right) \sin \left( \frac{n\pi x}{l} \right).$$

(14)

(iii) By using orthogonality (or identifying coefficients term by term), we get the ODE:

$$\begin{cases} 
    u''_{n}(t) + \left( \frac{n^2 c^2 \pi^2}{l^2} - u_{n}(t) \right) u_{n}(t) = f_{n}(t), \\
    u_{n}(0) = \phi_{n}, \quad u'_{n}(0) = \psi_{n}.
\end{cases}$$

(15)

Now using Duhamel’s principle, (or the method of variation of the parameters), we get the solution

$$u_{n}(t) = \psi_{n} \frac{l}{cn\pi} \sin \left( \frac{cn\pi t}{l} \right) + \phi_{n} \cos \left( \frac{cn\pi t}{l} \right) + \int_{0}^{t} \frac{1}{cn\pi} \sin \left( \frac{cn\pi (t-s)}{l} \right) f_{n}(s) ds.$$  

(16)

(iv) We have

$$w(x,t) = \sum_{n=1}^{+\infty} w_{n}(t) \sin \left( \frac{n\pi x}{l} \right),$$

(17)

and by integration by parts, we get that

$$w_{n}(t) = \frac{2}{l} \int_{0}^{l} u_{x}(x,t) \sin \left( \frac{n\pi x}{l} \right) dx$$

$$= -\frac{2n\pi}{l^2} \int_{0}^{l} u_{x}(x,t) \cos \left( \frac{n\pi x}{l} \right) dx$$

$$= -\frac{2n\pi}{l^2} \left( u_{x}(x,t) \cos \left( \frac{n\pi x}{l} \right) \bigg|_{x=0}^{x=l} + \frac{n\pi}{l} \int_{0}^{l} u(x,t) \sin \left( \frac{n\pi x}{l} \right) dx \right)$$

$$= -\frac{2n\pi}{l^2} \left( (-1)^n j(t) - h(t) \right) - \frac{n^2\pi^2}{l^2} u_{n}(t).$$

(18)
In the same fashion as previously, identifying the coefficients in the PDE, we get:

\[ u''_n(t) = c^2 w_n(t) + f_n(t) = f_n(t) - \frac{2nc^2 \pi}{l^2} ((-1)^n j(t) - h(t)) - \frac{n^2 c^2 \pi^2}{l^2} u_n(t). \]  \hspace{1cm} (19)

Therefore, the ODE to solve is now:

\[
\begin{align*}
  u''_n(t) + \left( \frac{n^2 c^2 \pi^2}{l^2} - u_n(t) \right) u_n(t) &= f_n(t) - \frac{2nc^2 \pi}{l^2} ((-1)^n j(t) - h(t)) - \frac{n^2 c^2 \pi^2}{l^2} u_n(t), \\
  u_n(0) &= \phi_n, \ u'_n(0) = \psi_n.
\end{align*}
\]  \hspace{1cm} (20)

Calling \( g_n(t) = f_n(t) - \frac{2nc^2 \pi}{l^2} ((-1)^n j(t) - h(t)) - \frac{n^2 c^2 \pi^2}{l^2} u_n(t) \), and using Duhamel’s principle, we finally get the solution

\[ u_n(t) = \psi_n \frac{l}{c n \pi} \sin \left( \frac{c n \pi t}{l} \right) + \phi_n \cos \left( \frac{c n \pi t}{l} \right) + \int_0^t \frac{l}{c n \pi} \sin \left( \frac{c n \pi (t-s)}{l} \right) g_n(s) ds. \]  \hspace{1cm} (21)

**Problem 7.**

(i) The eigenvalue/eigenfunction problem we have to solve reduces to:

\[ X''(x) = -\lambda X(x), \ X(l) = X(-l), \ X'(l) = X'(-l). \]  \hspace{1cm} (22)

For \( \lambda = 0 \), we have \( \lambda_0 = 0 \), and \( X_0(x) = 1 \).

For \( \lambda > 0 \), we have, after solving the ODE, \( \lambda_n = \left( \frac{n \pi}{l} \right)^2 \), and

\[ X_n(x) = A_n \cos \left( \frac{n \pi x}{l} \right) + B_n \sin \left( \frac{n \pi x}{l} \right), \ n \geq 1. \]  \hspace{1cm} (23)

For \( \lambda < 0 \), we get \( X(x) = 0 \). Therefore there is no strictly negative eigenvalue.

