Problem 1. Solve the wave equation on the line:
\[ u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \]
with
\[ \phi(x) = \begin{cases} 
0, & x < -1, \\
1 + x, & -1 < x < 0, \\
1 - x, & 0 < x < 1, \\
0, & x > 1. 
\end{cases} \]
and
\[ \psi(x) = \begin{cases} 
0, & x < -1, \\
2, & -1 < x < 1, \\
0, & x > 1. 
\end{cases} \]
Also describe in \( t > 0 \) where the solution vanishes, and where it is \( C^\infty \), and compare it with the general results discussed in lecture (Huygens’ principle and propagation of singularities).

Problem 2. Consider the PDE
\[ u_{tt} - \nabla \cdot (c^2 \nabla u) + qu = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \]
where \( c, q \geq 0 \), depend on \( x \) only, and \( c \) is bounded between positive constants, i.e. for some \( c_1, c_2 > 0, c_1 \leq c(x) \leq c_2 \) for all \( x \in \mathbb{R}^n \). Assume that \( u \) is \( C^2 \) throughout this problem, and \( u \) is real-valued. (All calculations would go through if one wrote \( |u_t|^2 \), etc., in the complex valued case.)

(i) Fix \( x_0 \in \mathbb{R}^n \) and \( R_0 > 0 \), and for \( t < \frac{R_0}{c_2} \), let
\[ E(t) = \int_{|x-x_0| < R_0 - c_2 t} (u_t^2 + c(x)^2 |\nabla u|^2 + q(x) u^2) \, dx. \]
Show that \( E \) is decreasing with \( t \) (i.e. non-increasing). (Hint: to make sure you don’t forget anything in the calculation, do it first on the line, when \( n = 1 \).)

(ii) Suppose that \( \text{supp} \phi, \text{supp} \psi \subset \{ |x| \leq R \} \), i.e. are 0 outside this ball. Show that \( u(x,t) = 0 \) if \( t \geq 0, |x| > R + c_2 t \), i.e. the wave indeed propagates at speed \( \leq c_2 \).

(iii) Show that there is at most one real-valued \( C^2 \) solution of (1).

Problem 3. Consider the wave equation on \( \mathbb{R}^n \):
\[ u_{tt} - c^2 \Delta u = f, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \]
and write \( x = (x', x_n) \) where \( x' = (x_1, \ldots, x_{n-1}) \)

(i) Show that if
\[ f(x', x_n, t) = f(x', -x_n, t), \quad \phi(x', x_n) = \phi(x', -x_n), \quad \psi(x', x_n) = \psi(x', -x_n) \]
These facts will enable us to solve the wave equation in the half space as shown in class for the ball \( \Omega = f \) with \( u \) following statement: Suppose that \( u(x', x_n, t) = u(x', -x_n, t) \), show that it solves the homogeneous wave equation with 0 initial conditions.)

(ii) Show that if
\[
f(x', x_n, t) = -f(x', -x_n, t), \quad \phi(x', -x_n) = -\phi(x', -x_n), \quad \psi(x', x_n) = -\psi(x', -x_n)
\]
for all \( x \) and \( t \), i.e. if \( f, \phi, \psi \) are all odd functions of \( x_n \), then \( u \) is an odd function of \( x_n \) as well.

(iii) If \( u \) is continuous, and is an odd function of \( x_n \), show that \( u(x', 0, t) = 0 \) for all \( x' \) and \( t \).

(iv) If \( u \) is a \( C^1 \) and is an even function of \( x_n \), show that \( \partial_{x_n} u(x', 0, t) = 0 \) for all \( x' \) and \( t \).

These facts will enable us to solve the wave equation in the half space \( x_n > 0 \) with Dirichlet or Neumann boundary conditions later in the course.

**Problem 4.** Use the maximum principle for Laplace’s equation on \( \mathbb{R}^n \) to show the following statement: Suppose that \( u \in C^2(\mathbb{R}^n) \) and \( \Delta u = 0 \). Suppose moreover that \( u(x) \to 0 \) at infinity uniformly in the following sense:
\[
\sup_{|x| > R} |u(x)| \to 0
\]
as \( R \to \infty \). Then \( u(x) = 0 \) for all \( x \in \mathbb{R}^n \). (Hint: Apply the maximum principle shown in class for the ball \( \Omega = \{x : |x| < R\} \) and for both \( u \) and \( -u \).

Use this to show that the solution of Laplace’s equation on \( \mathbb{R}^n \):
\[
\Delta u = f,
\]
with \( f \) given, is unique in the class of functions \( u \) such that \( u \in C^2(\mathbb{R}^n) \) and \( u(x) \to 0 \) at infinity uniformly.

**Problem 5.** (Do not hand in!) Show the following maximum principle for the heat equation on \( \mathbb{R}^n \): Suppose that \( u \in C^2(\mathbb{R}^n \times (0, T]) \cap C^0(\mathbb{R}^n \times [0, T]) \) and \( u_t = k\Delta u, k > 0 \). Suppose moreover that \( u(x, t) \to 0 \) at infinity uniformly in the following sense:
\[
\sup_{|x| > R, 0 \leq t \leq T} |u(x, t)| \to 0
\]
as \( R \to \infty \). Then
\[
\sup_{(x, t) \in \mathbb{R}^n \times [0, T]} u(x, t) = \max(0, \sup_{x \in \mathbb{R}^n} u(x, 0)).
\]
(Note that you can think of 0 as the ‘boundary value of \( u \) at infinity’, in analogy with the case \( \Omega \times [0, T] \) discussed in class, where the max would be the maximum of the initial and boundary values.)

Use this to show that the solution of the initial value problem for the heat equation on \( \mathbb{R}^n \times [0, T] \):
\[
u_t = k\Delta u, \quad u(x, 0) = \phi(x),
\]
with \( \phi \) given, is unique in the class of functions \( u \) such that \( u \in C^2(\mathbb{R}^n \times (0, T]) \cap C^0(\mathbb{R}^n \times [0, T]) \) and \( u(x, t) \to 0 \) at infinity uniformly.