Problem 1. Let $\psi \in C(\mathbb{R})$ be given by

$$
\psi(x) = \begin{cases} 
0, & x < -1, \\
1 + x, & -1 < x < 0, \\
1 - x, & 0 < x < 1, \\
0, & x > 1,
\end{cases}
$$

so $\psi \geq 0$, $\psi(x) = 0$ if $|x| \geq 1$ and $\int_{\mathbb{R}} \psi(x) \, dx = 1$. Let $\psi_j(x) = j \psi(jx)$, so $\psi_j(x) = 0$ if $|x| \geq 1/j$, and $\int_{\mathbb{R}} \psi_j(x) \, dx = 1$ for all $j$.

1. Show that $\psi_j \rightarrow \delta_0$ in $D'(\mathbb{R})$ (i.e., to be pedantic, $\iota \psi_j \rightarrow \delta_0$). (Hint: this is essentially written up in the notes on distributions, and was done in lecture!)

2. Let $\phi \in C^\infty_c(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| < 1$. Show that $\{\iota \psi_j (\phi)\}_{j=1}^\infty$ is not a convergent sequence in $\mathbb{R}$. Use this to conclude that $\{\iota \psi_j \}_{j=1}^\infty$ does not converge to any distribution.

3. Show that there is no continuous extension of the map $Q : f \mapsto f^2$ on $C(\mathbb{R})$ (so $Q : C(\mathbb{R}) \rightarrow C(\mathbb{R})$) to $D'(\mathbb{R})$, i.e. there is no map $\tilde{Q} : D'(\mathbb{R}) \rightarrow D'(\mathbb{R})$ such that
   - $\tilde{Q}(t \phi) = t Q f$ for every $f \in C(\mathbb{R})$ and
   - $u_j \rightarrow u$ in $D'(\mathbb{R})$ implies $\tilde{Q} u_j \rightarrow \tilde{Q} u$ in $D'(\mathbb{R})$.

Problem 2. Consider the conservation law

$$
u_t + (f(u))_x = 0, \quad u(x, 0) = \phi(x),
$$

with $f \in C^2(\mathbb{R})$. Let $v = f'(u)$. Show that if $f'' \neq 0$ and $u$ is continuous and is $C^1$ apart from jump discontinuities in its first derivatives then $v$ has the same properties and satisfies Burgers’ equation. (Note that the Rankine-Hugoniot condition is vacuous: there are no shocks.) If $f$ is strictly convex, i.e. $f'' > 0$, conclude that one can reduce general scalar conservation laws to Burgers’ equation, i.e. that one can find $u$ by solving for $v$ first.

Is the same true if $u$ has jump discontinuities, i.e. is it true that if $u$ satisfies the Rankine-Hugoniot conditions then $v$ does as well?

Problem 3. Consider Burgers’ equation

$$
u_t + uu_x = 0, \quad u(x, 0) = \phi(x),
$$

with initial condition

$$
\phi(x) = \begin{cases} 
0, & x < -1, \\
-1 - x, & -1 < x < 0, \\
-1 + x, & 0 < x < 1, \\
0, & x > 1.
\end{cases}
$$

1. Find the weak solution $u$ that satisfies the entropy condition (see Problem 4 on Problem Set 2).
(2) Show that \( \int u(x, t) \, dx \) is constant (independent of \( t \)).

(3) Show that \( \int u(x, t)^3 \, dx \) is constant until the time when a shock develops. What happens for later times? Explain this in terms of the results on Problem 2. (Hint for the last part: consider a conservation law \( v_t + g(v)_x = 0 \), so \( \int v(x, t) \, dx \) is conserved; find \( g \) so you can use this.)

**Problem 4.** Find the type (elliptic, hyperbolic, degenerate) of the following PDEs.

1. \( u_{xx} - u_{xy} - 2u_{yy} = 0 \).
2. \( u_{xx} - 2u_{xy} + u_{yy} = 0 \).
3. \( u_{xx} + 2u_{xy} + 2u_{yy} = 0 \).

**Problem 5.**

1. Find the general \( C^2 \) solution of the PDE
   \[
   u_{xx} - u_{xt} - 6u_{tt} = 0
   \]
   by reducing it to a system of first order PDEs.
2. Show that if \( f, g \in \mathcal{D}'(\mathbb{R}) \), and we define new distributions \( v, w \in \mathcal{D}'(\mathbb{R}^2) \) as in Problem 2 of Problem Set 2, i.e. formally \( v(x, t) = f(3x + t) \), \( w(x, t) = g(-2x + t) \), then \( u = v + w \) solves the PDE in (1). (Hint: use the result of Problem 2 of Problem Set 2, and factor our second order operator. This should only take a few lines.)

**Problem 6.** Solve \( u_{xx} + 3u_{xy} - 4u_{yy} = xy \), \( u(x, x) = \sin x \), \( u_x(x, x) = 0 \), \(-\infty < x, y < \infty\).