Problem 1. Let \( \psi \in C(\mathbb{R}) \) be given by:

\[
\psi(x) = \begin{cases} 
0, & x < -1, \\
1 + x, & -1 < x < 0, \\
1 - x, & 0 < x < 1, \\
0, & x > 1,
\end{cases}
\]  

(1)

so that it verifies \( \psi \geq 0 \) and \( \int_{\mathbb{R}} \psi(x) dx = 1 \).

Consider \((\psi_j)_{j \geq 1}\) constructed as \( \psi_j(x) = j \psi(jx) \), so that \( \psi_j(x) = 0 \) if \( |x| \geq 1/j \), and \( \int_{\mathbb{R}} \psi_j(x) dx = 1 \), for all \( j \geq 1 \).

Let us show that \( \iota \psi_j \rightarrow \delta_0 \) in \( D'(\mathbb{R}) \). By definition, it means that we need to prove that, for all \( \phi \in C_{c}^{\infty}(\mathbb{R}) \),

\[
\left| \int_{\mathbb{R}} \psi_j(x) \phi(x) dx - \phi(0) \right| < \epsilon.
\]  

(2)

Let \( \epsilon > 0 \) and \( \phi \in C_{c}^{\infty}(\mathbb{R}) \). Since \( \phi \) is continuous, there exists \( \eta > 0 \) such that for all \( |x| < \eta \), \( |\phi(x) - \phi(0)| < \epsilon \). Then consider \( j_0 \) such that \( 1/j_0 < \eta \).

Since we know that \( \psi_j(x) = 0 \) for all \( |x| \geq 1/j \), \( \int_{\mathbb{R}} \psi_j(x) dx = 1 \), and \( \psi_j(x) \geq 0 \) for all \( x \), for all \( j \geq j_0 \) we have:

\[
\left| \int_{\mathbb{R}} \psi_j(x) \phi(x) dx - \phi(0) \right| = \left| \int_{\mathbb{R}} \psi_j(x) \phi(x) dx - \int_{\mathbb{R}} \psi_j(x) \phi(0) dx \right|
\leq \int_{-1/j_0}^{-1/j} \psi_j(x) \left| \phi(x) - \phi(0) \right| dx
\leq \epsilon \int_{-1/j_0}^{-1/j} \psi_j(x) dx = \epsilon.
\]  

(3)

Let \( \phi \in C_{c}^{\infty}(\mathbb{R}) \) be such that \( \phi(x) = 1 \) for all \( |x| < 1 \). Then, for \( j \geq 1 \),

\[
\int_{\mathbb{R}} \psi_j(x)^2 \phi(x) dx = \int_{-1/j}^{0} j^2 (1 + jx)^2 dx + \int_{0}^{1/j} j^2 (1 - jx)^2 dx = \frac{2j}{3}.
\]  

(4)
Therefore, as \( j \to +\infty \), \( \int_{\mathbb{R}} \psi_j(x)^2 \phi(x) dx \to +\infty \), and \( \{t_{\psi_j^2}(\phi)\}_{j=1}^{+\infty} \) does not converge. Consequently, \( \{t_{\psi_j^2}(\phi)\}_{j=1}^{+\infty} \) does not converge to any distribution since \( \{t_{\psi_j^2}(\phi)\}_{j=1}^{+\infty} \) does not converge for the very \( \phi \) we exhibited.

(3) We have just shown that \( t_{\psi_j} \to \delta_0 \), but \( \{t_{\psi_j^2}(\phi)\}_{j=1}^{+\infty} \) does not converge to any distribution. Therefore there is no continuous extension of the map \( Q : f \mapsto f^2 \) on \( C(\mathbb{R}) \) to \( D'(\mathbb{R}) \).

Problem 2. We consider the conservation law:
\[
 u_t + (f(u))_x = 0, \quad u(x,0) = \phi(x),
\]  
with \( f \in C^2(\mathbb{R}) \).

Since \( u \) is continuous and \( f \) is \( C^2 \), \( v = f'(u) \) is also continuous. Since \( u \) is \( C^1 \) apart from jump discontinuities in its first derivatives, away from the jumps, \( u_t \) (resp. \( u_x \)) is perfectly defined and continuous. Therefore, since \( f' \) is \( C^1 \), and away from the discontinuities, \( v_t = f''(u)u_t \) (resp. \( v_x = f''(u)u_x \)) is also continuous, \( v \) is \( C^1 \) apart from jump discontinuities in its first derivatives (the same ones as \( u \)). Therefore \( v \) has the same properties as \( u \). Moreover, we have, away from discontinuities:
\[
 v_t + v v_x = f''(u)u_t + f'(u)f''(u)u_x = f''(u)(u_t + f'(u)u_x) = f''(u)(u_t + (f(u))_x) = 0.
\]  
So \( v \) verifies the Burger's equation (the Rankine-Hugoniot condition is vacuous: there are no shock since \( v \) is continuous).

