Problem 1. Solve the inhomogeneous heat equation on the half-line for Dirichlet boundary conditions:

\[ u_t - ku_{xx} = f, \quad u(x,0) = \phi(x), \quad u(0,t) = 0, \]

in two different ways:

(i) Using Duhamel’s principle, and the solution formula for the homogeneous equation derived in class (i.e. with \( f = 0 \)) on the half line.

(ii) Using the appropriate extension of \( f \) and \( \phi \) to the whole real line and solving the inhomogeneous PDE on the real line.

Problem 2. Derive Duhamel’s principle for the wave equation on \( \mathbb{R} \)

\[ u_{tt} - c^2 \partial_x^2 u = f, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \]

by setting up a first order system for \( U = \begin{bmatrix} u \\ v \end{bmatrix} \), \( v = u_t \), namely

\[
\begin{align*}
  u_t - v &= 0, \quad u(x,0) = \phi(x), \\
  v_t - c^2 \partial_x^2 u &= f, \quad v(x,0) = \psi(x).
\end{align*}
\]

Thus, one has

\[
\partial_t U - AU = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad U(0,x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix},
\]

where

\[
A = \begin{bmatrix} 0 & \text{Id} \\ c^2 \partial_x^2 & 0 \end{bmatrix}.
\]

This is now a first order equation in time, so Duhamel’s principle for first order equations is applicable, and gives the solution of the inhomogeneous equation as

\[
U(x,t) = S(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix}(x) + \int_0^t S(t-s) \begin{bmatrix} 0 \\ f_s \end{bmatrix}(x) \, ds,
\]

where \( S \) is the solution operator for the homogeneous problem \( \partial_t U - AU = 0 \). You need to work this out explicitly, in particular what \( S \) is, to derive the solution of the wave equation.

Problem 3. (i) Consider the following eigenvalue problem on \([0,\ell]\):

\[-X'' = \lambda X, \quad X(0) = 0, \quad X'(\ell) = 0.\]

Find all eigenvalues and eigenfunctions.

(ii) Using separation of variables, find the general ‘separated’ solution of the wave equation

\[ u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0, \quad u(x,\ell,t) = 0. \]

(iii) Solve the wave equation with initial conditions

\[ u(x,0) = \sin(3\pi x/(2\ell)) - 2 \sin(5\pi x/(2\ell)), \quad u_t(x,0) = 0. \]

(iv) Using separation of variables, find the general ‘separated’ solution of the heat equation

\[ u_t = ku_{xx}, \quad u(0,t) = 0, \quad u(x,\ell,t) = 0, \]

here \( k > 0 \) constant.
Problem 4.  (i) Using the general ‘separated’ solution you found in Problem 3, solve the wave equation
\[ u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0, \]
with initial conditions
\[ u(x,0) = 0, \quad u_x(x,0) = x(x - \ell)^2. \]

(ii) Using the general ‘separated’ solution you found in Problem 3, solve the heat equation
\[ u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(\ell,t) = 0, \quad u(x,0) = x(x - \ell)^2. \]

You may assume throughout that the generalized Fourier series you construct converge to the function they are supposed to represent.

Problem 5. (Optional!) The goal of this problem is to show that if \( u \in \mathcal{D}'(\mathbb{R}^3) \) and \( \Delta u = f \) satisfies \( x_0 \notin \text{sing supp } f \), i.e. \( f \) is \( C^\infty \) near \( x_0 \), then \( u \) is \( C^\infty \) near \( x_0 \).

This is called elliptic regularity: \( \Delta \) is elliptic, and for an elliptic operator \( P \) if \( Pu \) is \( C^\infty \) near some \( x_0 \) then so is \( u \).

We achieve this as follows.

(i) First suppose that \( u \) is a \( C^2 \) function. Let \( \phi \in C^\infty_c(\mathbb{R}^3) \) be identically 1 near \( x_0 \) such that \( f \) is \( C^\infty \) on \( \text{supp } \phi \). Then show that \( \Delta(\phi u) = \phi \Delta u + v \), where \( v \) is a compactly supported distribution that vanishes near \( x_0 \). Now as \( w = \phi u \) is compactly supported,
\[ w(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x - y|} \Delta_y(\phi(y)u(y)) \, dy. \]

(ii) Expand \( \Delta_y(\phi(y)u(y)) \) as above. To analyze
\[ \int_{\mathbb{R}^3} \frac{1}{4\pi|x - y|} v(y) \, dy \]
for \( x \) near \( x_0 \), note that if \( x \) is near \( x_0 \) and \( y \in \text{supp } v \) then \( x \neq y \), so \( |x - y|^{-1} \) is \( C^\infty \). On the other hand, \( \phi \Delta u \) is \( C^\infty \) by assumption. Write the corresponding part of the convolution as
\[ \int_{\mathbb{R}^3} \frac{1}{4\pi|x - y|} \phi(x - y)(\Delta u)(x - y) \, dy, \]
and deduce that it is \( C^\infty \).

(iii) Suppose now that \( u \in \mathcal{D}' \). Proceed as above, writing
\[ w = -\frac{1}{4\pi|x|} * (\Delta(\phi u)), \]
convolution in the sense of distributions (so \( w \) is merely a distribution), and show that both parts are \( C^\infty \) near \( x_0 \). You do not have to be very careful in writing up this part; there are some technicalities, but the point is to get the main idea.