Problem 1. Show that the only solution $u \in \mathcal{D}'(\mathbb{R})$ of $u' = 0$ is $u = c$, $c$ a constant function.

$u' = 0$ in $\mathcal{D}'(\mathbb{R})$ means that $u(\phi') = 0$, $\forall \phi \in \mathcal{C}^\infty_c(\mathbb{R})$. And we want to show that:

$$\exists c \in \mathbb{R}, \forall \psi \in \mathcal{C}^\infty_c(\mathbb{R}), u(\psi) = \int_{\mathbb{R}} c \psi(x) dx. \quad (1)$$

Let’s choose a $\phi_0 \in \mathcal{C}^\infty_c(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi_0 \neq 0$. For an arbitrary $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$, we want to write it as $\psi = a_\psi \phi_0 + \phi'_\psi$, with $a_\psi \in \mathbb{R}$ and $\phi'_\psi \in \mathcal{C}^\infty_c(\mathbb{R})$. So we need to specify $a_\psi$ and $\phi'_\psi$.

We directly have that $\phi'_\psi = \psi - a_\psi \phi_0$, and with the fundamental theorem of calculus, if say $\text{supp} \psi \subset [-N, N]$ with some $N \geq 0$, then

$$\int_{\mathbb{R}} (\psi - a_\psi \phi_0) = \int_{\mathbb{R}} \phi'_\psi = \phi_\psi(N) - \phi_\psi(-N) = 0 \quad (2)$$

which therefore gives: $a_\psi = \frac{\int_{\mathbb{R}} \psi}{\int_{\mathbb{R}} \phi_0}$ (that is why we required the denominator to be non-zero). Now of course with this choice of $a_\psi$, we can then define $\phi'_\psi$ as being:

$$\phi'_\psi(x) = \int_{-\infty}^{x} (\psi(y) - a_\psi \phi_0(y)) dy, \quad (3)$$

and $\phi'_\psi \in \mathcal{C}^\infty_c(\mathbb{R})$, $\psi = a_\psi \phi_0 + \phi'_\psi$.

Now, let $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$. We’ve just shown that there exist $a_\psi \in \mathbb{R}$ and $\phi'_\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ such that $\psi = a_\psi \phi_0 + \phi'_\psi$.

Therefore

$$u(\psi) = u(a_\psi \phi_0 + \phi'_\psi) = a_\psi u(\phi_0) + u(\phi'_\psi) \quad \text{(linearity)}$$

$$= a_\psi u(\phi_0) \quad \text{(assumption on } u)$$

$$= \int_{\mathbb{R}} \psi \frac{u(\phi_0)}{\int_{\mathbb{R}} \phi_0} \quad \text{(4)}$$

$$= \left( \frac{u(\phi_0)}{\int_{\mathbb{R}} \phi_0} \right) \int_{\mathbb{R}} \psi$$

$$= c \int_{\mathbb{R}} \psi,$$
with \( c = \frac{u(\phi_0)}{\int_{\mathbb{R}} \phi_0} \), which is indeed a constant. The only remaining detail is to show that \( c \) does not depend on the choice of \( \phi_0 \). But this is straightforward by considering a \( \psi \) such that \( \int_{\mathbb{R}} \psi \neq 0 \).

**Problem 2.** We consider here the PDE (transport equation):

\[
au_x + u_y = 0.
\]

(1) Suppose \( f \) is merely piecewise continuous (or even only locally integrable (in \( L^1_{\text{loc}}(\mathbb{R}) \))). We are asked to verify that \( u(x, y) = f(x - ay) \) is a solution of the PDE (5) in the sense of distribution. Since \( f \in L^1_{\text{loc}}(\mathbb{R}) \), we can write, for any \( \phi \in C^\infty_c(\mathbb{R}^2) \):

\[
\iota(au_x + u_y)(\phi) = -\iota(a\phi_x + \phi_y) \text{ (derivative)}
\]

\[
= -\int_{\mathbb{R}^2} f(x - ay) (a\phi_x + \phi_y) (x, y) dx \, dy,
\]

\[
= -\int_{\mathbb{R}^2} f(\xi) \phi_\eta(\xi, \eta) d\xi \, d\eta,
\]

\[
= -\int_{\mathbb{R}} f(\xi) \left( \int_{\mathbb{R}} \phi_\eta(\xi, \eta) d\eta \right) d\xi
\]

\[
= 0 \quad (\phi \in C^\infty_c(\mathbb{R}^2))
\]

where we did the change of variables \( \xi = x - ay \) and \( \eta = y \), and used the Fubini theorem to separate the integrals (can be applied here since \( f \in L^1_{\text{loc}}(\mathbb{R}) \)).