If \( f'' > 0 \), \( f \) is strictly convex and \( f' \) is strictly increasing and therefore the inverse function \( (f')^{-1} \) exists. We can therefore first solve for \( v \) from the Burger’s equation:
\[
 v_t + v v_x = 0, \quad v(x,0) = f'(\phi(x)),
\]  
and then \( u = (f')^{-1}(v) \) is solution of the original PDE.

Suppose now that \( u \) has a jump discontinuity. Then, according to the Rankine-Hugoniot condition:
\[
 \xi'(t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.
\]  
If \( v \) could have been defined as previously, \( v \) would have the same discontinuity \( (v = f'(u)) \) and again by Rankine-Hugoniot:
\[
 \xi'(t) = \frac{v_+^2 - v_-^2}{v_+ - v_-} = \frac{1}{2}(v_+ + v_-) = \frac{1}{2}(f'(u_+) + f'(u_-)).
\]
But in general,

\[
\frac{f(u_+) - f(u_-)}{u_+ - u_-} \neq \frac{1}{2} (f'(u_+) + f'(u_-))
\]  \hspace{1cm} (10)

(consider for instance \(f(x) = e^x\)) and the statement is FALSE.

**Problem 3.** Consider Burger’s equation

\[
u_t + uu_x = 0, \quad u(x,0) = \phi(x), \hspace{1cm} (11)
\]

with initial condition

\[
\phi(x) = \begin{cases} 
0, & x < -1, \\
-1 - x, & -1 < x < 0, \\
-1 + x, & 0 < x < 1, \\
0, & x > 1.
\end{cases}  \hspace{1cm} (12)
\]

(1) To build the weak solution, it is very convenient to draw the characteristic curves.

![Characteristic curves](image)

Figure 1: Characteristic curves. We can see that we have 4 different behaviors of the solutions (limit regime in green), and some characteristic curves intersect each others: it is at this moment that a jump discontinuity appears and the classic solution is no longer acceptable.

In the Burger case, we know that the solution \(u\) is constant along the characteristic curves \(x_r(t) = \phi(r)t + r\). As long as the characteristic curves don’t intersect, we have:
• If \( r < -1 \), then \( \phi(r) = 0 \), which implies that the characteristic curves are

\[
x_r(t) = r, \quad r < -1,
\]

and the solution \( u(x, t) = 0 \) along those curves,

• If \(-1 < r < 0\), then \( \phi(r) = -1 - r \), which implies that the characteristic curves are

\[
x_r(t) = (-1 - r)t + r, \quad -1 < r < 0,
\]

and the solution \( u(x, t) = -1 - r = -1 - \frac{x + t}{1 - t} = \frac{x + 1}{t - 1} \) along those curves.

• If \( 0 < r < 1 \), then \( \phi(r) = -1 + r \), which implies that the characteristic curves are

\[
x_r(t) = (-1 + r)t + r, \quad 0 < r < 1,
\]

and the solution \( u(x, t) = -1 + r = -1 + \frac{x + t}{1 + t} = \frac{x - 1}{t + 1} \) along those curves.

• If \( r > 1 \), then \( \phi(r) = 0 \), which implies that the characteristic curves are

\[
x_r(t) = r, \quad r > 1,
\]

and the solution \( u(x, t) = 0 \) along those curves.

To sum up, we have that, for \( t \) small,

\[
u(x, t) = \begin{cases} 
0, & x < -1, \\
\frac{x + 1}{t - 1}, & -1 < x < -t, \quad (-1 < \frac{x + t}{1 - t} < 0) \\
\frac{x - 1}{t + 1}, & -1 < x < -t, \quad (0 < \frac{x + t}{1 + t} < 1) \\
0, & x > 1.
\end{cases}
\]

Now, from the sketch of the characteristic curves and/or the condition \(-1 < x < -t\) of the solution, we can see that the characteristic curves don’t intersect while \( t < 1 \). Therefore the above solution is valid for \( t < 1 \).
Figure 2: Characteristic curves. Domains where everything is fine in color, domain of intersection in white: a jump discontinuity appears.