Then we can solve for \( T_n \) for each value of \( \lambda_n \). We have the ODE

\[ T''_n(t) = -c^2 \lambda_n T_n(t), \]  \hspace{1cm} (24)

which gives the solutions:

\[ T_0(t) = C_0 + D_0 t, \]

\[ T_n(t) = C_n \cos \left( \frac{n c \pi t}{l} \right) + D_n \sin \left( \frac{n c \pi t}{l} \right), \ n \geq 1. \]  \hspace{1cm} (25)

Therefore the general separated solution is which gives the solutions:

\[
\begin{align*}
  u(x, t) &= A_0 + D_0 t + \sum_{n=1}^{+\infty} A_n \cos \left( \frac{n c \pi t}{l} \right) \cos \left( \frac{n \pi x}{l} \right) \\
  &\quad + \sum_{n=1}^{+\infty} B_n \cos \left( \frac{n c \pi t}{l} \right) \sin \left( \frac{n \pi x}{l} \right) + \sum_{n=1}^{+\infty} C_n \sin \left( \frac{n c \pi t}{l} \right) \cos \left( \frac{n \pi x}{l} \right) + \sum_{n=1}^{+\infty} D_n \sin \left( \frac{n c \pi t}{l} \right) \sin \left( \frac{n \pi x}{l} \right).
\end{align*}
\]  \hspace{1cm} (26)
(with arbitrary $A_n$, $B_n$, $C_n$ and $D_n$.)

(ii) Initial condition $u(x,0) = 0$ gives $A_n = B_n = 0$ for all $n \in \mathbb{N}$.
Initial condition $u_t(x,0) = \cos \left( \frac{2\pi x}{l} \right) - \sin \left( \frac{\pi x}{l} \right)$ implies that
$D_0 = 0$, $D_1 = -\frac{l}{c\pi}$, $C_2 = \frac{1}{2c\pi}$ and $C_n = D_n = 0$ for the other $n$'s.
Hence the solution is
$$u(x,t) = \frac{l}{2c\pi} \sin \left( \frac{2c\pi t}{l} \right) \cos \left( \frac{2\pi x}{l} \right) - \frac{l}{c\pi} \sin \left( \frac{ct}{l} \right) \sin \left( \frac{\pi x}{l} \right). \quad (27)$$

(iii) Consider $\tilde{u}$, the $2l$-periodic extension of $u$. Then $\tilde{u}$ verifies the following PDE:
$$\tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0, \quad \tilde{u}(x,0) = 0, \quad \tilde{u}_t(x,0) = \cos \left( \frac{2\pi x}{l} \right) - \sin \left( \frac{\pi x}{l} \right). \quad (28)$$
By d’Alembert’s formula, we know that
$$\tilde{u}(x,t) = \frac{l}{2c\pi} \int_{x-ct}^{x+ct} \left( \cos \left( \frac{2\pi y}{l} \right) - \sin \left( \frac{\pi y}{l} \right) \right) dy$$
$$= \frac{l}{4c\pi} \left( \sin \left( \frac{2\pi y}{l} \right) \bigg|_{x-ct}^{x+ct} \right) + \frac{l}{2c\pi} \cos \left( \frac{\pi y}{l} \right) \bigg|_{x-ct}^{x+ct}$$
$$= \frac{l}{4c\pi} \left( \sin \left( \frac{2\pi (x+ct)}{l} \right) - \sin \left( \frac{2\pi (x-ct)}{l} \right) \right) + \frac{l}{2c\pi} \left( \cos \left( \frac{\pi (x+ct)}{l} \right) - \cos \left( \frac{\pi (x-ct)}{l} \right) \right)$$
$$= \frac{l}{2c\pi} \sin \left( \frac{2c\pi t}{l} \right) \cos \left( \frac{2\pi x}{l} \right) - \frac{l}{c\pi} \sin \left( \frac{ct}{l} \right) \sin \left( \frac{\pi x}{l} \right). \quad (29)$$

(iv) $u$ will have singularities on lines $x \pm ct = x_0 + 2kl$, where $k \in \mathbb{Z}$.
Consider the strip to be a circle. In other words, the head and the tail are connected. The singularity at the beginning will propagate at speed $c$ in both directions. At time $t$, it has run a distance of $ct$, so it has reached $x_0 \pm ct$. $u$ has singularities at $(x_0 \pm ct \mod 2l, t)$, where $y \mod 2l$ means the unique point $z \in [-l,l)$ such that $y - z$ is a multiple of $2l$. In other words, singularities will keep traveling around the ring at speed $c$. And since the ring has perimeter $2l$, after traveling for time $\frac{2l}{c}$, or any integer multiple of this, they are back to where they started. From the perspective of the method of images (going to the reals, i.e. the universal cover), one has a singularity in the initial data at $x_0$ plus integer multiples of $2l$; these propagate in the standard manner for the wave equation on the real line (at speed $c$); if one takes $[-l,l]$ as the interval representing the circle, when one of these is in this interval (at some time), it gives rise to a singularity you observe.
Here is a sketch of what is going on:

Figure 1: Left: circular representation (perimeter is $2l$). Right: representation on the real line ($2l$-periodic function). $m \in \mathbb{Z}$, $k \in \mathbb{Z}$, $-l \leq x_0 \leq l$