(2) We proceed in the same fashion as we did in class to handle the cases of derivative and multiplication. Meaning that we are going to start with \( f \) merely piecewise continuous (or locally integrable) and see how it works, then infer what the distribution is when \( f \) is a distribution.

First, to emphasize the analogy, let us define the rule sending \( f \) to \( u \) as a map \( \tau_a : f \mapsto u \):

\[
\tau_a f(x, y) = f(x - ay),
\]

Then we need to express the distribution \( \iota_{\tau_a f} \) associated to \( \tau_a f \) in terms of \( \iota_f \), i.e. \( f \) should only enter on the right hand side via \( \iota_f \).

For any test function \( \psi \in C^\infty_c(\mathbb{R}^2) \), we have

\[
\iota_{\tau_a f}(\psi) = \int_{\mathbb{R}^2} (\tau_a f)(x, y) \psi(x, y) dx \, dy
\]

\[
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x - ay) \psi(x, y) dx \right) dy.
\]

Again, by the change of variables \( \xi = x - ay \), \( \eta = y \), we get
where $\Phi_{-a}\psi \in C_\infty_c(\mathbb{R}^2)$ is the test function given by

$$\Phi_{-a}\psi(\xi) = \int_{\mathbb{R}} \psi(\xi + a\eta, \eta) d\eta.$$  

(10)

So now we make the definition that if $f$ is replaced by a distribution, then for any test functions $\psi \in C_\infty_c(\mathbb{R}^2)$,

$$(\tau_a f)(\psi) = f(\Phi_{-a}\psi),$$

where $\Phi_{-a}\psi(\xi) = \int_{\mathbb{R}} \psi(\xi + a\eta, \eta) d\eta$.  

(11)

(3) By definition, $u_x + au_y$ is the distribution such that, for all $\psi \in C_\infty_c(\mathbb{R}^2)$,

$$(u_x + au_y)(\psi) = -u(\psi_x) - au(\psi_y) = -u(\psi_x + a\psi_y).$$

So we just need to prove that this vanishes for all $\psi$ when $u = \tau_a f$.

By definition of $\tau_a f$, we have

$$u(\psi_x + a\psi_y) = f(\Phi_{-a}(\psi_x + a\psi_y)).$$  

(12)

But by definition of $\Phi_{-a}$,

$$\Phi_{-a}(\psi_x + a\psi_y) = \int_{\mathbb{R}} (\psi_x + a\psi_y)(\xi + a\eta, \eta) d\eta.$$  

(13)

But now observe that if we define $\phi(\xi, \eta) = \psi(\xi + a\eta, \eta)$, then $\phi \in C_\infty_c(\mathbb{R})$ and

$$\phi_\eta(\xi, \eta) = a\psi_x(\xi + a\eta) + \psi_y(\xi + a\eta).$$  

(14)

by the chain rule. Therefore

$$\Phi_{-a}(\psi_x + a\psi_y) = \int_{\mathbb{R}} (\psi_x + a\psi_y)(\xi + a\eta, \eta) d\eta = \int_{\mathbb{R}} \phi_\eta(\xi, \eta) d\eta = 0.$$  

(15)

by the fundamental theorem of calculus and the fact that $\phi$ is compactly supported.

Therefore, eventually, for all $\psi$ test function,

$$(u_x + au_y)(\psi) = -u(\psi_x + a\psi_y)$$

$$= -f(\Phi_{-a}(\psi_x + a\psi_y))$$

$$= -f(0) = 0.$$  

(16)

which exactly means that $u_x + au_y = 0$ in the sense of distribution.
Problem 3. In \( t \geq 0 \), we consider the conservation law
\[
 u_t + f(u)_x = 0, \quad u(x,0) = \phi(x),
\] (17)
and suppose that \( u \) is \( C^1 \) except that it has a jump discontinuity along a \( C^1 \)-edge given by \( x = \xi(t) \).

Let us show that \( u \) is a weak solution of the PDE (17) if and only if \( u_+ \) and \( u_- \) solve the PDE in the classical sense.