Figure 3: Characteristic curves. Sketch of the characteristic field also valid for time $t > 1$ for weak solution.

The curves intersect at $t = 1$ Beyond that time, we therefore consider a weak solution satisfying the Rankine-Hugoniot condition. At the level of the discontinuity $\xi(t)$, we have that $u_-(\xi(t),t) = 0$ and $u_+(\xi(t),t) = \frac{\xi(t) - 1}{t + 1}$, and $\xi(1) = -1$. Therefore the Rankine-Hugoniot condition is
\[ \xi'(t) = \frac{0 - \frac{1}{2} \left( \frac{\xi(t)-1}{t+1} \right)^2}{0 - \frac{\xi(t)-1}{t+1}} = \frac{1}{2} \frac{\xi(t) - 1}{t+1}. \]  

(18)

Hence, \( \xi \) verifies the following ODE:

\[
\begin{cases}
\xi'(t) = \frac{1}{2} \frac{\xi(t) - 1}{t+1}, \\
\xi(1) = -1.
\end{cases}
\]

(19)

Solve it (say, by separation of variables) and you get \( \xi(t) = 1 - \sqrt{2(1+t)} \).

Therefore, for \( t \geq 1 \), the solution is

\[
\begin{cases}
0, & -1 < x < 1 - \sqrt{2(1+t)}, \\
\frac{x - 1}{t + 1}, & 1 - \sqrt{2(1+t)} < x < 1, \\
0, & x > 1.
\end{cases}
\]

(20)

(2) There are two cases for \( t \) to consider.

For \( 0 \leq t < 1 \),

\[
\int_{\mathbb{R}} u(x,t) \, dx = \int_{-1}^{-t} \frac{x + 1}{t - 1} \, dx + \int_{-t}^{1} \frac{x - 1}{t + 1} \, dx = -1.
\]

(21)

And for \( t > 1 \),

\[
\int_{\mathbb{R}} u(x,t) \, dx = \int_{1}^{\sqrt{2(1+t)}} \frac{x - 1}{t + 1} \, dx = -1.
\]

(22)

Therefore \( \int_{\mathbb{R}} u(x,t) \, dx \) is indeed constant.

(3) Let us call \( E(t) = \int_{\mathbb{R}} w(x,t) \, dx \), where \( w = w^3 \).

For \( 0 \leq t < 1 \), we have

\[
E(t) = \int_{\mathbb{R}} w(x,t) \, dx = \int_{-1}^{-t} \frac{x + 1}{t - 1} \, dx + \int_{-t}^{1} \frac{x - 1}{t + 1} \, dx = -\frac{1}{2}.
\]

(23)

So \( E(t) \) is indeed constant before a shock develops, but for \( t > 1 \),

\[
E(t) = \int_{\mathbb{R}} w(x,t) \, dx = \int_{1}^{\sqrt{2(1+t)}} \frac{x - 1}{t + 1} \, dx = -\frac{1}{t + 1},
\]

(24)

and \( E(t) \) is no longer a constant.
Let’s now explain what is going on here.

Following Problem 2, if we define \( g(x) = \frac{3}{4} x^{4/3} \), we have that \( u = g'(w) \).

From Problem 2, we know that \( w_t + (g(w))_x = 0 \). After the shock develops, we have for \( u \):

\[
\frac{d}{dt} \int_{\mathbb{R}} u(x,t) dx = \frac{d}{dt} \int_{\xi(t)}^1 u(x,t) dx \\
= \int_{\xi(t)}^1 \left((u_+)_t - \xi'(t)u_+(\xi(t), t)\right) dx \\
= \int_{\xi(t)}^1 \left(\frac{1}{2} u^2_+\right)_x - \xi'(t)u_+(\xi(t), t) dx \\
= \frac{1}{2} u_+(\xi(t), t)^2 - \xi'(t)u_+(\xi(t), t) = 0, 
\]

from the Rankine-Hugoniot jump condition (remember \( u_\pm = 0 \)).

Meanwhile for \( w \) we have:

\[
\frac{d}{dt} \int_{\mathbb{R}} w(x,t) dx = \frac{d}{dt} \int_{\xi(t)}^1 w(x,t) dx \\
= \int_{\xi(t)}^1 \left((w_+)_t - \xi'(t)w_+(\xi(t), t)\right) dx \\
= \int_{\xi(t)}^1 (g(w_+))_x - \xi'(t)w_+(\xi(t), t) dx \\
= g(w_+(\xi(t), t)) - \xi'(t)w_+(\xi(t), t) \neq 0, 
\]

because \( w \) does not satisfy the same Rankine-Hugoniot condition.

