MATH 220: PROBLEM SET 1, SOLUTIONS
DUE THURSDAY, OCTOBER 1, 2009

Problem 1. *Classify the following PDEs by degree of non-linearity (linear, semilinear, quasilinear, fully nonlinear):*

(1) $(\cos x)\, u_x + u_y = u^2$.
(2) $u\, u_{tt} = u_{xx}$.
(3) $u_x - e^x u_y = \cos x$.
(4) $u_{tt} - u_{xx} + e^u u_x = 0$.

Solution. They are: (1) semilinear, (2) quasilinear, (3) linear, (4) semilinear.

Problem 2.

(1) *Solve*

$$u_x + (\sin x) u_y = y, \; u(0, y) = 0.$$

(2) *Sketch the projected characteristic curves for this PDE.*

Solution. The characteristic ODEs are

$$\frac{dx}{ds} = 1, \; \frac{dy}{ds} = \sin x, \; \frac{dz}{ds} = y.$$

We first solve the x ODE, substitute the solution into the y ODE, and then substitute the solution into the z ODE. So:

$x(r, s) = s + c_1(r)$

$\dfrac{dy}{ds} = \sin(s + c_1(r)) \Rightarrow y(r, s) = -\cos(s + c_1(r)) + c_2(r)$

$\dfrac{dz}{ds} = -cos(s + c_1(r)) + c_2(r) \Rightarrow z(r, s) = -\sin(s + c_1(r)) + c_2(r)s + c_3(r)$.

The initial condition is that the characteristic curves go through

$$\{(0, r, 0) : \; r \text{ arbitrary}\}$$

at $s = 0$, i.e. that

$$(0, r, 0) = (c_1(r), -\cos(c_1(r)) + c_2(r), -\sin(c_1(r)) + c_3(r)).$$

Thus, $c_1(r) = 0$, $-1 + c_2(r) = r$. i.e. $c_2(r) = r + 1$, and $c_3(r) = 0$, so the solution of the characteristic ODEs satisfying the initial conditions is

$$(x, y, z) = (s, -\cos s + r + 1, -\sin s + (r + 1)s).$$

We need to invert the map $(r, s) \mapsto (x(r, s), y(r, s))$, i.e. express (r, s) in terms of (x, y). This gives $s = x$, and $r = y + \cos s - 1 = y + \cos x - 1$. The solution of the PDE is thus

$$u(x, y) = z(r(x, y), s(x, y)) = -\sin x + (y + \cos x)x.$$

The projected characteristic curves are the curves along which r is constant, i.e. they are $y = -\cos x + C$, C a constant (namely $r + 1$).

Problem 3.

(1) *Solve*

$$yu_x + xu_y = 0,\ u(0,y) = e^{-y^2}.$$

(2) *In which region is u uniquely determined?*

Solution. This is a homogeneous linear PDE with no first order term, so its solutions are functions which are constant along the projected characteristic curves, i.e. the integral curves of the vector field $V(x,y) = (y,x)$. Note also that the initial curve, the y-axis, is characteristic at exactly one point, namely the origin, where V vanishes. Elsewhere along the y axis $V(0,y) = (y,0)$ which is not tangent to the y-axis.

The characteristic equations in this case are

$$\frac{dx}{ds} = y,\ \frac{dy}{ds} = x,\ \frac{dz}{ds} = 0.$$

The z ODE is trivial: $z = c_3(r)$. One can find the solution of the (x,y) ODEs either by obtaining a second order ODE for x:

$$\frac{d^2x}{ds^2} = \frac{dy}{ds} = x,$$

whose solutions are $x = c_1(r)e^s + c_2(r)e^{-s}$. As $y = \frac{dx}{ds}$, this gives

$$(x,y,z) = (c_1(r)e^s + c_2(r)e^{-s}, c_1(r)e^s - c_2(r)e^{-s}, c_3(r)).$$

Thus, $x+y = 2c_1(r)e^s$, $x-y = 2c_2(r)e^{-s}$, so $x^2-y^2 = (x+y)(x-y) = 4c_1(r)c_2(r)$, i.e. is a constant along the projected characteristic curves. In other words, the projected characteristic curves are $x^2 - y^2 = C$, C a constant, and the solution is a function that is constant along these. One has to be slightly careful, as the same value of C corresponds to two characteristic curves, see the argument two paragraphs below concerning the sign of r. In particular, any function f of $x^2 - y^2$ will solve the PDE. As we want $f(x^2 - y^2) = u(x,y)$ to satisfy $u(0,y) = e^{-y^2}$, we deduce that $f(-y^2) = e^{-y^2}$ for all real y, i.e. $f(t) = e^t$ for $t \leq 0$. Note that $f(t)$ is not defined by this restriction for $t > 0$. So one obtains that $u(x,y) = f(x^2 - y^2)$ solves the PDE where $f(t) = e^t$ for $t \leq 0$, $f(t)$ arbitrary for $t > 0$.