First, let’s suppose that \( u \) is a weak solution. Then, for all \( \psi \in C_c^\infty (\mathbb{R} \times [0, +\infty)) \)
\[
\int_0^{+\infty} \int_{-\infty}^{\infty} ( u \psi_t + f(u) \psi_x ) \, dx \, dt + \int_{-\infty}^{+\infty} \phi(x) \psi(0,x) \, dx = 0. 
\] (18)

Now let choose \( \psi \) such that \( \psi(x,0) = 0 \), and break up the first integral into the regions \( \Omega_+ \) and \( \Omega_- \). We then get
\[
0 = \int_0^{+\infty} \int_{-\infty}^{\infty} ( u \psi_t + f(u) \psi_x ) \, dx \, dt \\
= \int \int_{\Omega_-} ( u_- \psi_t + f(u_-) \psi_x ) \, dx \, dt + \int \int_{\Omega_+} ( u_+ \psi_t + f(u_+) \psi_x ) \, dx \, dt \\
\] (19)

Now since \( u_\pm \) is \( C^1 \) on \( \Omega_\pm \), we can use the classical divergence theorem and the fact that \( \psi \) has compact support and \( \psi(x,0) = 0 \) to write
\[
\int \int_{\Omega_-} ( u_- \psi_t + f(u_-) \psi_x ) \, dx \, dt = - \int \int_{\Omega_+} ( (u_\pm)_t + (f(u_\pm)_x) ) \psi \, dx \, dt \\
\pm \int_{x=\xi(t)} ( u_- \psi \nu_2 + f(u_-) \psi \nu_1 ) \, ds,
\] (20)

where \( \nu = (\nu_1, \nu_2) \) is the outward unit normal to \( \Omega_- \) (so also the opposite of the outward normal to \( \Omega_+ \)).

Since \( u \) is a weak solution, and \( u \) is smooth on either side of the curve \( x = \xi(t) \), \( u_\pm \) is a strong solution of (17) in \( \Omega_\pm \) and
\[
\int \int_{\Omega_\pm} ( (u_\pm)_t + (f(u_\pm)_x) \psi ) \, dx \, dt = 0.
\] (21)

Therefore, for all smooth function \( \psi \), we have
\[
\int_{x=\xi(t)} ( u_- \psi \nu_2 + f(u_-) \psi \nu_1 ) \, ds - \int_{x=\xi(t)} ( u_+ \psi \nu_2 + f(u_+) \psi \nu_1 ) \, ds = 0, 
\] (22)

which leads to
\[
u \left. \begin{array}{c}
( u_- (\xi(t),t) \nu_2 + f(u_- (\xi(t),t)) \nu_1 = u_+ (\xi(t),t) \nu_2 + f(u_+ (\xi(t),t)) \nu_1 \\
\end{array} \right)
\] (23)

and
\[
\left. \frac{\nu_2}{\nu_1} ( u_- (\xi(t),t) - u_+ (\xi(t),t) ) \right) = \xi' (t) ( u_- (\xi(t),t) - u_+ (\xi(t),t) ),
\] (24)
which is exactly the Rankine-Hugoniot jump condition. Hence $u_+$ and $u_-$ solve the PDE in the classical sense.

Conversely, let’s suppose that $u_+$ and $u_-$ solve the PDE in the classical sense. Then (21) and (24) hold. By performing the previous operations in reverse order and using similar arguments, we get that $u$ is indeed a weak solution.

Now let’s look at the Burgers’ equation with particular initial condition:

$$
\begin{cases}
  u_t + uu_x = 0, \ x \in \mathbb{R}, \ t > 0, \\
  u(x, 0) = \phi(x) = \begin{cases} 
1, & x < 0 \\
0, & x > 0
\end{cases}
\end{cases}
$$

and let’s find a weak solution using the previous analysis. From the method of characteristics for first-order quasi-linear PDEs, we know that we have an implicit solution where $u$ is smooth, namely $u = \phi(x - ut)$. Moreover $u$ is constant along the projected characteristic curves given by $x_r(t) = \phi(r)t + r$.

- If $r < 0$, then $\phi(r) = 1$, which implies that the characteristic curves are
  $$
x_r(t) = t + r, \ r < 0,
$$
  and the solution $u = u_-$ should equal 1 along those curves,

- If $r > 0$, then $\phi(r) = 0$, which implies that the characteristic curves are
  $$
x_r(t) = r, \ r > 0,
$$
  and the solution $u = u_+$ should equal 0 along those curves.