**Problem 4.**

(1) \[ u_{xx} - u_{xy} - 2u_{yy} = 0. \] \hspace{1cm} (27)

We have \( A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & -2 \end{pmatrix} \). Therefore \( \det(A) = -2 - 1/4 = -9/4 < 0 \), and \( Tr(A) = 1 + 2 = 3 < 0 \). Therefore the eigenvalues of \( A \) are non zero and of opposite signs: Hyperbolic PDE.

(2) \[ u_{xx} - 2u_{xy} + u_{yy} = 0. \] \hspace{1cm} (28)

We have \( A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). Therefore \( \det(A) = 0 \). Therefore at least one of the eigenvalues of \( A \) is zero: Degenerate PDE.
\( u_{xx} + 2u_{xy} + 2u_{yy} = 0. \)  

(29)

We have \( A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \). Therefore \( \det(A) = 2 - 1 = 1 > 0 \) and \( \text{Tr}(A) = 3 > 0 \). Therefore the eigenvalues of \( A \) are non zero and of the same sign: Elliptic PDE.

**Problem 5.**

(1) Let’s find the general \( C^2 \) solution of the PDE

\[ u_{xx} - u_{xt} - 6u_{tt} = 0, \quad (30) \]

by reducing it to a system of first order PDEs (by the way, this is an elliptic PDE).

We are looking for \( a, b, c, d \) that formally verify:

\[ \partial_{xx} - \partial_{xt} - 6\partial_{tt} = (a\partial_x + b\partial_t)(c\partial_x + d\partial_t) = ac\partial_{xx} + (ad + bc)\partial_{xt} + bd\partial_{tt}. \]  

(31)

So we get the (under-determined) system:

\[
\begin{cases}
ac = 1, \\
ad + bc = -1, \\
bd = -6.
\end{cases}
\]

(32)

From the first equation, let us simply take \( a = c = 1 \). Then the system reduces to

\[
\begin{cases}
d + b = -1, \\
bd = -6,
\end{cases}
\]

(33)

which gives \( b = 2 \) and \( d = -3 \).

Therefore we can write \( u_{xx} - u_{xt} - 6u_{tt} = 0 \) as \( (\partial_x + 2\partial_t)(\partial_x - 3\partial_t) u = 0. \)

Now let \( v = (\partial_x - 3\partial_t) u \). Then \( v \) verifies \( v_x + 2v_t = 0 \) (first order linear PDE!).

And we know that the solution writes \( v(x, t) = h(t - 2x) \) for some \( h \in C^1 \).

Now for \( u \) we have the system:

\[ u_x - 3u_t = h(t - 2x). \]  

(34)

Using the method of characteristics, we get the following equations:

\[
\begin{cases}
x'_r(s) = 1, & x_r(0) = 0, \\
t'_r(s) = -3, & t_r(0) = r, \\
v'_r(s) = h(t_r(s) - 2x_r(s)), & v_r(0) = \phi(r),
\end{cases}
\]

(35)

for some function \( \phi \in C^2 \).

Therefore we have \( x_r(s) = s, \ t_r(s) = -3s + r \), and

\[ v'_r(s) = h(-3s + r - 2s) = h(-5s + r), \]

(36)
and by integrating from \( s = 0 \), we get:

\[
v_r(s) = \int_0^s h(-5s' + r) ds' + \phi(r) = \frac{1}{5} \int_r^{s+5r} h(y) dy + \phi(r),
\]

after a change of variables. Now, since we have \( s = x \) and \( r = t + 3x \), we finally get:

\[
u(x,t) = \frac{1}{5} \int_{t-2x}^{t+3x} h(y) dy + \phi(t + 3x) = f(t + 3x) + g(t - 2x),
\]

for some \( f,g \in C^2 \).

Reciprocally, we verify that \( u \) of the form \( u(x,t) = f(t + 3x) + g(t - 2x) \) for \( f,g \in C^2 \) indeed solves the PDE.