In particular, the solution is *not unique* where $x^2 - y^2 > 0$, i.e. where $|x| > |y|$. This is exactly the region in which the characteristic curves do not approach the y axis.

To see how our usual method of substituting in the initial conditions works, note that the initial data curve is $(0, r, e^{-r^2})$, so at $s = 0$ we get $c_1(r) + c_2(r) = 0$, $c_1(r) - c_2(r) = r$, $c_3(r) = e^{-r^2}$, so the solution of the characteristic ODEs taking into account the initial conditions is

$$(x,y,z) = (\frac{r}{2}(e^s - e^{-s}), \frac{r}{2}(e^s + e^{-s}), e^{-r^2}) = (r\sinh s, r\cosh s, e^{-r^2}).$$

As $\cosh^2 s - \sinh^2 s = 1$, we deduce that $y^2 - x^2 = r^2$ along the projected characteristic curves. This gives that $|y| \geq |x|$ in the region where the projected characteristic curves crossing the y axis reach. In this region, $r = \pm\sqrt{y^2 - x^2}$, with the the sign $\pm$ agreeing with the sign of y (i.e. is $+$ where $y > 0$). In any case, the solution is $u(x,y) = e^{-r^2} = e^{x^2-y^2}$ in $|y| \geq |x|$. Note that this method does *not* give the solution in the region $|y| < |x|$, as the projected characteristic curves never reach the region. Note also that there is no neighborhood of the origin in which this method gives u; this is because the y-axis is characteristic for this PDE at the origin.

A simpler way of finding the projected characteristic curves is to parameterize them by x or y. In the former case, one gets

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{x}{y},$$

so $\int y\,dy = \int x\,dx$, i.e. $y^2 = x^2 + C$. Again, C is a parameter.

Problem 4.

(1) *Solve* $u_x + u_t = u^2,\ u(x,0) = e^{-x^2}$.
(2) *Show that there is* $T > 0$ *such that* u *blows up at time* T, *i.e.* u *is continuously differentiable for* $t \in [0,T)$, x *arbitrary, but for some* x_0, $|u(x_0,t)| \to \infty$ *as* $t \to T-$. *What is* T*?*

Solution. The characteristic ODEs are

$$\frac{dx}{ds} = 1,\ \frac{dt}{ds} = 1,\ \frac{dz}{ds} = z^2.$$

The solution is

$$\begin{aligned} &x(r,s) = s + c_1(r),\\ &t(r,s) = s + c_2(r),\\ &-z^{-1} = s + c_3(r) \Rightarrow z = \frac{-1}{s + c_3(r)}. \end{aligned}$$

The initial conditions give that at $s = 0$, $(x,t,z) = (r, 0, e^{-r^2})$, so $c_1(r) = r$, $c_2(r) = 0$, $c_3(r) = -e^{r^2}$. Thus,

$$(x,t,z) = (s + r, s, \frac{-1}{s - e^{r^2}}).$$

Inverting the map $(r,s) \mapsto (x(r,s), t(r,s))$ yields $s = t$, $r = x - s = x - t$, so

$$u(x,t) = z(r(x,t), s(x,t)) = \frac{-1}{t - e^{(x-t)^2}}.$$

Note that the denominator vanishes only if $t = e^{(x-t)^2}$, and $(x-t)^2 \geq 0$, so the denominator can only vanish if $t \geq 1$. In particular, u is a C^1, indeed C^∞, function on $\mathbb{R}_x \times [0,1)_t$. On the other hand, for $x = 1$, as $t \to 1-$, $u(x,t) = \frac{-1}{t - e^{(1-t)^2}} \to +\infty$, i.e. the solution blows up at $T = 1$ (at $x_0 = 1$).

Problem 5. *Solve*

$$u_t + uu_x = 0,\ u(x,0) = -x^2$$

for $|t|$ *small.*

Solution. We parameterize the x-axis as $\Gamma(r) = (r,0)$, and note that the vector field $(z,1)$ is not tangent to Γ at any point regardless of the value of z, so this is a non-characteristic initial value problem. The characteristic equations are

$$\begin{aligned} &\frac{\partial t}{\partial s} = 1,\ t(r,0) = 0,\\ &\frac{\partial x}{\partial s} = z,\ x(r,0) = r,\\ &\frac{\partial z}{\partial s} = 0,\ z(r,0) = -r^2. \end{aligned}$$

The solution is

$$t(r,s) = s,$$
$$z(r,s) = -r^2,$$
$$x(r,s) = -r^2 s + r.$$

Thus, $s = t$, and $tr^2 - r + x = 0$, so if $t = 0$ then $r = x$, and if $t \neq 0$ then r solves

$$r = \frac{1 \pm \sqrt{1-4tx}}{2t}.$$

The choice of the sign is dictated by $r = x$ when $t = 0$ (i.e. by taking the limit as $t \to 0$ using, say, L'Hospital's rule), so one needs the negative sign, and

$$r = \frac{1 - \sqrt{1-4tx}}{2t}.$$

The solution is then

$$u(x,t) = -R(x,t)^2 = -\frac{(1-\sqrt{1-4tx})^2}{4t^2},\ t \neq 0,$$

and $u(x,0) = -x^2$.