Furthermore, at the jump, we have the Rankine-Hugoniot jump condition:

$$
\frac{(u_-)^2}{2} - \frac{(u_+)^2}{2} = \xi'(t)(u_- - u_+),
$$

which reduces to

$$
\xi'(t) = \frac{1}{2}.
$$

Moreover, the curve $x = \xi(t)$ contains the point $(x, t) = (0, 0)$ (we indeed have a discontinuity for the initial function at $x = 0$). Therefore the curve of discontinuity is given by $x = \frac{t}{2}$ and the weak solution is:

$$
u(x, t) = \begin{cases} 
1, & x < \frac{t}{2} \\
0, & x > \frac{t}{2}
\end{cases}
$$

which means that the discontinuity (shock) is moving at speed $1/2$. 

5
Problem 4. Consider the Burgers’ equation
\[
\begin{cases}
u_t + uu_x = 0, & x \in \mathbb{R}, \; t > 0, \\
u(x,0) = \phi(x) = \begin{cases} 0, & x < 0 \\
\frac{x}{\epsilon}, & 0 < x < \epsilon \\
1, & x > \epsilon
\end{cases}
\end{cases}
\] (31)

Let’s solve the equation. Once again, we know that $u$ is constant along the projected characteristic curves $x_r(t) = \phi(t)t + r$.

- If $r < 0$, then $\phi(r) = 0$, which implies that the characteristic curves are
  \[x_r(t) = r, \; r < 0,\] (32)
  and the solution $u(x,t) = 0$ along those curves,

- If $0 < r < \epsilon$, then $\phi(r) = \frac{r}{\epsilon}$, which implies that the characteristic curves are
  \[x_r(t) = \frac{r}{\epsilon} + r, \; 0 < r < \epsilon,\] (33)
  and the solution $u(x,t) = \frac{r}{\epsilon} = \frac{x}{t + \epsilon}$ along those curves,

- If $r > \epsilon$, then $\phi(r) = 1$, which implies that the characteristic curves are
  \[x_r(t) = t + r, \; r > \epsilon,\] (34)
  and the solution $u(x,t) = 1$ along those curves.
For $t \geq 0$, the characteristic curves do not intersect one another and the solution is then defined as

$$u(x,t) = \begin{cases} 
0, & x < 0 \\
\frac{x}{t + \epsilon}, & 0 < x < t + \epsilon \\
1, & x > t + \epsilon
\end{cases}$$  \hspace{1cm} (35)$$

Figure 2: Characteristic curves do not intersect

Now as $\epsilon \to 0$, it converges to

$$v(x,t) = \begin{cases} 
0, & x < 0 \\
\frac{x}{t}, & 0 < x < t \\
1, & x > t
\end{cases}$$  \hspace{1cm} (36)$$
Figure 3: Characteristic curves do not intersect, rarefaction wave

$v$ is indeed a weak solution of the Burgers’ equation with $\phi(x) = H(x)$. To verify this, we use the characterization given in Problem 3., meaning that we consider here the domains $\Omega_1 = x < 0$, $\Omega_2 = 0 < x < t$ and $\Omega_3 = x > t$, corresponding to the domains where $v$ is $C^1$. We easily verify that the PDE holds pointwise in $\Omega_1 \cup \Omega_2 \cup \Omega_3$, that the initial condition holds at $t = 0$ and that we have the Rankine-Hugoniot jump condition at each interface ($v$ is continuous here, so there is nothing to verify).

If we are looking for another weak solution which is piecewise constant, using the same procedure as in Problem 3., we get another solution given by

$$w(x, t) = \begin{cases} 
0, & x < \frac{t}{2} \\
1, & x > \frac{t}{2}
\end{cases} \quad (37)$$
However, $w$ does not satisfy the entropy condition since

$$
\begin{align*}
  f'(w_-) &= w_- = 0, \\
  f'(w_+) &= w_+ = 1, \\
  \text{and} \\
  f'(u_-) &= u_- = 1, \\
  f'(u_+) &= u_+ = 0,
\end{align*}
$$

while the solution $u$ of Problem 3. satisfies it since

$$
\begin{align*}
  \frac{f(w_-) - f(w_+)}{w_- - w_+} &= \frac{1}{2}, \\
  \frac{f(u_-) - f(u_+)}{u_- - u_+} &= \frac{1}{2}.
\end{align*}
$$