For an arbitrary \( \phi \in C^\infty_c(\mathbb{R}^2) \) we have to show that

\[
u(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) + w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = 0.
\]

But from Problem 2 of Pset 2, we have:

\[
v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v((\partial_x - 3\partial_t)(\phi_x + 2\phi_t)) = 0,
\]

and similarly,

\[
w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = w((\partial_x + 2\partial_t)(\phi_x - 3\phi_t)) = 0.
\]

**Problem 6.** Let us solve (in the strong sense):

\[
\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \begin{cases} u_{xx} + 3u_{xy} - 4u_{yy} = xy, \\ u(x, x) = \sin x, \ u_x(x, x) = 0. \end{cases}
\]

Same strategy here, we reduce it to a system of first order PDEs (by the way, this is an hyperbolic PDE!).

We are looking for \( a, b, c, d \) that formally verify:

\[
\partial_{xx} + 3\partial_{xy} - 4\partial_{yy} = (a\partial_x + b\partial_y)(c\partial_x + d\partial_y) = ac\partial_{xx} + (ad + bc)\partial_{xy} + bd\partial_{yy}.
\]

So we get the (under-determined) system:

\[
\begin{cases} ac = 1 \\ ad + bc = 3 \\ bd = -4. \end{cases}
\]
From the first equation, let us simply take \( a = c = 1 \). Then the system reduces to

\[
\begin{align*}
    d + b &= 3 \\
    bd &= -4,
\end{align*}
\]

which gives, say, \( b = -1 \) and \( d = 4 \).

Therefore we can write \( u_{xx} + 3u_{xy} - 4u_{yy} = 0 \) as \((\partial_x - \partial_y)(\partial_x + 4\partial_y)\) \( u = 0 \). Now let \( v = (\partial_x + 4\partial_y)u \). Then \( v \) verifies \( v_x - v_y = xy \) (first order semi-linear PDE!), and \((u_x + u_y)|(x,x) = \sin'(x) = \cos(x)\), \( u_x(x,x) = 0 \) imply that \( u_y(x,x) = \cos(x) \) and \( v(x,x) = 4\cos(x) \). Therefore \( v \) satisfies the following PDE:

\[
\begin{align*}
    v_x - v_y &= xy, \\
    v(x,x) &= 4\cos(x).
\end{align*}
\]

The ODEs for the characteristics are then:

\[
\begin{align*}
    x_r'(s) &= 1, & x_r(0) &= r, \\
    y_r'(s) &= -1, & y_r(0) &= r, \\
    v_r'(s) &= x_r(s)y_r(s), & v_r(0) &= 4\cos(r).
\end{align*}
\]

After solving, we get:

\[
\begin{align*}
    x_r(s) &= s + r, \\
    y_r(s) &= r - s, \\
    v_r(s) &= v^2s - \frac{s^3}{3} + 4\cos(r).
\end{align*}
\]

Therefore we get the PDE for \( u \):

\[
\begin{align*}
    u_x + 4u_y &= v(x,y) = \left(\frac{x+y}{2}\right)^2 \left(\frac{x-y}{2}\right) + \frac{1}{3} \left(\frac{x-y}{2}\right)^3 + 4\cos \left(\frac{x+y}{2}\right), \\
    u(x,x) &= \sin(x).
\end{align*}
\]

Writing once again the characteristic ODEs for the PDE, we get:

\[
\begin{align*}
    x_r'(s) &= 1, & x_r(0) &= r, \\
    y_r'(s) &= 4, & y_r(0) &= r, \\
    v_r'(s) &= v(x_r(s), y_r(s)), & v_r(0) &= \sin(r).
\end{align*}
\]

After solving, we get:

\[
\begin{align*}
    x_r(s) &= s + r, \\
    y_r(s) &= 4s + r, \\
    v_r(s) &= \sin(r) + \int_0^s \left(-\frac{1}{3} - \frac{3}{2}r\right)^3 - \left(\frac{5r + 2r}{2}\right)^2 \frac{3r}{2} + 4\cos \left(\frac{5r + 2r}{2}\right) \right) dr \\
    &= \frac{9}{32}s^4 - \frac{75}{32}s^4 - \frac{5}{2}s^3r - \frac{3}{4}s^2r^2 + \frac{8}{5}\sin \left(\frac{5s + 2r}{2}\right) - \frac{3}{5}\sin(r).
\end{align*}
\]
Therefore (and finally):

\[ u(x,t) = -\frac{1}{16} \left( \frac{y-x}{3} \right)^2 \left( \frac{5x+y}{3} \right) \left( \frac{5y+13x}{3} \right) + \frac{8}{5} \sin \left( \frac{x+y}{2} \right) - \frac{3}{5} \sin \left( \frac{4x-y}{3} \right) . \]