Problem 6. *Consider the PDE*

$$u_t + uu_x = 0,\ u(x,0) = \phi(x).$$

Suppose that $\phi' \geq -C$, *where* $C > 0$. *Show that the PDE has a* C^1 *solution on* $\mathbb{R}_x \times [0, \frac{1}{C})_t$. *Show also that for* $t \in [0, \frac{1}{C})$, u_x *satisfies the estimate*

$$u_x(x,t) \geq \frac{1}{t - C^{-1}}.$$

(Note that the right hand side is negative!) (Hint: Consider the difference quotients $\frac{u(\xi_2(t),t)-u(\xi_1(t),t)}{\xi_2(t)-\xi_1(t)}$, *where* $x = \xi_j(t)$ *are the projected characteristic curves emanating from the point* x_j *on the* x *axis.)*

Solution. As discussed in class and in the lecture notes, the classical solutions of Burger's equations are constant along the projected characteristic curves, and the latter are given implicitly by $x = \phi(r)t + r$. In particular, the classical (C^1) solutions exist until two characteristic curves intersect. To see when this happens, consider the characteristic curves given by $x = \xi_j(t)$, $j = 1, 2$, with $\xi_j(t) = \phi(r_j)t + r_j$, i.e. these are the curves through $(r_j, 0)$. If these intersect at (x,t) then $\xi_1(t) = \xi_2(t)$, i.e.

$$(\phi(r_2) - \phi(r_1))t = r_1 - r_2.$$

By the mean value theorem, $\phi(r_2) - \phi(r_1) = (r_2 - r_1)\phi'(r_0)$ for some $r_0 \in (r_1, r_2)$, so at the point of intersection $-\phi'(r_0)t = 1$ (as $r_1 \neq r_2$). So $\phi'(r_0) \neq 0$ and $t = \frac{1}{-\phi'(r_0)}$. As we want $t \geq 0$, we must have $-\phi'(r_0) > 0$ and as $-\phi' \leq C$, we deduce that $t \geq C^{-1}$, as claimed. So the solution exists on $\mathbb{R}_x \times [0, \frac{1}{C})_t$.

For the second part, consider $t \in [0, \frac{1}{C})$. Then

$$\frac{u(\xi_2(t),t) - u(\xi_1(t),t)}{\xi_2(t) - \xi_1(t)} = \frac{\phi(r_2) - \phi(r_1)}{(\phi(r_2) - \phi(r_1))t + r_2 - r_1} = \frac{1}{t + \frac{r_2 - r_1}{\phi(r_2) - \phi(r_1)}}$$

if $\phi(r_2) \neq \phi(r_1)$, and 0 otherwise. But by the mean value theorem, as above, $\phi(r_2) - \phi(r_1) = (r_2 - r_1)\phi'(r_0)$ for some $r_0 \in (r_1, r_2)$.

Suppose $r_2 > r_1$, and $\phi(r_2) \geq \phi(r_1)$. Then

$$0 \leq \frac{u(\xi_2(t),t) - u(\xi_1(t),t)}{\xi_2(t) - \xi_1(t)} \leq \frac{\phi(r_2) - \phi(r_1)}{r_2 - r_1},$$

while if $r_2 > r_1$ and $\phi(r_2) < \phi(r_1)$ then $\frac{r_2 - r_1}{\phi(r_2) - \phi(r_1)} = \frac{1}{\phi'(r_0)}$, $\phi'(r_0) \in [-C, 0)$, so $\frac{r_2 - r_1}{\phi(r_2) - \phi(r_1)} \leq -C^{-1}$, and so

$$0 > \frac{u(\xi_2(t),t) - u(\xi_1(t),t)}{\xi_2(t) - \xi_1(t)} \geq \frac{1}{t - C^{-1}}.$$

Thus, whether $\phi(r_2) > \phi(r_1)$, $\phi(r_2) = \phi(r_1)$ or $\phi(r_2) < \phi(r_1)$, we deduce that

$$\frac{u(\xi_2(t),t) - u(\xi_1(t),t)}{\xi_2(t) - \xi_1(t)} \geq \frac{1}{t - C^{-1}}.$$

But

$$\frac{\partial u}{\partial x}(\xi_1(t),t) = \lim_{r_2 \to r_1} \frac{u(\xi_2(t),t) - u(\xi_1(t),t)}{\xi_2(t) - \xi_1(t)},$$

so $\frac{\partial u}{\partial x}(\xi_1(t),t) \geq \frac{1}{t - C^{-1}}$ as claimed.